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Finite Gabor systems and uncertainty principle for block sliding discrete Fourier transform

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Abstract. In this paper, we study the finite Gabor system for oversampling schemes. A characterization of dual finite Gabor tight frame using discrete time Zak transform is given. Also, a method to calculate the coefficients of the finite Gabor system expansion in the case of oversampling and a necessary and sufficient condition for the existence of biorthogonal pair of Riesz basis in $l^2(\mathbb{Z}_{\mathfrak{L}})$ is given. Further, we introduce the notion of block sliding discrete Fourier transform (BSDFT) which reduces the computational complexity and give uncertainty principle for BSDFT. An uncertainty principle for two finite Parseval Gabor frames in terms of sparse representations is given. Finally, using the notion of numerical sparsity, an uncertainty principle for finite Gabor frames is given.

1. Introduction

The most extensively used tool in signal processing is Fourier transform which helps in analyzing the frequency information of a time series $\mathfrak{x}(t)$ by transforming it from the time domain into the frequency domain. In 1822, Fourier defined the notion of Fourier transform, a representation for non-periodic signals in terms of an integral of weighted sine and cosine functions. Since signals $\mathfrak{x}_{\mathfrak{k}}$ received with the help of a data acquisition system are primarily sampled at discrete time intervals, rather than continuous time intervals, $\mathfrak{x}_{\mathfrak{k}}$ can be transformed into the frequency domain using discrete time Fourier transform (DTFT). For details on Fourier transform, one may read [20, 47].
Let $\mathfrak{L} \in \mathbb{N}$ and define $\mathcal{E}_{\mathfrak{j}}(\mathfrak{m}) = \frac{1}{\sqrt{\mathfrak{L}}} e^{2\pi i \mathfrak{j}\mathfrak{m}/\mathfrak{L}} = w_{\mathfrak{m},\mathfrak{j}}$, $0 \leq \mathfrak{j}, \mathfrak{m} \leq \mathfrak{L} - 1$. Then discrete Fourier matrix is $\mathfrak{F} = [\mathcal{E}_0\, \mathcal{E}_1\, \mathcal{E}_2 \ldots \mathcal{E}_{\mathfrak{L}-1}]$ with discrete Fourier transform (DFT) is given by $\mathfrak{F}^*\mathfrak{z}(\mathfrak{j}) = \langle \mathfrak{z}, \mathcal{E}_{\mathfrak{j}} \rangle = \frac{1}{\sqrt{\mathfrak{L}}} \sum_{\mathfrak{m}=0}^{\mathfrak{L}-1} \mathfrak{z}(\mathfrak{m}) e^{-2\pi i \mathfrak{j}\mathfrak{m}/\mathfrak{L}}$, $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$ and the inverse DFT is defined as $\mathfrak{F}\mathfrak{z}(\mathfrak{m}) = \frac{1}{\sqrt{\mathfrak{L}}} \sum_{\mathfrak{j}=0}^{\mathfrak{L}-1} \mathfrak{z}(\mathfrak{j}) e^{2\pi i \mathfrak{j}\mathfrak{m}/\mathfrak{L}}$, $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$.

Note that $\{\mathcal{E}_{\mathfrak{j}}\}$ is an orthonormal basis of $l^2(\mathbb{Z}_{\mathfrak{L}})$ and reconstruction property of $\mathfrak{F}$ is $\mathfrak{z} = \mathfrak{F}\mathfrak{F}^*\mathfrak{z} = \sum_{\mathfrak{j}=0}^{\mathfrak{L}-1} \langle \mathfrak{z}, \mathcal{E}_{\mathfrak{j}} \rangle \mathcal{E}_{\mathfrak{j}}$ with $\|\mathfrak{z}\|^2 = \sum_{\mathfrak{j}=0}^{\mathfrak{L}-1} |\langle \mathfrak{z}, \mathcal{E}_{\mathfrak{j}} \rangle|^2$. We shall denote discrete Fourier transform by $\mathcal{F} = \mathfrak{F}^*$.

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Zak transform was first appeared in the work on eigenfunction expansions related to Schrödinger operators with periodic potentials done by Gel'fand. This was the reason why Zak transform was known by Gel'fand mapping. In 1967, Zak [51] rediscovered this transform in the form of $k - q$ representation in order to construct a quantum mechanical representation for the motion of a Bloch electron in the presence of a magnetic or electric field. As far as applications are concerned, Zak transform is used in the study of coherent states representation in quantum field theory [32] and in digital data transmission [10]. Further, it is useful in a time-frequency representation for time-continuous signals, particularly in signal theory. In 1991, Zeevi and Gertner [49] employed finite Zak transform for the analysis of spatially nonstationary images. Zak transform was also used in the context of Gabor representation problem in order to study the orthogonality and the completeness of the Gabor frames when $ab = 1$. Further, Heil [14] defined discrete time Zak transform (DTZT) and analyze discrete time Weyl-Heisenberg frames in $l^2(\mathbb{Z})$. In 2019, Poumai et al. [42] introduced the notion of multidimensional discrete time Zak transform (MDTZT) which is used in discrete sampling of multivariate discrete time signals. For more details on Zak transform, one may refer [8, 11, 25, 31, 50] and for Gabor systems, one may see [19, 26].
Let $\mathfrak{M}, \mathfrak{N} \in \mathbb{N}$ and $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$. Finite Zak transform on $l^2(\mathbb{Z}_{\mathfrak{L}})$ is a map $\mathcal{Z} : l^2(\mathbb{Z}_{\mathfrak{L}}) \to l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}})$ given by $\mathcal{Z}\mathfrak{z}(\mathfrak{m}, \mathfrak{n}) = \sum_{\mathfrak{r}=0}^{\mathfrak{N}-1} \mathfrak{z}(\mathfrak{m} + \mathfrak{r}\mathfrak{M})e^{-2\pi i \mathfrak{r}\mathfrak{n}/\mathfrak{N}}$, where $(\mathfrak{m}, \mathfrak{n}) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}$ and $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Indeed, finite Zak transform $\mathcal{Z}$ is a square matrix of order $\mathfrak{L} \times \mathfrak{L}$. Note that $\frac{1}{\sqrt{\mathfrak{N}}}\mathcal{Z}$ is a unitary operator. For example, for $\mathfrak{N} = 2$ and $\mathfrak{M} = 3$, the finite Zak transform is given by

$$\mathcal{Z} = \begin{pmatrix} w_{0,0} & 0 & 0 & w_{0,1} & 0 & 0 \\ w_{1,0} & 0 & 0 & w_{1,1} & 0 & 0 \\ 0 & w_{0,0} & 0 & 0 & w_{0,1} & 0 \\ 0 & w_{1,0} & 0 & 0 & w_{1,1} & 0 \\ 0 & 0 & w_{0,0} & 0 & 0 & w_{0,1} \\ 0 & 0 & w_{1,0} & 0 & 0 & w_{1,1} \end{pmatrix}.$$

The process of obtaining a digital signal from an analog signal is known as sampling. The sampled version of an analog signal $\mathfrak{f} \in L^2(\mathbb{R})$ can be seen as a sequence of complex numbers $\mathfrak{z}(\mathfrak{n}) = \mathfrak{f}(\mathfrak{n}\mathcal{P}_\mathfrak{s})$, for $\mathfrak{n} \in \mathbb{Z}$, where $\mathcal{P}_\mathfrak{s}$ is a positive constant known as sampling period which gives the amount of time between two samples and $\mathfrak{f}_\mathfrak{s} = \frac{1}{\mathcal{P}_\mathfrak{s}}$ is known as sampling frequency. Among various ways of filling or interpolating the missing information between the sample points, a significant one is given by $\mathfrak{f}(t) = \sum_{\mathfrak{n}\in\mathbb{Z}} \mathfrak{z}(\mathfrak{n})\mathfrak{g}(t - \mathfrak{n}\mathcal{P}_s)$ with impulse function $\mathfrak{g}(t)$, $t \in \mathbb{R}$. Multirate digital signal processing is an important aspect of signal processing which lies in the fact that prior to conversion of a digital signal to analog signal, it is essential to alter the sampling rate of a signal successively so as to upsurge the efficacy of several operations of signal processing. For details, one may invoke [46, 47]. The process of multirate digital signal processing consists of decimation and expansion. The reduction in sampling rate by factor $\mathfrak{N} \in \mathbb{N}$ is known as decimation or downsampling.

Let $\mathfrak{L}, \mathfrak{N}, \mathfrak{M} \in \mathbb{N}$ with $\mathfrak{L} = \mathfrak{N}\times\mathfrak{M}$. Then a map $D_\mathfrak{N} : l^2(\mathbb{Z}_\mathfrak{L}) \to l^2(\mathbb{Z}_\mathfrak{M})$ defined by $D_\mathfrak{N}\mathfrak{z}(\mathfrak{n}) = \mathfrak{z}(\mathfrak{n}\mathfrak{N})$, $\mathfrak{z} \in l^2(\mathbb{Z}_\mathfrak{L})$, $\mathfrak{n} \in \mathbb{Z}_\mathfrak{M}$ is known as downsampling operator. In general, the downsampled signal does not hold the entire information about the original signal $\mathfrak{z}$ and is exercised substantially in filter banks. Downsampling is often preceded by filtering in order to extract the conformant frequency bands. Circular convolution with respect to filter $\nu \in l^2(\mathbb{Z}_\mathfrak{L})$ is an operator $G_\nu : l^2(\mathbb{Z}_\mathfrak{L}) \to l^2(\mathbb{Z}_\mathfrak{L})$ given by $G_\nu(\mathfrak{z})(\mathfrak{n}) = \mathfrak{z} * \nu(\mathfrak{n}) = \sum_{k=0}^{\mathfrak{L}-1} \mathfrak{z}(k)\nu(\mathfrak{n} - k)$. The adjoint of G_ν is given by $G_\nu^*\mathfrak{z} = G_{\tilde{\nu}}\mathfrak{z} = \mathfrak{z} * \tilde{\nu}$, where $\tilde{\nu}(\mathfrak{n}) = \overline{\nu(-\mathfrak{n})}$. Consider a sequence $\nu \in l^2(\mathbb{Z}_\mathfrak{L})$ and let $w(k) = \mathfrak{z} * \tilde{\nu}(k\mathfrak{N}) = D_\mathfrak{N}(\mathfrak{z} * \tilde{\nu})(k) = D_\mathfrak{N}G_{\tilde{\nu}}\mathfrak{z}(k)$. The operator $G_\chi : l^2(\mathbb{Z}_\mathfrak{L}) \to l^2(\mathbb{Z}_\mathfrak{L})$ given by $G_\chi(\mathfrak{z}) = \mathfrak{z} * \chi$ ($\chi \in l^2(\mathbb{Z}_\mathfrak{L})$) is a convolution operator with impulse response χ which is usually known as filter and the convolution with the impulse response is known as filtering. Note that in discrete sampling, filtering with the impulse response $\tilde{\nu}$ is followed by downsampling by factor $\mathfrak{N}$, whereas the sampling rate increases by factor $\mathfrak{N} \in \mathbb{N}$ in filling or upsampling. A map $U_\mathfrak{N} : l^2(\mathbb{Z}_\mathfrak{M}) \to l^2(\mathbb{Z}_\mathfrak{L})$ given by $U_\mathfrak{N}y(\mathfrak{n}) = \begin{cases} y(\frac{\mathfrak{n}}{\mathfrak{N}}), & \text{if } \mathfrak{N}|\mathfrak{n} \ (i.e.\ \mathfrak{N} \text{ divides } \mathfrak{n}), \\ 0, & \text{if } \mathfrak{N} \nmid \mathfrak{n} \ (i.e.\ \mathfrak{N} \text{ does not divide } \mathfrak{n}), \end{cases}$
where $y \in l^2(\mathbb{Z}_\mathfrak{M})$, $\mathfrak{n} \in \mathbb{Z}_\mathfrak{L}$ is said to be upsampling operator.
In this case, we successively put $\mathfrak{N} - 1$ zeros amongst successive values of the input $\{y(\mathfrak{n})\}$ and upsampling is often followed by filtering. In discrete interpolation, we obtain a sequence $w \in l^2(\mathbb{Z}_\mathfrak{L})$ through upsampling

by factor $\mathfrak{N}$ which is followed by filtering with impulse response ν. That is, $w(\mathfrak{l}) = G_\nu U_{\mathfrak{N}} y(\mathfrak{l}) = \nu * U_{\mathfrak{N}} y(\mathfrak{l})$, $y \in l^2(\mathbb{Z}_{\mathfrak{M}})$ and $\mathfrak{l} \in \mathbb{Z}_{\mathfrak{L}}$. Sampling followed by interpolation can be explained as $x = G_\nu U_{\mathfrak{N}} D_{\mathfrak{N}} G_{\tilde{\nu}} \mathfrak{z} = \nu * U_{\mathfrak{N}} D_{\mathfrak{N}} \mathfrak{z} * \tilde{\nu}$, for $x, \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Whereas, interpolation followed by sampling can be described as $f = D_{\mathfrak{N}} G_{\tilde{\nu}} G_\nu U_{\mathfrak{N}} g = D_{\mathfrak{N}} \tilde{\nu} * \nu * U_{\mathfrak{N}} g$, for $f, g \in l^2(\mathbb{Z}_{\mathfrak{M}})$. As usual, delta function is given as

$$\delta(\mathfrak{n}) = \begin{cases} 1, & \text{if } \mathfrak{n} = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{n} \in \mathbb{Z}_{\mathfrak{L}}$. The matrix representation of downsampling operator is $D_{\mathfrak{N}} = \left[d_{\mathfrak{m},\mathfrak{l}}\right] = \left[\delta(\mathfrak{m}\mathfrak{N} - \mathfrak{l})\right]$, where $0 \le \mathfrak{m} \le \mathfrak{M} - 1$ and $0 \le \mathfrak{l} \le \mathfrak{L} - 1$. Upsampling matrix $U_{\mathfrak{N}} = D_{\mathfrak{N}}^*$ (that is the adjoint of $D_{\mathfrak{N}}$) is given by $U_{\mathfrak{N}} = \left[u_{\mathfrak{l},\mathfrak{m}}\right] = \left[\delta(\mathfrak{l} - \mathfrak{m}\mathfrak{N})\right]$, where $0 \le \mathfrak{m} \le \mathfrak{M} - 1$ and $0 \le \mathfrak{l} \le \mathfrak{L} - 1$. Next, we give the product of $D_{\mathfrak{N}}$ and $U_{\mathfrak{N}}$ as $D_{\mathfrak{N}} U_{\mathfrak{N}} = \left[\sum_{\mathfrak{l}=0}^{\mathfrak{L}-1} \delta(\mathfrak{m}\mathfrak{N} - \mathfrak{l})\delta(\mathfrak{l} - \mathfrak{s}\mathfrak{N})\right] = \mathbf{I}_{\mathfrak{N}}$, where $0 \le \mathfrak{m}, \mathfrak{s} \le \mathfrak{M} - 1$. Moreover, $U_{\mathfrak{N}} D_{\mathfrak{N}} = \left[\sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \delta(\mathfrak{l} - \mathfrak{m}\mathfrak{N})\delta(\mathfrak{m}\mathfrak{N} - \mathfrak{j})\right]$, where $0 \le \mathfrak{l}, \mathfrak{j} \le \mathfrak{L} - 1$. Let $\nu \in l^2(\mathbb{Z}_{\mathfrak{L}})$ and $\mathfrak{s} \in \mathbb{Z}_{\mathfrak{L}}$. The translation operator $T_{\mathfrak{s}} : l^2(\mathbb{Z}_{\mathfrak{L}}) \to l^2(\mathbb{Z}_{\mathfrak{L}})$ is given by $T_{\mathfrak{s}}\nu(\mathfrak{l}) = \nu(\mathfrak{l} - \mathfrak{s})$, $0 \le \mathfrak{l} \le \mathfrak{L} - 1$. And the modulation operator $E_{\frac{\mathfrak{s}}{\mathfrak{L}}} : l^2(\mathbb{Z}_{\mathfrak{L}}) \to l^2(\mathbb{Z}_{\mathfrak{L}})$ is given by $E_{\frac{\mathfrak{s}}{\mathfrak{L}}}\nu(\mathfrak{l}) = e^{2\pi i \frac{\mathfrak{s}\mathfrak{l}}{\mathfrak{L}}}\nu(\mathfrak{l})$, $0 \le \mathfrak{l} \le \mathfrak{L} - 1$. Notice that

$$\frac{1}{\mathfrak{N}} \sum_{\mathfrak{j}=0}^{\mathfrak{N}-1} E_{\frac{-\mathfrak{j}}{\mathfrak{N}}} \mathfrak{z}(\mathfrak{n}) = U_{\mathfrak{N}} D_{\mathfrak{N}} \mathfrak{z}(\mathfrak{n}), \ \mathfrak{z} \in l^2(\mathbb{Z}).$$

For more details on sampling theory, see [1, 2]. Let $\mathfrak{D}$ be a set, characteristic function $\chi_{\mathfrak{D}}$ is given by

$$\chi_{\mathfrak{D}}(\mathfrak{t}) = \begin{cases} 1, & \text{if } \mathfrak{t} \in \mathfrak{D}, \\ 0, & \text{if } \mathfrak{t} \notin \mathfrak{D}. \end{cases}$$

2. Finite Gabor System

For a continuous-time function $\mathfrak{f}$, Gabor expansion is given by $\mathfrak{f}(\tau) = \sum_{\mathfrak{m},\mathfrak{k}} \alpha_{\mathfrak{m},\mathfrak{k}}\, w_{\mathfrak{m},\mathfrak{k}}(\tau)$, with $w_{\mathfrak{m},\mathfrak{k}}(\tau) \triangleq w(\tau - \mathfrak{m}P)e^{i\mathfrak{k}\Theta\tau}$, $\mathfrak{m}, \mathfrak{k} = 0, \pm 1, \pm 2, \ldots$, where $w(\tau)$ is a window function with an effective width P_1, P is a shift parameter which controls the window's discrete shift along τ and $e^{i\mathfrak{k}\Theta\tau}$ is a Fourier kernel sampled at a constant frequency interval Θ. The choice of parameters P_1, P and Θ is very imperative which can influence the existence, uniqueness, convergence properties and the numerical stability of the resulting expansion directly. Gabor [22] proposed the classical constraints $\Theta P = 2\pi$ and $P_1 = P$ which happened to be optimal in terms of minimum sampling rate and numerical stability. In 1990, Wexler and Raz [48] converted the Gabor representation into a discrete and finite format which was best suited for numerical implementation. Heil [27] employed discrete Zak transform in order to analyze and construct discrete time Weyl-Heisenberg (DTWH) frames which are bases and are generated by sequences with good decay. In 2019, Poumai et al. [41] used discrete time Zak transform (DTZT) in order to characterize the dual discrete time Weyl-Heisenberg tight (DDTWHT) frames, DTWH frames and tight DTWH frames based on oversampling schemes. We begin this section with the definition of finite Gabor frame for $l^2(\mathbb{Z}_{\mathfrak{L}})$.

Let $\mathfrak{M}, \mathfrak{M}_1, \mathfrak{N}, \mathfrak{N}_1 \in \mathbb{N}$ and $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}_1 = \mathfrak{M}_1 \times \mathfrak{N}$. Define $\nu_{\mathfrak{m},\mathfrak{k}} = E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{k}\mathfrak{M}_1}\nu$, $(\mathfrak{m}, \mathfrak{k}) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}$ and $\nu \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Family $\{\nu_{\mathfrak{m},\mathfrak{k}}\}$ is the finite Gabor system on $l^2(\mathbb{Z}_{\mathfrak{L}})$. Let $\mathscr{H}$ be a real (or complex) separable Hilbert space with inner product $\langle ., . \rangle$.

For a countable or finite set Ξ, a sequence $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ in a Hilbert space $\mathscr{H}$ is said to be a frame for $\mathscr{H}$, if there exist numbers $\mathscr{A}, \mathscr{B} > 0$ such that

$$\mathscr{A}\|\mathfrak{f}\|^2 \le \sum_{\xi\in\Xi} |\langle \mathfrak{f}, \mathfrak{f}_\xi \rangle|^2 \le \mathscr{B}\|\mathfrak{f}\|^2, \text{ for all } \mathfrak{f} \in \mathscr{H}, \tag{1}$$

where the scalars (not necessarily unique) $\mathscr{A}$ and $\mathscr{B}$ are called the frame bounds. Further, $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ is tight frame if $\mathscr{A} = \mathscr{B}$ and Parseval frame if $\mathscr{A} = \mathscr{B} = 1$.

An operator $\mathcal{T} : l^2(\Xi) \to \mathcal{H}$ defined by $\mathcal{T}\{c(\xi)\}_{\xi\in\Xi} = \sum_{\xi\in\Xi} c(\xi)\mathfrak{f}_\xi$, $\{c(\xi)\}_{\xi\in\Xi} \in l^2(\Xi)$, is called the pre-frame operator (or synthesis operator) with adjoint operator $\mathcal{T}^* : \mathcal{H} \to l^2(\Xi)$ given by $\mathcal{T}^*(\mathfrak{f}) = \{\langle \mathfrak{f}, \mathfrak{f}_\xi\rangle\}$, for all $\mathfrak{f} \in \mathcal{H}$ is called as analysis operator. Furthermore, an operator $\mathcal{S} = \mathcal{T}\mathcal{T}^* : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{S}(\chi) = \sum_{\xi\in\Xi}\langle \mathfrak{f}, \mathfrak{f}_\xi\rangle \mathfrak{f}_\xi$, for all $\mathfrak{f} \in \mathcal{H}$ is called as frame operator which happens to be a positive, self-adjoint, bounded and invertible operator on $\mathcal{H}$.
On the other hand, a sequence $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ is complete in $\mathcal{H}$ and there exist constants $\mathcal{A}, \mathcal{B} > 0$ such that
$\mathcal{A}\sum_{\xi\in\Xi} |\alpha(\xi)|^2 \leq \| \sum_{\xi\in\Xi} \alpha(\xi)\mathfrak{f}_\xi\|^2 \leq \mathcal{B}\sum_{\xi\in\Xi} |\alpha(\xi)|^2$, for all $\{\alpha(\xi)\}_{\xi\in\Xi} \in l^2(\Xi)$.
A pair of families $(\{\mathfrak{f}_\xi\}_{\xi\in\Xi}, \{\mathfrak{g}_\xi\}_{\xi\in\Xi})$ in a Hilbert space $\mathcal{H}$ is a biorthogonal pair of Riesz bases if

(i) $\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ and $\{\mathfrak{g}_\xi\}_{\xi\in\Xi}$ are Riesz bases for $\mathcal{H}$.
(ii) $\langle \mathfrak{f}_\xi, \mathfrak{g}_{\xi'}\rangle = \delta(\xi - \xi')$, for $\xi, \xi' \in \Xi$.

For details on frame theory, one may see [3, 14, 16, 39, 44, 45].

Definition 2.1. *A system $\{\nu_{\mathfrak{m},\mathfrak{k}}\}$ is a finite Gabor frame for $l^2(\mathbb{Z}_{\mathfrak{L}})$ if there exist positive constants $0 < \mathcal{A} \leq \mathcal{B} < \infty$ such that*

$$\mathcal{A}\|\mathfrak{z}\|^2 \leq \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathfrak{k}=0}^{\mathfrak{N}-1} |\langle \mathfrak{z}, \nu_{\mathfrak{m},\mathfrak{k}}\rangle|^2 \leq \mathcal{B}\|\mathfrak{z}\|^2, \textit{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Decomposition of frame operator is stated as

$$\mathcal{S}(\mathfrak{z}) = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathfrak{k}=0}^{\mathfrak{N}-1}\langle \mathfrak{z}, \nu_{\mathfrak{m},\mathfrak{k}}\rangle \nu_{\mathfrak{m},\mathfrak{k}} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} E_{\frac{\mathfrak{m}}{\mathfrak{M}}} G_\nu U_{\mathfrak{M}_1} D_{\mathfrak{M}_1} G_{\tilde{\nu}} E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z},\ \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Next, we give definition of dual finite Gabor tight frame for $l^2(\mathbb{Z}_{\mathfrak{L}})$.

Definition 2.2. *Let $\mathfrak{M}, \mathfrak{M}_1, \mathfrak{N}, \mathfrak{N}_1 \in \mathbb{N}$ such that $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}_1 = \mathfrak{M}_1 \times \mathfrak{N}$. Let $\nu, w \in l^2(\mathbb{Z}_{\mathfrak{L}})$ and define $\nu_{\mathfrak{m},\mathfrak{k}} = E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{k}\mathfrak{M}_1}\nu$ and $w_{\mathfrak{m},\mathfrak{k}} = E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{k}\mathfrak{M}_1} w$ where $0 \leq \mathfrak{m} \leq \mathfrak{M} - 1$, $0 \leq \mathfrak{k} \leq \mathfrak{N} - 1$. A pair $(\{\nu_{\mathfrak{m},\mathfrak{k}}\}, \{w_{\mathfrak{m},\mathfrak{k}}\})$ is a dual finite Gabor tight frame for $l^2(\mathbb{Z}_{\mathfrak{L}})$ if*

$$\mathfrak{z} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathfrak{k}=0}^{\mathfrak{N}-1}\langle \mathfrak{z}, w_{\mathfrak{m},\mathfrak{k}}\rangle \nu_{\mathfrak{m},\mathfrak{k}}, \textit{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Decomposition of dual finite Gabor tight frame is given by

$$\mathfrak{z} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathfrak{k}=0}^{\mathfrak{N}-1}\langle \mathfrak{z}, w_{\mathfrak{m},\mathfrak{k}}\rangle \nu_{\mathfrak{m},\mathfrak{k}} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} E_{\frac{\mathfrak{m}}{\mathfrak{M}}} G_\nu U_{\mathfrak{M}_1} D_{\mathfrak{M}_1} G_{\tilde{w}} E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z},\ \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

The mutual cohorence between the two finite Gabor frames $V = \{\nu_{\mathfrak{m},\mathfrak{k}}\}$ and $W = \{w_{\mathfrak{m},\mathfrak{k}}\}$ is defined as

$$\mu = \sup |\langle \nu_{\mathfrak{m},\mathfrak{k}}, w_{\mathfrak{m},\mathfrak{k}}\rangle|.$$

Let $\mathfrak{L}, \mathfrak{N}_1, \mathfrak{M}, \mathfrak{N}, \mathfrak{M}_1 \in \mathbb{N}$ such that $\mathfrak{L} = \mathfrak{N}_1 \times \mathfrak{M} = \mathfrak{N} \times \mathfrak{M}_1$ and $\nu \in l^2(\mathbb{Z}_{\mathfrak{L}})$.
In our result, we use the following version of Zak transform

$$\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}) = \sum_{\mathfrak{r}=0}^{\mathfrak{N}-1} \nu(\mathfrak{n} + \mathfrak{r}\mathfrak{M}_1)e^{-2\pi i\mathfrak{r}\mathfrak{p}/\mathfrak{N}},\ 0 \leq \mathfrak{n} \leq \mathfrak{M}_1 - 1,\ 0 \leq \mathfrak{p} \leq \mathfrak{N} - 1.$$

Using above notations, we state a result in the form of a lemma which will used in our main results.

Lemma 2.3. *Let $\chi, y \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Then*

(a) $\mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\chi(\mathfrak{n},\mathfrak{p}) = e^{2\pi i\frac{\mathfrak{m}}{\mathfrak{M}}\mathfrak{n}}\mathcal{Z}\chi(\mathfrak{n},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1),\ 0\le\mathfrak{m}\le\mathfrak{M}-1.$

(b) $\mathcal{Z}E_{-\frac{\mathfrak{j}}{\mathfrak{M}_1}}\chi(\mathfrak{n},\mathfrak{p}) = e^{-2\pi i\frac{\mathfrak{j}}{\mathfrak{M}_1}\mathfrak{n}}\mathcal{Z}\chi(\mathfrak{n},\mathfrak{p}),\ 0\le\mathfrak{j}\le\mathfrak{M}_1-1.$

(c) $\mathcal{Z}\tilde{\chi}(\mathfrak{n},\mathfrak{p}) = \overline{\mathcal{Z}\chi(-\mathfrak{n},\mathfrak{p})}.$

(d) $\mathcal{Z}(\chi * y)(\mathfrak{n},\mathfrak{p}) = \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}\mathcal{Z}\chi(\mathfrak{n}-\mathfrak{s},\mathfrak{p})\mathcal{Z}y(\mathfrak{s},\mathfrak{p}) = \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}\mathcal{Z}\chi(\mathfrak{s},\mathfrak{p})\mathcal{Z}y(\mathfrak{n}-\mathfrak{s},\mathfrak{p}).$

Proof. Straightforward. □

In the given result, we give a characterization of dual finite Gabor tight frame using discrete time Zak transform.

Theorem 2.4. *Let $\mathfrak{L} = \mathfrak{M}\times\mathfrak{N}_1 = \mathfrak{M}_1\times\mathfrak{N}$ and let $\mathfrak{P}\in\mathbb{N}$ be such that $\mathfrak{M} = \mathfrak{P}\times\mathfrak{M}_1$ and $\nu, w\in l^2(\mathbb{Z}_{\mathfrak{L}})$. Then a pair $(\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{k\mathfrak{M}_1}\nu\},\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{k\mathfrak{M}_1}w\})$ is a dual finite Gabor tight frame if and only if*

$$\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)\overline{\mathcal{Z}w(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} = \mathfrak{M}_1^{-1},\ \textit{for all}\ (\mathfrak{n},\mathfrak{p})\in\mathbb{Z}_{\mathfrak{M}_1}\times\mathbb{Z}_{\mathfrak{N}}.$$

Proof. Note that

$$\begin{aligned}
\mathcal{S}(\mathfrak{z}) &= \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}\langle\mathfrak{z}, w_{\mathfrak{m},k}\rangle\nu_{\mathfrak{m},k}\\
&= \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu * E_{\frac{\mathfrak{m}}{\mathfrak{M}}}(U_{\mathfrak{M}_1}D_{\mathfrak{M}_1}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w}))\\
&= \frac{1}{\mathfrak{M}_1}\sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu * E_{\frac{\mathfrak{m}}{\mathfrak{M}}}(\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}E_{-\frac{\mathfrak{j}}{\mathfrak{M}_1}}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w})).
\end{aligned}$$

Let $(\mathfrak{n},\mathfrak{p})\in\mathbb{Z}_{\mathfrak{M}_1}\times\mathbb{Z}_{\mathfrak{N}}$. Then, we have

$$\begin{aligned}
&\mathcal{Z}[E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu * E_{\frac{\mathfrak{m}}{\mathfrak{M}}}(\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}E_{-\frac{\mathfrak{j}}{\mathfrak{M}_1}}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w}))](\mathfrak{n},\mathfrak{p})\\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}\mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu(\mathfrak{n}-\mathfrak{s},\mathfrak{p})\mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}(\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}E_{-\frac{\mathfrak{j}}{\mathfrak{M}_1}}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w}))(\mathfrak{s},\mathfrak{p})\\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}e^{2\pi i\frac{\mathfrak{m}}{\mathfrak{M}}(\mathfrak{n}-\mathfrak{s})}\mathcal{Z}\nu(\mathfrak{n}-\mathfrak{s},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)e^{2\pi i\frac{\mathfrak{m}}{\mathfrak{M}}\mathfrak{s}}\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}\mathcal{Z}E_{-\frac{\mathfrak{j}}{\mathfrak{M}_1}}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w})(\mathfrak{s},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)\\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}e^{-2\pi i\frac{\mathfrak{m}}{\mathfrak{M}}\mathfrak{n}}\mathcal{Z}\nu(\mathfrak{n}-\mathfrak{s},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)e^{-2\pi i\frac{\mathfrak{j}}{\mathfrak{M}_1}\mathfrak{s}}\mathcal{Z}(E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z} * \tilde{w})(\mathfrak{s},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)\\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1}\sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1}e^{2\pi i\frac{\mathfrak{m}}{\mathfrak{M}}\mathfrak{n}}\mathcal{Z}\nu(\mathfrak{n}-\mathfrak{s},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)e^{-2\pi i\frac{\mathfrak{j}}{\mathfrak{M}_1}\mathfrak{s}}\\
&\quad\times\sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1}\mathcal{Z}E_{-\frac{\mathfrak{m}}{\mathfrak{M}}}\mathfrak{z}(\mathfrak{r},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)\mathcal{Z}\tilde{w}(\mathfrak{s}-\mathfrak{r},\mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)
\end{aligned}$$

$$= \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1} \sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1} e^{2\pi i \frac{\mathfrak{m}}{\mathfrak{M}} \mathfrak{n}} \mathcal{Z}\nu(\mathfrak{n}-\mathfrak{s}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1) e^{-2\pi i \frac{\mathfrak{j}}{\mathfrak{M}_1}\mathfrak{s}}$$

$$\times \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} e^{-2\pi i \frac{\mathfrak{m}}{\mathfrak{M}} \mathfrak{r}} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p}) \overline{\mathcal{Z}w(\mathfrak{r}-\mathfrak{s}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)}$$

$$= \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \sum_{\mathfrak{s}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}-\mathfrak{s}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}-\mathfrak{s}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p})$$

$$\times e^{-2\pi i \frac{\mathfrak{m}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})} \sum_{\mathfrak{j}=0}^{\mathfrak{M}_1-1} e^{-2\pi i \frac{\mathfrak{j}}{\mathfrak{M}_1}\mathfrak{s}}$$

$$= \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p}) e^{-2\pi i \frac{\mathfrak{m}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})} \mathfrak{M}_1.$$

Thus, we get

$$\mathcal{Z}\mathcal{S}(\mathfrak{z})(\mathfrak{n}, \mathfrak{p}) = \frac{1}{\mathfrak{M}_1} \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}, \mathfrak{p}-\mathfrak{m}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p}) e^{-2\pi i \frac{\mathfrak{m}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})} \mathfrak{M}_1.$$

Given that $\mathfrak{M} = \mathfrak{P} \times \mathfrak{M}_1$, we have

$$\mathfrak{M} \times \mathfrak{N} = \mathfrak{P} \times \mathfrak{M}_1 \times \mathfrak{N} = \mathfrak{P} \times \mathfrak{M} \times \mathfrak{N}_1 \Rightarrow \mathfrak{N} = \mathfrak{P} \times \mathfrak{N}_1.$$

Let $\mathfrak{m} = \mathfrak{m}_1\mathfrak{P} + \mathfrak{l}$ for $0 \le \mathfrak{l} \le \mathfrak{P}-1$ and $0 \le \mathfrak{m}_1 \le \mathfrak{M}_1 - 1$.
Then, we compute

$$\mathcal{Z}\mathcal{S}(\mathfrak{z})(\mathfrak{n}, \mathfrak{p})$$

$$= \sum_{\mathfrak{m}_1=0}^{\mathfrak{M}_1-1} \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{m}_1\mathfrak{N}-\mathfrak{l}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}, \mathfrak{p}-\mathfrak{m}_1\mathfrak{N}-\mathfrak{l}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p})$$

$$\times e^{-2\pi i \frac{\mathfrak{m}_1}{\mathfrak{M}_1}(\mathfrak{r}-\mathfrak{n})} e^{-2\pi i \frac{\mathfrak{l}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})}$$

$$= \sum_{\mathfrak{m}_1=0}^{\mathfrak{M}_1-1} \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p})$$

$$\times e^{-2\pi i \frac{\mathfrak{l}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})} e^{-2\pi i \frac{\mathfrak{m}_1}{\mathfrak{M}_1}(\mathfrak{r}-\mathfrak{n})}$$

$$= \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} \sum_{\mathfrak{r}=0}^{\mathfrak{M}_1-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{r}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{r}, \mathfrak{p})$$

$$\times e^{-2\pi i \frac{\mathfrak{l}}{\mathfrak{M}}(\mathfrak{r}-\mathfrak{n})} \sum_{\mathfrak{m}_1=0}^{\mathfrak{M}_1-1} e^{-2\pi i \frac{\mathfrak{m}_1}{\mathfrak{M}_1}(\mathfrak{r}-\mathfrak{n})}$$

$$= \mathfrak{M}_1 \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} \mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1) \overline{\mathcal{Z}w(\mathfrak{n}, \mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} \mathcal{Z}\mathfrak{z}(\mathfrak{n}, \mathfrak{p}).$$

Since the pair $(\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathcal{k}\mathfrak{M}_1}\nu\}, \{E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathcal{k}\mathfrak{M}_1} w\})$ constitutes a dual finite Gabor tight frame,

$$\mathfrak{z} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \sum_{\mathcal{k}=0}^{\mathfrak{N}-1} \langle \mathfrak{z}, w_{\mathfrak{m},\mathcal{k}} \rangle \nu_{\mathfrak{m},\mathcal{k}}, \text{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

So, taking finite Zak transform, we have

$$\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p}) = \mathfrak{M}_1\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p})\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)\overline{\mathcal{Z}w(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)}.$$

Thus, we get $\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)\overline{\mathcal{Z}w(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} = \mathfrak{M}_1^{-1}$.
Conversely, we have

$$\mathcal{Z}\mathcal{S}\mathfrak{z}(\mathfrak{n},\mathfrak{p}) = \mathfrak{M}_1\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p})\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)\overline{\mathcal{Z}w(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)} = \mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p}).$$

Hence

$$\mathfrak{z} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathcal{k}=0}^{\mathfrak{N}-1}\langle\mathfrak{z}, w_{\mathfrak{m},\mathcal{k}}\rangle\nu_{\mathfrak{m},\mathcal{k}},\ \text{for all } \mathfrak{z}\in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

□

In Theorem 2.4, if we take $\nu = w$, then we obtain the following result.

Corollary 2.5. $\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathcal{k}\mathfrak{M}_1}\nu\}$ *is finite Gabor Parseval frame if and only if*

$$\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}|\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)|^2 = \mathfrak{M}_1^{-1},\ \textit{for all } (\mathfrak{n},\mathfrak{p})\in\mathbb{Z}_{\mathfrak{M}_1}\times\mathbb{Z}_{\mathfrak{N}}.$$

In the following result, we give a necessary and sufficient condition for the existence of finite Gabor frames which serves as a method to calculate the coefficients of the finite Gabor system expansion in the case of oversampling.

Theorem 2.6. *Let* $\mathfrak{L} = \mathfrak{N}_1\times\mathfrak{M} = \mathfrak{N}\times\mathfrak{M}_1$ *and* $\mathfrak{M} = \mathfrak{P}\times\mathfrak{M}_1$. *Let* $\nu_{\mathfrak{m},\mathcal{k}} = E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathcal{k}\mathfrak{M}_1}\nu$, *where* $0\le\mathfrak{m}<\mathfrak{M}-1$ *and* $0\le\mathcal{k}\le\mathfrak{N}-1$. *Then* $\{\nu_{\mathfrak{m},\mathcal{k}}\}$ *is a finite Gabor frame with bounds* $\mathcal{A}$ *and* $\mathcal{B}$ *if and only if*

$$\mathcal{A}\le\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}|\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)|^2\mathfrak{M}_1\le\mathcal{B},\ \textit{for all } (\mathfrak{n},\mathfrak{p})\in\mathbb{Z}_{\mathfrak{M}_1}\times\mathbb{Z}_{\mathfrak{N}}.$$

Proof. Take

$$\mathcal{S}(\mathfrak{z}) = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{\mathcal{k}=0}^{\mathfrak{N}-1}\langle\mathfrak{z},\nu_{\mathfrak{m},\mathcal{k}}\rangle\nu_{\mathfrak{m},\mathcal{k}},\ \text{for all } \mathfrak{z}\in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

As proved in Theorem 2.4, we have

$$\mathcal{Z}\mathcal{S}(\mathfrak{z})(\mathfrak{n},\mathfrak{p}) = \mathfrak{M}_1\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}|\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)|^2\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p}).$$

Also, we compute

$$\begin{aligned}
\langle\mathcal{S}\mathfrak{z},\mathfrak{z}\rangle &= \frac{1}{\mathfrak{N}}\langle\mathcal{Z}\mathcal{S}\mathfrak{z},\mathcal{Z}\mathfrak{z}\rangle\\
&= \frac{1}{\mathfrak{N}}\sum_{\mathfrak{n}=0}^{\mathfrak{M}_1-1}\sum_{\mathfrak{p}=0}^{\mathfrak{N}-1}\mathcal{Z}\mathcal{S}\mathfrak{z}(\mathfrak{n},\mathfrak{p})\overline{\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p})}\\
&= \frac{1}{\mathfrak{N}}\sum_{\mathfrak{n}=0}^{\mathfrak{M}_1-1}\sum_{\mathfrak{p}=0}^{\mathfrak{N}-1}|\mathcal{Z}\mathfrak{z}(\mathfrak{n},\mathfrak{p})|^2\mathfrak{M}_1\sum_{\mathfrak{l}=0}^{\mathfrak{P}-1}|\mathcal{Z}\nu(\mathfrak{n},\mathfrak{p}-\mathfrak{l}\mathfrak{N}_1)|^2.
\end{aligned}$$

If $\{\nu_{\mathfrak{m},\mathfrak{k}}\}$ is a frame, then

$$\mathcal{A}\|\mathcal{Z}\mathfrak{z}\|^2 \leq \langle \mathcal{Z}\mathcal{S}\mathfrak{z}, \mathcal{Z}\mathfrak{z}\rangle \leq \mathcal{B}\|\mathcal{Z}\mathfrak{z}\|^2.$$

This gives

$$\mathcal{A} \leq \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} |\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p} - \mathfrak{l}\mathfrak{N}_1)|^2 \mathfrak{M}_1 \leq \mathcal{B}.$$

Conversely, we have

$$\mathcal{A}\|\mathcal{Z}\mathfrak{z}\|^2 \leq \|\mathcal{Z}\mathfrak{z}\|^2 \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} |\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p} - \mathfrak{l}\mathfrak{N}_1)|^2 \mathfrak{M}_1 \leq \mathcal{B}\|\mathcal{Z}\mathfrak{z}\|^2$$

and

$$\begin{aligned} \|\mathcal{Z}\mathfrak{z}\|^2 \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} |\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p} - \mathfrak{l}\mathfrak{N}_1)|^2 \mathfrak{M}_1 &= \sum_{\mathfrak{n}=0}^{\mathfrak{M}_1-1} \sum_{\mathfrak{p}=0}^{\mathfrak{N}-1} |\mathcal{Z}\mathfrak{z}(\mathfrak{n}, \mathfrak{p})|^2 \sum_{\mathfrak{l}=0}^{\mathfrak{P}-1} |\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p} - \mathfrak{l}\mathfrak{N}_1)|^2 \mathfrak{M}_1 \\ &= \langle \mathcal{Z}\mathcal{S}\mathfrak{z}, \mathcal{Z}\mathfrak{z}\rangle. \end{aligned}$$

Thus

$$\mathcal{A}\|\mathfrak{z}\|^2 \leq \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \sum_{\mathfrak{k}=0}^{\mathfrak{N}-1} |\langle \mathfrak{z}, \nu_{\mathfrak{m},\mathfrak{k}}\rangle|^2 \leq \mathcal{B}\|\mathfrak{z}\|^2, \text{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

□

Next, we give necessary and sufficient conditions for the existence of Riesz basis and orthonormal basis for finite Gabor frame.

Corollary 2.7. (a) *Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$. Then $\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{k}M}\nu\}$ is a Riesz basis in $l^2(\mathbb{Z}_{\mathfrak{L}})$ if and only if*

$$\mathcal{A} \leq \mathfrak{M}|\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p})|^2 \leq \mathcal{B}, \textit{ for all } (\mathfrak{n}, \mathfrak{p}) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}.$$

(b) *Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$. Then $\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{k}\mathfrak{M}}\nu\}$ is an orthonormal basis in $l^2(\mathbb{Z}_{\mathfrak{L}})$ if and only if*

$$|\mathcal{Z}\nu(\mathfrak{n}, \mathfrak{p})|^2 = \mathfrak{M}^{-1}, \textit{ for all } (\mathfrak{n}, \mathfrak{p}) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}.$$

In the following result, we give necessary and sufficient conditions for the existence of biorthogonal pair of Riesz bases in $l^2(\mathbb{Z}_{\mathfrak{L}})$. Before proving the result, we first give a lemma which will help us in writing the proof of the main result.

Lemma 2.8. *Let $\mathscr{H}$ be a Hilbert space. A pair of families $\{\mathfrak{f}_\xi\}_{\xi\in\Xi} \subseteq \mathscr{H}$ and $\{\mathfrak{g}_\xi\}_{\xi\in\Xi} \subseteq \mathscr{H}$ is a biorthogonal pair of Riesz bases if and only if*

(i) *$\{\mathfrak{f}_\xi\}_{\xi\in\Xi}$ and $\{\mathfrak{g}_\xi\}_{\xi\in\Xi}$ are Bessel sequences for $\mathscr{H}$.*
(ii) *$\mathfrak{f} = \sum_{\xi\in\Xi} \langle \mathfrak{f}, \mathfrak{f}_\xi\rangle \mathfrak{g}_\xi$, for all $\mathfrak{f} \in \mathscr{H}$.*
(iii) *$\langle \mathfrak{f}_\xi, \mathfrak{g}_{\xi'}\rangle = \delta(\xi - \xi')$, for $\xi, \xi' \in \Xi$.*

Proof. Here (Only if part) is trivial. So we will prove for (If part). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be synthesis operators of Bessel sequences $\{\mathfrak{f}_\xi\}$ and $\{\mathfrak{g}_\xi\}$ respectively. From the given condition, we have $\mathcal{T}_2\mathcal{T}_1^* = I_{\mathscr{H}}$. So, it is clear that $\{\mathfrak{f}_\xi\}$ is a frame for $\mathscr{H}$. Let $\{\alpha(\xi)\} \in l^2(\Xi)$ and note that

$$\mathcal{T}_1^*\mathcal{T}_2(\{\alpha(\xi)\}) = \{\sum_{\xi\in\Xi} \alpha(\xi)\langle \mathfrak{g}_\xi, f_{\xi'}\rangle\}_{\xi'\in\Xi} = \{\alpha(\xi)\}.$$

This gives $\mathcal{T}_1^*$ is invertible. So $\{\mathfrak{f}_\xi\}$ is a Riesz basis for $\mathscr{H}$. Also, we have $\mathcal{T}_1\mathcal{T}_2^* = I_{\mathscr{H}}$. Similarly, we can prove that $\{\mathfrak{g}_\xi\}$ is a Riesz basis for $\mathscr{H}$. □

Theorem 2.9. *Let $\mathfrak{L} = \mathfrak{M}\times\mathfrak{N}$. For $0 \leq \mathfrak{m} \leq \mathfrak{M}-1$ and $0 \leq \mathfrak{k} \leq \mathfrak{N}-1$, a pair $(\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}\nu\}, \{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}w\})$ is biorthogonal pair of Riesz bases in $l^2(\mathbb{Z}_{\mathfrak{L}})$ if and only if*

(a) $(\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}\nu\}, \{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}w\})$ *is a dual finite Gabor tight frame in* $l^2(\mathbb{Z}_{\mathfrak{L}})$.

(b) $\sum_{\mathfrak{s}=0}^{\mathfrak{M}-1} \mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu(\mathfrak{s},\mathfrak{p})\overline{\mathcal{Z}E_{\frac{\mathfrak{m}'}{\mathfrak{M}}}w(\mathfrak{s},\mathfrak{p})} = \delta(\mathfrak{m}-\mathfrak{m}')\mathfrak{M}$ *and* $0 \leq \mathfrak{p} \leq \mathfrak{N}-1$.

Proof. Note that $\langle E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}\nu, E_{\frac{\mathfrak{m}'}{M}}T_{\mathfrak{k}'\mathfrak{M}}w\rangle = \delta(\mathfrak{m}-\mathfrak{m}')\delta(\mathfrak{k}-\mathfrak{k}')$ if and only if $\langle E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu, T_{\mathfrak{k}\mathfrak{M}}E_{\frac{\mathfrak{m}'}{M}}w\rangle = \delta(\mathfrak{m}-\mathfrak{m}')\delta(\mathfrak{k})$ if and only if $\langle \nu, E_{\frac{\mathfrak{m}'-\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}w\rangle = \delta(\mathfrak{m}-\mathfrak{m}')\delta(\mathfrak{k})$ if and only if $E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}(\mathfrak{k}\mathfrak{M}) = \delta(\mathfrak{m}-\mathfrak{m}')\delta(\mathfrak{k})$. But

$$\frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} E_{-\frac{\mathfrak{l}}{\mathfrak{M}}}\Big(E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}\Big)(\mathfrak{p}) = \begin{cases} E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}(\mathfrak{p}), & \text{if } \mathfrak{M}|\mathfrak{p} \\ 0, & \text{elsewhere.}\end{cases}$$

So $\frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} E_{-\frac{\mathfrak{l}}{\mathfrak{M}}}\Big(E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}\Big) = \delta(\mathfrak{m}-\mathfrak{m}')\delta$. By taking Zak Transform, we get

$$\mathcal{Z}\Big(\frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} E_{-\frac{\mathfrak{l}}{\mathfrak{M}}}\Big(E_{\frac{\mathfrak{m}-\mathfrak{m}'}{M}}\nu * \tilde{w}\Big)\Big)(\mathfrak{n},\mathfrak{p}) = \delta(\mathfrak{m}-\mathfrak{m}')\mathfrak{M}.$$

We compute

$$\begin{aligned}
&\mathcal{Z}\Big(\frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} E_{-\frac{\mathfrak{l}}{\mathfrak{M}}}\Big(E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}\Big)\Big)(\mathfrak{n},\mathfrak{p}) \\
&= \frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} e^{-2\pi i\frac{\mathfrak{l}}{\mathfrak{M}}\mathfrak{n}}\mathcal{Z}\Big(E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu * \tilde{w}\Big)(\mathfrak{n},\mathfrak{p}) \\
&= \frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1} e^{-2\pi i\frac{\mathfrak{l}}{\mathfrak{M}}\mathfrak{n}}\sum_{\mathfrak{s}=0}^{\mathfrak{M}-1}\mathcal{Z}E_{\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}}\nu(\mathfrak{s},\mathfrak{p})\mathcal{Z}\tilde{w}(\mathfrak{n}-\mathfrak{s},\mathfrak{p}) \\
&= \frac{1}{\mathfrak{M}}\sum_{\mathfrak{s}=0}^{\mathfrak{M}-1}\mathcal{Z}\nu(\mathfrak{s},\mathfrak{p})e^{2\pi i\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}\mathfrak{s}}\overline{\mathcal{Z}w(\mathfrak{s}-\mathfrak{n},\mathfrak{p})}\sum_{\mathfrak{l}=0}^{\mathfrak{M}-1}e^{-2\pi i\frac{\mathfrak{l}}{\mathfrak{M}}\mathfrak{n}} \\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}-1}\mathcal{Z}\nu(\mathfrak{s},\mathfrak{p})\overline{\mathcal{Z}w(\mathfrak{s}-\mathfrak{n},\mathfrak{p})}e^{2\pi i\frac{\mathfrak{m}-\mathfrak{m}'}{\mathfrak{M}}\mathfrak{s}} \\
&= \sum_{\mathfrak{s}=0}^{\mathfrak{M}-1}\mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu(\mathfrak{s},\mathfrak{p})\overline{\mathcal{Z}E_{\frac{\mathfrak{m}'}{\mathfrak{M}}}w(\mathfrak{s},\mathfrak{p})}.
\end{aligned}$$

Thus $(\{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}\nu\}, \{E_{\frac{\mathfrak{m}}{\mathfrak{M}}}T_{\mathfrak{k}\mathfrak{M}}w\})$ is biorthogonal if and only if

$$\sum_{\mathfrak{s}=0}^{\mathfrak{M}-1}\mathcal{Z}E_{\frac{\mathfrak{m}}{\mathfrak{M}}}\nu(\mathfrak{s},\mathfrak{p})\overline{\mathcal{Z}E_{\frac{\mathfrak{m}'}{\mathfrak{M}}}w(\mathfrak{s},\mathfrak{p})} = \delta(\mathfrak{m}-\mathfrak{m}')\mathfrak{M},\ \text{for } 0 \leq \mathfrak{p} \leq \mathfrak{N}-1.$$

Hence the proof follows immediately from Lemma 2.8. □

Frazier [20] defined first stage wavelet system $\mathscr{B} = \{T_{2\mathfrak{k}}\nu\}_{\mathfrak{k}\in\mathbb{Z}} \cup \{T_{2\mathfrak{k}}w\}_{\mathfrak{k}\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$ and observed that $\mathscr{B}$ is a basis (orthonormal) for $l^2(\mathbb{Z})$. He gave various characterizations of such systems in connection with the reconstruction property. Frazier et al. [21] defined dual wavelet frames and gave various characterizations of dual wavelet frames. For further readings on wavelet tight frames, one may see [5, 18] and for more details on wavelet theory, see [33–35]. In the next result, we give the definitions of dual discrete time wavelet tight frame (DDTWT frame) and discrete time wavelet frame (DTW frame) of scale $\mathfrak{N}$ in $l^2(\mathbb{Z}_{\mathfrak{L}})$.

Definition 2.10. *Let* $\mathfrak{M}, \mathfrak{N}, \mathfrak{K} \in \mathbb{N}$ *and* $\mathfrak{L} = \mathfrak{N} \times \mathfrak{K}$. *Let* $\mathcal{V} = \{\nu_1, \nu_2, ..., \nu_{\mathfrak{M}}\}, \mathcal{W} = \{w_1, w_2, \ldots, w_{\mathfrak{M}}\} \subseteq l^2(\mathbb{Z}_{\mathfrak{L}})$. *Family* $\mathcal{V}$ *with* $\mathcal{W}$ *generate a dual discrete time wavelet tight frame (DDTWT frame) of scale* $\mathfrak{N}$ *if*

$$\mathfrak{z} = \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \sum_{\mathfrak{k}=0}^{\mathfrak{K}-1} \langle \mathfrak{z}, T_{\mathfrak{k}\mathfrak{N}} w_{\mathfrak{j}} \rangle T_{\mathfrak{k}\mathfrak{N}} \nu_{\mathfrak{j}}, \textit{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Definition 2.11. *Let* $\mathfrak{M}, \mathfrak{N}, \mathfrak{K} \in \mathbb{N}$ *and* $\mathfrak{L} = \mathfrak{N} \times \mathfrak{K}$. *A family* $\{\nu_1, \nu_2, ..., \nu_{\mathfrak{M}}\} \subseteq l^2(\mathbb{Z}_{\mathfrak{L}})$ *generates a discrete time wavelet frame (DTW frame) of scale* $\mathfrak{N}$ *in* $l^2(\mathbb{Z}_{\mathfrak{L}})$ *if there exist constants* $0 < \mathcal{A} \leq \mathcal{B} < \infty$ *such that*

$$\mathcal{A}\|\mathfrak{z}\|^2 \leq \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \sum_{\mathfrak{k}=0}^{\mathfrak{K}-1} |\langle \mathfrak{z}, T_{\mathfrak{k}\mathfrak{N}} \nu_{\mathfrak{j}} \rangle|^2 \leq \mathcal{B}\|\mathfrak{z}\|^2, \textit{ for all } \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Decomposition of DTW frame operator:

$$\mathcal{S}(\mathfrak{z}) = \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \sum_{\mathfrak{k}=0}^{\mathfrak{K}-1} \langle \mathfrak{z}, T_{\mathfrak{k}\mathfrak{N}} \nu_{\mathfrak{j}} \rangle T_{\mathfrak{k}\mathfrak{N}} \nu_{\mathfrak{j}} = \sum_{\mathfrak{j}=1}^{\mathfrak{M}} G_{\nu_{\mathfrak{j}}} U_{\mathfrak{N}} D_{\mathfrak{N}} G_{\tilde{\nu}_{\mathfrak{j}}} \mathfrak{z}, \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

In the given result, we give a characterization in the form of construction of DDTWT frames in $l^2(\mathbb{Z}_{\mathfrak{L}})$.

Theorem 2.12. *Let* $\mathfrak{N}, \mathfrak{K}, \mathfrak{M} \in \mathbb{N}$ *such that* $\mathfrak{L} = \mathfrak{N} \times \mathfrak{K}$ *and* $\mathfrak{M} \geq \mathfrak{N}$. *A family* $\{\nu_1, \nu_2, \ldots, \nu_{\mathfrak{M}}\} \subseteq l^2(\mathbb{Z}_{\mathfrak{L}})$ *with* $\{w_1, w_2, \ldots, w_{\mathfrak{M}}\} \subseteq l^2(\mathbb{Z}_{\mathfrak{L}})$ *generate DDTWT frame of scale* $\mathfrak{N}$ *if and only if*

(a) $\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_j(\mathfrak{p} + \mathfrak{s}\mathfrak{K})\overline{\mathcal{F}w_{\mathfrak{j}}(\mathfrak{p} + \mathfrak{s}\mathfrak{K})} = \frac{\mathfrak{N}}{\mathfrak{L}}$, *for all* $\mathfrak{p} \in \{0, 1, \ldots, \mathfrak{K} - 1\}, \mathfrak{s} \in \{0, 1, \ldots, \mathfrak{N} - 1\}$.

(b) $\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p} + \mathfrak{s}\mathfrak{K})\overline{\mathcal{F}w_{\mathfrak{j}}(\mathfrak{p} + \mathfrak{n}\mathfrak{K})} = 0$, *for all* $\mathfrak{p} \in \{0, 1, \ldots, \mathfrak{K} - 1\}, \mathfrak{s}, \mathfrak{n} \in \{0, 1, \ldots, \mathfrak{N} - 1\}$ *and* $\mathfrak{s} \neq \mathfrak{n}$.

Proof. We know that

$$\begin{aligned}\mathcal{S}(\mathfrak{z}) &= \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \sum_{\mathfrak{k}=0}^{\mathcal{K}-1} \langle \mathfrak{z}, T_{\mathfrak{k}\mathfrak{N}} w_{\mathfrak{j}} \rangle T_{\mathfrak{k}\mathfrak{N}} \nu_{\mathfrak{j}} \\ &= \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \nu_{\mathfrak{j}} * U_{\mathfrak{N}} D_{\mathfrak{N}} (\mathfrak{z} * \tilde{w}_{\mathfrak{j}}) \\ &= \frac{1}{\mathfrak{N}} \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \nu_{\mathfrak{j}} * \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} E_{-\frac{\mathfrak{n}}{\mathfrak{N}}} (\mathfrak{z} * \tilde{w}_{\mathfrak{j}}).\end{aligned}$$

This gives

$$\begin{aligned}\mathcal{F}\mathcal{S}(\mathfrak{z})(\mathfrak{p}) &= \frac{1}{\mathfrak{N}} \mathcal{F}\Big[\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \nu_{\mathfrak{j}} * \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} E_{-\frac{\mathfrak{n}}{\mathfrak{N}}} (\mathfrak{z} * \tilde{w}_{\mathfrak{j}}) \Big](\mathfrak{p}) \\ &= \frac{\sqrt{\mathfrak{L}}}{\mathfrak{N}} \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}) \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} \mathcal{F}\Big[E_{-\frac{\mathfrak{n}}{\mathfrak{N}}} (\mathfrak{z} * \tilde{w}_{\mathfrak{j}}) \Big](\mathfrak{p}) \\ &= \frac{\mathfrak{L}}{\mathfrak{N}} \sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}) \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} \mathcal{F}\mathfrak{z}(\mathfrak{p} + \mathfrak{n}\mathfrak{K}) \overline{\mathcal{F}w_{\mathfrak{j}}(\mathfrak{p} + \mathfrak{n}\mathfrak{K})}.\end{aligned}$$

This yields

$$\mathcal{F}\mathcal{S}(\mathfrak{z})(\mathfrak{p}) = \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p})\,\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p})}\,\mathcal{F}\mathfrak{z}(\mathfrak{p}) + \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{n}=1}^{\mathfrak{N}-1}\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{n}\mathfrak{K})\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p})\,\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{n}\mathfrak{K})}.$$

(⇐) If conditions (a) and (b) are satisfied and by taking $\mathfrak{s} = 0$, we have

$$\mathcal{F}\mathcal{S}(\mathfrak{z})(\mathfrak{p}) = \mathcal{F}\mathfrak{z}(\mathfrak{p}).$$

Thus $\mathfrak{z} = \sum_{\mathfrak{j}=1}^{\mathfrak{M}}\sum_{\mathfrak{k}=0}^{\mathfrak{K}-1}\langle \mathfrak{z}, T_{\mathfrak{k}\mathfrak{N}}\mathfrak{w}_{\mathfrak{j}}\rangle T_{\mathfrak{k}\mathfrak{N}}\nu_{\mathfrak{j}}$.
(⇒) We have

$$\mathcal{F}\mathfrak{z}(\mathfrak{p}) = \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p})\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p})}\,\mathcal{F}\mathfrak{z}(\mathfrak{p}) + \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{n}=1}^{\mathfrak{N}-1}\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{n}\mathfrak{K})\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p})\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{n}\mathfrak{K})} \quad (2)$$

Also, for $0 \le \mathfrak{s} \le \mathfrak{N} - 1$, we have

$$\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{s}\mathfrak{K}) = \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{j}=1}^{\mathfrak{M}} \mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})}\,\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{s}\mathfrak{K}) + \frac{\mathfrak{L}}{\mathfrak{N}}\sum_{\mathfrak{n}=1}^{\mathfrak{N}-1}\mathcal{F}\mathfrak{z}(\mathfrak{p}+(\mathfrak{n}+\mathfrak{s})\mathfrak{K})\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+(\mathfrak{s}+\mathfrak{n})\mathfrak{K})}.$$

Let $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$ be such that $\mathcal{F}\mathfrak{z} = \chi_{\{\mathfrak{s}\mathfrak{K},\mathfrak{s}\mathfrak{K}+1,\dots,(\mathfrak{s}+1)\mathfrak{K}-1\}}$. So $\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{s}\mathfrak{K}) = 1$ and $\mathcal{F}\mathfrak{z}(\mathfrak{p}+(\mathfrak{n}+\mathfrak{s})\mathfrak{K}) = 0$, for all $\mathfrak{p} \in \{0,1,\dots,\mathfrak{K}-1\}$ and $\mathfrak{n} = 1,2,\dots,\mathfrak{N}-1$.
So $\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})} = \mathfrak{N}/\mathfrak{L}$, for all $\mathfrak{s} \in \{0,1,\dots,\mathfrak{N}-1\}$.
Let $\mathfrak{r} \in \{0,1,\dots,\mathfrak{s}-1,\mathfrak{s}+1,\dots,\mathfrak{N}-1\}$. Let $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$ such that $\mathcal{F}\mathfrak{z} = \chi_{\{\mathfrak{r}\mathfrak{K},\mathfrak{r}\mathfrak{K}+1,\dots,(\mathfrak{r}+1)\mathfrak{K}-1\}}$. Then, for $\mathfrak{p} \in \{0,1,\dots,\mathfrak{K}-1\}$, we have

$$\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{r}\mathfrak{K}) = 1,\ \mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{s}\mathfrak{K}) = 0 \text{ and}$$
$$\mathcal{F}\mathfrak{z}(\mathfrak{p}+\mathfrak{m}\mathfrak{K}) = 0, \text{ for all } \mathfrak{m} \in \{0,1,\dots,\mathfrak{N}-1\} \text{ and } \mathfrak{m} \ne \mathfrak{r}.$$

Thus $\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})\,\overline{\mathcal{F}\,\mathfrak{w}_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{r}\mathfrak{K})} = 0$, for all $\mathfrak{p} \in \{0,1,\dots,\mathfrak{K}-1\}, \mathfrak{s} \ne \mathfrak{r}$. □

Next, we give an interesting result for the existence of discrete time wavelet Parseval frame (DTW Parseval frame) of scale $\mathfrak{N}$ in $l^2(\mathbb{Z}_{\mathfrak{L}})$ in the form of a corollary.

Corollary 2.13. *Let $\mathfrak{N},\mathfrak{K},\mathfrak{M} \in \mathbb{N}$ such that $\mathfrak{L} = \mathfrak{N}\times\mathfrak{K}$ and $\mathfrak{M} \ge \mathfrak{N}$. A family $\{\nu_1,\nu_2,\dots,\nu_{\mathfrak{M}}\} \subseteq l^2(\mathbb{Z}_{\mathfrak{L}})$ generates a discrete time wavelet Parseval frame (DTW frame) of scale $\mathfrak{N}$ in $l^2(\mathbb{Z}_{\mathfrak{L}})$ if and only if*

(a) $\sum_{\mathfrak{j}=1}^{\mathfrak{M}}|\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})|^2 = \frac{\mathfrak{N}}{\mathfrak{L}}$, *for all* $\mathfrak{p} \in \{0,1,\dots,\mathfrak{K}-1\}, \mathfrak{s} \in \{0,1,\dots,\mathfrak{N}-1\}$.

(b) $\sum_{\mathfrak{j}=1}^{\mathfrak{M}}\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{s}\mathfrak{K})\overline{\mathcal{F}\nu_{\mathfrak{j}}(\mathfrak{p}+\mathfrak{n}\mathfrak{K})} = 0$, *for all* $\mathfrak{p} \in \{0,1,\dots,\mathfrak{K}-1\}, \mathfrak{s},\mathfrak{n} \in \{0,1,\dots,\mathfrak{N}-1\}$ *and* $\mathfrak{s} \ne \mathfrak{n}$.

3. Uncertainity Principle

For $\mathfrak{f} \in L^2(\mathbb{R})$ with $||\mathfrak{f}|| = 1$, we have $\triangle_\tau^2 \triangle_{\hat{\mathfrak{f}}r}^2 \geq \frac{1}{16\pi^2}$, where $\triangle_\tau$ is time spread of a signal $\mathfrak{f}$, given by $\triangle_\tau^2 = \frac{1}{||\mathfrak{f}||^2}\int_{\mathbb{R}}(\tau - \mu_\tau)^2|\mathfrak{f}(\tau)|^2 d\tau$ with time center $\mu_\tau = \frac{1}{||\mathfrak{f}||^2}\int_{\mathbb{R}}\tau|\mathfrak{f}(\tau)|^2 d\tau$ and $\triangle_{\hat{\mathfrak{f}}r}$ is frequency spread of a signal $\mathfrak{f}$, given by $\triangle_{\hat{\mathfrak{f}}r}^2 = \frac{1}{||\mathfrak{f}||^2}\int_{\mathbb{R}}(\nu - \mu_{\hat{\mathfrak{f}}r})^2|\hat{\mathfrak{f}}(\nu)|^2 d\nu$ with frequency center $\mu_{\hat{\mathfrak{f}}r} = \frac{1}{||\mathfrak{f}||^2}\int_{\mathbb{R}}\nu|\hat{\mathfrak{f}}(\nu)|^2 d\nu$. Here $\hat{\mathfrak{f}}$ stands for the Fourier transform of $\mathfrak{f}$. The above result is known as the classical uncertainty principle. For details, one may see [9, 25, 30]. Analogously, one can see uncertainty principle for $\mathbf{k} \in l^2(\mathbb{Z})$ with a condition $\hat{\mathbf{k}}(1) = 0$, where $\triangle_\tau$ is time spread of a signal $\mathbf{k}$, given by $\triangle_\tau^2 = \frac{1}{||\mathbf{k}||^2}\sum_{\mathfrak{m}\in\mathbb{Z}}(\mathfrak{m} - \mu_\tau)^2|\mathbf{k}(\mathfrak{m})|^2$ with time center $\mu_\tau = \frac{1}{||\mathbf{k}||^2}\sum_{\mathfrak{m}\in\mathbb{Z}}\mathfrak{m}|\mathbf{k}(\mathfrak{m})|^2$ and $\triangle_{\hat{\mathfrak{f}}r}$ is frequency spread of a signal $\mathbf{k}$, given by $\triangle_{\hat{\mathfrak{f}}r}^2 = \frac{1}{||\mathbf{k}||^2}\sum_{\mathfrak{m}\in\mathbb{Z}}(\mathfrak{m} - \mu_\tau)^2|\mathbf{k}(\mathfrak{m})|^2$ with frequency center $\mu_{\hat{\mathfrak{f}}r} = \frac{1}{||\mathbf{k}||^2}\int_0^1 \nu|\mathcal{F}\mathbf{k}(\nu)|^2 d\nu$. For details, see [47]. Other versions of the uncertainty principle for sequences are also available that uses a different measure of frequency spread and which even drops the condition of $\hat{\mathbf{k}}(1) = 0$ (see [40]). Donoho and Stark [15] further generalized the uncertainty principle significantly and explain interesting phenomena in signal recovery problems where there is an interplay of missing data, sparsity, and bandlimiting. It states that for Fourier transform pair $(\mathfrak{f}, \hat{\mathfrak{f}})$ with $\mathfrak{f}$ practically zero outside a measurable set R and $\hat{\mathfrak{f}}$ practically zero outside the measurable set S, we have $|R||S| \geq 1 - \rho$, where $|R|$ and $|S|$ denote the measures of the sets and ρ is a small number depending on the phrase "practically zero" (in L^2 or L^1 sense). In case of sequences, if $\{\mathfrak{y}_r\}_{r=0}^{V-1}$ is a sequence of length V with discrete Fourier transform $\{\hat{\mathfrak{y}}_s\}_{s=0}^{V-1}$ and if $\{\mathfrak{y}_r\}$ is not zero at V_r points and $\{\hat{\mathfrak{y}}_s\}$ is not zero at V_s points. Then $V_r V_s \geq V$. The underdetermined linear system of equations $\mathbf{Py} = \mathbf{q}$ (where $\mathbf{P} \in \mathbb{R}^{i\times j}$ is a full-rank matrix) has infinitely many solutions. In order to obtain one well-defined solution, sparsity optimization problem is considered. A vector $\mathbf{y}$ is said to be sparse if it consists of only few nonzero entries. In other words, $\mathbf{y}$ is sparse if $||\mathbf{y}||_0 \ll j$, where $||\cdot||_0$ denotes the l_0 "norm" defined as $||\mathbf{y}||_0 = \sharp\{\mathfrak{k} : \mathfrak{y}_{\mathfrak{k}} \neq 0\}$. Thus, a better representation method which leads to more sparsity is practically a preferable way to tackle such problems. Now, consider a nonzero vector (or a signal) $\mathbf{u} \in \mathbb{R}^i$ and let $\mathbf{\Upsilon}$ and $\mathbf{\Omega}$ be two orthobases. Then $\mathbf{u}$ can be represented as $\mathbf{u} = \mathbf{\Upsilon}\gamma = \mathbf{\Omega}\delta$, where γ and δ are uniquely defined. If we take $\mathbf{\Upsilon}$ to be the identity matrix and $\mathbf{\Omega}$ to be the matrix of the Fourier transform, then the time-domain and frequency-domain representations of $\mathbf{u}$ are γ and δ, respectively. One may note that for particular choice of pairs of bases $\mathbf{\Upsilon}$ and $\mathbf{\Omega}$, either γ can be sparse, or δ can be sparse. Mathematically, we have

$$||\gamma||_0 + ||\delta||_0 \geq 2/\mu(\mathbf{P}), \tag{3}$$

where $\mu(\mathbf{P})$ is the mutual coherence matrix $\mathbf{P}$, i.e., the largest absolute normalized inner product between different columns from $\mathbf{P}$ which characterizes the dependence between columns of the matrix $\mathbf{P}$. Above inequality is also known as uncertainty principle 1. One may note that if the mutual coherence of two bases is small, γ and δ cannot both be very sparse. In contrast to classical uncertainty principle, (3) gives a lower bound on the sum of the nonzeros. Moving from uncertainty to uniqueness leads to one of its another version, i.e., uncertainty principle 2. Let $\mathbf{y}_0$ and $\mathbf{y}_1$ be two solutions of linear system $\mathbf{Py} = [\mathbf{\Upsilon}, \mathbf{\Omega}]\mathbf{y} = \mathbf{q}$ of which one is very sparse. Then according to uncertainty principle 2, any two distinct solutions of the linear system $[\mathbf{\Upsilon}, \mathbf{\Omega}]\mathbf{y} = \mathbf{q}$ cannot both be very sparse. Mathematically, it is given by

$$||\mathbf{y_0}||_0 + ||\mathbf{y_1}||_0 \geq ||\mathbf{e}||_0 \geq 2/\mu(\mathbf{P}), \tag{4}$$

where $\mathbf{e} = \mathbf{y_0} - \mathbf{y_1}$. For more details on uncertainty principle, one may read [4, 6, 15, 17, 23, 37, 43].

Discrete Fourier transform is the conventional technique for spectrum analysis which is usually executed using a fast Fourier transform algorithm. However, spectrum analysis exclusively over a subset of the L center frequencies of an L-point discrete Fourier transform can be done by computing a single complex DFT spectral bin value for every L input time samples. The involved method is Goertzel algorithm. Later, Jacobsen and Lyons [28, 29] constructed sliding discrete Fourier transform (SDFT), a recursive algorithm for computation of discrete Fourier transform on a sample-by-sample basis. For real-time spectral analysis, it was found that SDFT requires fewer computations over Goertzel algorithm.

Executing the intermediate computations by adopting the techniques of FFT motivated the block processing approach. Burrus [12, 13] proposed an exquisite method for block processing which employ the use of the matrix representation of the convolution and helped in representing periodically time-varying digital filters using a time-in-variant digital filter. This approach was further beneficial in the FFT implementation of multirate digital filters. An interesting characteristic that lies in the block implementation method introduced by Burrus is that while analyzing the eigenvalues of a specific $\mathfrak{M} \times \mathfrak{M}$ matrix with block length $\mathfrak{M}$, stability of block structure can dexterously be examined and in fact, if the original scalar transfer function is stable, then the block structure is also stable. For more details, see [36, 38]. Thus, the introduction of block structures reduces the computational complexity and block implementations of digital filters endorses efficacious use of parallel processors which results in high speed. This led us to define the notion of block sliding discrete Fourier transform.

Definition 3.1. *Let $\mathfrak{M}, \mathfrak{N} \in \mathbb{N}$ and $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$. Block sliding discrete Fourier transform (BSDFT) on $l^2(\mathbb{Z}_{\mathfrak{L}})$ is a map $\mathcal{F}_{\mathfrak{b}} : l^2(\mathbb{Z}_{\mathfrak{L}}) \to l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}})$ given by*

$$\mathcal{F}_{\mathfrak{b}}\mathfrak{z}(\mathfrak{m}, \mathfrak{n}) = \frac{1}{\sqrt{\mathfrak{M}}} \sum_{\mathfrak{r}=0}^{\mathfrak{M}-1} \mathfrak{z}(\mathfrak{r} + \mathfrak{n}\mathfrak{M}) e^{-2\pi i \mathfrak{r}\mathfrak{m}/\mathfrak{M}},$$

where $(\mathfrak{m}, \mathfrak{n}) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}$ and $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$.

Indeed, BSDFT $\mathcal{F}_{\mathfrak{b}}$ is a square matrix of order $\mathfrak{L} \times \mathfrak{L}$. For example, let $\mathfrak{N} = 2, \mathfrak{M} = 3$ and BSDFT $\mathcal{F}_{\mathfrak{b}}$ is given by

$$\mathcal{F}_{\mathfrak{b}} = \frac{1}{\sqrt{3}} \begin{pmatrix} w_{0,0} & w_{0,1} & w_{0,2} & 0 & 0 & 0 \\ w_{1,0} & w_{1,1} & w_{1,2} & 0 & 0 & 0 \\ w_{2,0} & w_{2,1} & w_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{0,0} & w_{0,1} & w_{0,2} \\ 0 & 0 & 0 & w_{1,0} & w_{1,1} & w_{1,2} \\ 0 & 0 & 0 & w_{2,0} & w_{2,1} & w_{2,2} \end{pmatrix}.$$

Let $\tau_{\mathfrak{m},\mathfrak{n}} = \frac{1}{\sqrt{\mathfrak{M}}} E_{\frac{\mathfrak{m}}{\mathfrak{M}}} T_{\mathfrak{n}\mathfrak{M}} \chi_{\mathbb{Z}_{\mathfrak{M}}}$, where $0 \le \mathfrak{m} \le \mathfrak{M} - 1$ and $0 \le \mathfrak{n} \le \mathfrak{N} - 1$. One can notice that $\{\tau_{\mathfrak{m},\mathfrak{n}}\}$ is an orthonormal basis of $l^2(\mathbb{Z}_{\mathfrak{L}})$. In fact, BSDFT $\mathcal{F}_{\mathfrak{b}}$ is the analysis operator of orthonormal basis of $\{\tau_{\mathfrak{m},\mathfrak{n}}\}$ from $l^2(\mathbb{Z}_{\mathfrak{L}})$ onto $l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}})$, that is $\mathcal{F}_{\mathfrak{b}}\mathfrak{z}(\mathfrak{m}, \mathfrak{n}) = \langle \mathfrak{z}, \tau_{\mathfrak{m},\mathfrak{n}} \rangle = \frac{1}{\sqrt{\mathfrak{M}}} \sum_{\mathfrak{r}=0}^{\mathfrak{M}-1} \mathfrak{z}(\mathfrak{r} + \mathfrak{n}\mathfrak{M}) e^{-2\pi i \mathfrak{r}\mathfrak{m}/\mathfrak{M}}$, $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$.
Using frame theory and above discussion, we can summarize it as given below.

Proposition 3.2. *BSDFT $\mathcal{F}_{\mathfrak{b}}$ is a unitary operator from $l^2(\mathbb{Z}_{\mathfrak{L}})$ onto $l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}})$.*

The inversion of BSDFT $\mathcal{F}_{\mathfrak{b}}$ is a map $\mathcal{F}_{\mathfrak{b}}^* : l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}) \to l^2(\mathbb{Z}_{\mathfrak{L}})$ given by

$$\mathcal{F}_{\mathfrak{b}}^*(\{\alpha(\mathfrak{m}, \mathfrak{n})\}) = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} \alpha(\mathfrak{m}, \mathfrak{n}) \tau_{\mathfrak{m},\mathfrak{n}}, \ \{\alpha(\mathfrak{m}, \mathfrak{n})\} \in l^2(\mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}).$$

The reconstruction of signals from BSDFT $\mathcal{F}_{\mathfrak{b}}$ is given by

$$\mathfrak{z} = \mathcal{F}_{\mathfrak{b}}^* \mathcal{F}_{\mathfrak{b}} \mathfrak{z} = \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1} \sum_{\mathfrak{n}=0}^{\mathfrak{N}-1} \langle \mathfrak{z}, \tau_{\mathfrak{m},\mathfrak{n}} \rangle \tau_{\mathfrak{m},\mathfrak{n}}, \ \mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}}).$$

Next, we give an uncertainty principle for BSDFT. Recall that the uncertainty principle for DFT is $\|\mathfrak{x}\|_0 \|\mathcal{F}\mathfrak{x}\|_0 \ge \mathfrak{L}$, for $\mathfrak{x} \ne 0$, $\mathfrak{x} \in l^2(\mathbb{Z}_{\mathfrak{L}})$. In this direction, we show BSDFT is sparse than DFT.

Theorem 3.3. *Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$ and $\mathfrak{x} \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Then $\|\mathfrak{x}\|_0 \|\mathcal{F}_{\mathfrak{b}}\mathfrak{x}\|_0 \ge \mathfrak{M}$, $\mathfrak{x} \ne 0$.*

Proof. Let $\|\mathcal{F}_{\mathfrak{b}}\mathfrak{x}\|_\infty = |\langle \mathfrak{x}, \tau_{\mathfrak{m},\mathfrak{k}} \rangle|$, for some $\mathfrak{m} \in \mathbb{Z}_{\mathfrak{M}}$ and $\mathfrak{k} \in \mathbb{Z}_{\mathfrak{N}}$. Then

$$\|\mathcal{F}_{\mathfrak{b}}\mathfrak{x}\|_\infty = |\sum_{\mathfrak{j}=0}^{\mathfrak{L}-1} \mathfrak{x}(\mathfrak{j})\, \tau_{\mathfrak{m},\mathfrak{k}}(\mathfrak{j})| \le \frac{1}{\sqrt{\mathfrak{M}}} \sum_{\mathfrak{j}=0}^{\mathfrak{L}-1} |\mathfrak{x}(\mathfrak{j})| = \frac{1}{\sqrt{\mathfrak{M}}} \|\mathfrak{x}\|_1.$$

Also, let $\|x\|_\infty = |x(\mathfrak{s})|$, for some $\mathfrak{s} \in \mathbb{Z}_{\mathfrak{L}}$. So, we have

$$\|x\|_\infty = |\sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}\langle x, \tau_{\mathfrak{m},k}\rangle\tau_{\mathfrak{m},k}(\mathfrak{s})| \leq \frac{1}{\sqrt{\mathfrak{M}}}\sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}|\langle x, \tau_{\mathfrak{m},k}\rangle| = \frac{1}{\sqrt{\mathfrak{M}}}\|\mathcal{F}_{\mathfrak{b}}x\|_1.$$

But $\|x\|_1 \leq \|x\|_0\|x\|_\infty$ and $\|\mathcal{F}_{\mathfrak{b}}x\|_1 \leq \|\mathcal{F}_{\mathfrak{b}}x\|_0\|\mathcal{F}_{\mathfrak{b}}x\|_\infty$. Note that

$$\|x\|_1 \leq \|x\|_0\frac{1}{\sqrt{\mathfrak{M}}}\|\mathcal{F}_{\mathfrak{b}}x\|_1 \leq \frac{1}{\sqrt{\mathfrak{M}}}\|x\|_0\|\mathcal{F}_{\mathfrak{b}}x\|_0\|\mathcal{F}_{\mathfrak{b}}x\|_\infty \leq \frac{1}{\mathfrak{M}}\|x\|_0\|\mathcal{F}_{\mathfrak{b}}x\|_0\|x\|_1.$$

This gives $\|x\|_0\|\mathcal{F}_{\mathfrak{b}}x\|_0 \geq \mathfrak{M}$. □

Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}$ and $x \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Let $\mathcal{Z}x(\mathfrak{n}, k) = \sum_{\mathfrak{j}=0}^{\mathfrak{N}-1} x(\mathfrak{n} + \mathfrak{j}\mathfrak{M})\ e^{-2\pi i\frac{\mathfrak{j}k}{\mathfrak{N}}}, (\mathfrak{n}, k) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}$ and $x(\mathfrak{s}) = \frac{1}{\mathfrak{N}}\sum_{k=0}^{\mathfrak{N}-1}\mathcal{Z}x(\mathfrak{s}, k)$.

In the following result, we give an uncertainty Principle in sparsity for finite Zak transform. Here also we show, finite Zak transform is sparse than DFT.

Theorem 3.4. *Let $x \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Then $\|x\|_0\|\mathcal{Z}x\|_0 \geq \mathfrak{N},\ x \neq 0$.*

Proof. Let $\|x\|_\infty = |x(\mathfrak{n})|$, for some $\mathfrak{n} \in \mathbb{Z}_{\mathfrak{L}}$. Then

$$\|x\|_\infty = |\frac{1}{\mathfrak{N}}\sum_{k=0}^{\mathfrak{N}-1}\mathcal{Z}x(\mathfrak{n}, k)| \leq \frac{1}{\mathfrak{N}}\sum_{\mathfrak{n}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}|\mathcal{Z}x(\mathfrak{n}, k)| = \frac{1}{\mathfrak{N}}\|\mathcal{Z}x\|_1.$$

Similarly, let $\|\mathcal{Z}x\|_\infty = |\mathcal{Z}x(\mathfrak{n}, k)|$, for some $(\mathfrak{n}, k) \in \mathbb{Z}_{\mathfrak{M}} \times \mathbb{Z}_{\mathfrak{N}}$. Then

$$\begin{aligned}\|\mathcal{Z}x\|_\infty &= |\sum_{\mathfrak{j}=0}^{\mathfrak{N}-1} x(\mathfrak{n} + \mathfrak{j}\mathfrak{M})e^{-2\pi i\mathfrak{j}\frac{k}{\mathfrak{N}}}| \\ &= |\frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{L}-1} x(\mathfrak{l})e^{-2\pi i k\frac{\mathfrak{l}-\mathfrak{n}}{\mathfrak{L}}}\sum_{\mathfrak{r}=0}^{\mathfrak{M}-1} e^{-2\pi i\frac{\mathfrak{l}-\mathfrak{n}}{\mathfrak{M}}\mathfrak{r}}| \\ &\leq \frac{1}{\mathfrak{M}}\sum_{\mathfrak{l}=0}^{\mathfrak{L}-1}|x(\mathfrak{l})| \times \mathfrak{M} = \|x\|_1.\end{aligned}$$

Note that $\|x\|_1 \leq \|x\|_0\|x\|_\infty$ and $\|\mathcal{Z}x\|_1 \leq \|\mathcal{Z}x\|_0\|\mathcal{Z}x\|_\infty$.
Also $\|x\|_1 \leq \|x\|_0\frac{1}{\mathfrak{N}}\|\mathcal{Z}x\|_1 \leq \frac{1}{\mathfrak{N}}\|x\|_0\|\mathcal{Z}x\|_0\|\mathcal{Z}x\|_\infty \leq \frac{1}{\mathfrak{N}}\|x\|_0\|\mathcal{Z}x\|_0\|x\|_1$.
This gives $\|x\|_0\|\mathcal{Z}x\|_0 \geq \mathfrak{N}$. □

Analogous to uncertainty principle 1, we give an uncertainty principle for two finite Gabor Parseval frames in terms of sparse representations.

Theorem 3.5. *Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}_1 = \mathfrak{M}_1 \times \mathfrak{N}$. Let $V = \{\nu_{\mathfrak{m},k}\}$, $W = \{w_{\mathfrak{m},k}\}$ be finite Gabor Parseval frames for $l^2(\mathbb{Z}_{\mathfrak{L}})$. Then for every $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$, $\|\Phi^*\mathfrak{z}\|_0\|\Psi^*\mathfrak{z}\|_0 \geq \frac{1}{\mu^2}$, where Φ^*, Ψ^* are the analysis operators of $\{\nu_{\mathfrak{m},k}\}$, $\{w_{\mathfrak{m},k}\}$ and μ is mutual coherence of V and W.*

Proof. Let $\|\Phi^*\mathfrak{z}\|_\infty = |\langle\mathfrak{z}, \nu_{\mathfrak{n},\mathfrak{s}}\rangle|$, for some $\mathfrak{n} \in \mathbb{Z}_{\mathfrak{M}}, \mathfrak{s} \in \mathbb{Z}_{\mathfrak{N}}$. Then

$$\begin{aligned}\|\Phi^*\mathfrak{z}\|_\infty &= |\sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}\langle\mathfrak{z}, w_{\mathfrak{m},k}\rangle\langle w_{\mathfrak{m},k}, \nu_{\mathfrak{n},\mathfrak{s}}\rangle| \\ &\leq \sum_{\mathfrak{m}=0}^{\mathfrak{M}-1}\sum_{k=0}^{\mathfrak{N}-1}|\langle\mathfrak{z}, w_{\mathfrak{m},k}\rangle|\ \mu \\ &= \|\Psi^*\mathfrak{z}\|_1\ \mu.\end{aligned}$$

Similarly, $||\Psi^* \mathfrak{z}||_\infty \leq ||\Phi^* \mathfrak{z}||_1 \ \mu$ and so

$$||\Phi^* \mathfrak{z}||_1 \leq ||\Phi^* \mathfrak{z}||_0 ||\Phi^* \mathfrak{z}||_\infty \leq ||\Phi^* \mathfrak{z}||_0 ||\Psi^* \mathfrak{z}||_1 \ \mu.$$

Thus

$$\begin{aligned} ||\Psi^* \mathfrak{z}||_1 &\leq ||\Psi^* \mathfrak{z}||_0 ||\Psi^* \mathfrak{z}||_\infty \\ &\leq ||\Psi^* \mathfrak{z}||_0 ||\Phi^* \mathfrak{z}||_1 \ \mu \\ &\leq ||\Psi^* \mathfrak{z}||_0 ||\Phi^* \mathfrak{z}||_0 ||\Psi^* \mathfrak{z}||_1 \ \mu^2. \end{aligned}$$

Hence $||\Phi^* \mathfrak{z}||_0 ||\Psi^* \mathfrak{z}||_0 \geq \frac{1}{\mu^2}$. □

The concept of numerical sparsity appeared around 1978 in the field of geophysics. Recently, it has been widely used as sparse representation in various signal processing applications.

Definition 3.6. *[7] Let $\mathfrak{z} \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Numerical sparsity of $\mathfrak{z}$ is defined as*

$$NS(\mathfrak{z}) = \frac{||\mathfrak{z}||_1^2}{||\mathfrak{z}||_2^2}.$$

Finally, using the notion of numerical sparsity, we provide uncertainty principle for finite Gabor frame.

Theorem 3.7. *Let $\mathfrak{L} = \mathfrak{M} \times \mathfrak{N}_1 = \mathfrak{M}_1 \times \mathfrak{N}$. Let $\{\nu_{m,k}\}$ is a finite Gabor frame with bounds $\mathcal{A}$ and $\mathcal{B}$. Let $0 \neq \chi \in l^2(\mathbb{Z}_{\mathfrak{L}})$. Then $NS(\mathfrak{z})NS(\Phi^* \mathfrak{z}) \geq \frac{\mathcal{A}}{||\Phi||_2^2}$, where $\Phi^*$is the analysis operator of $\{\nu_{m,k}\}$ and $||.||_2$ is the induced norm of matrix.*

Proof. We have

$$\begin{aligned} NS(\mathfrak{z})\ NS(\Phi^* \mathfrak{z}) &= \frac{||\mathfrak{z}||_1^2}{||\mathfrak{z}||_2^2} \times \frac{||\Phi^* \mathfrak{z}||_1^2}{||\Phi^* \mathfrak{z}||_2^2} \geq \frac{||\mathfrak{z}||_2^2}{||\mathfrak{z}||_2^2} \times \mathcal{A} \frac{||\mathfrak{z}||_2^2}{||\Phi^* \mathfrak{z}||_2^2} \\ &= \mathcal{A} \frac{||\mathfrak{z}||_2^2}{||\Phi^* \mathfrak{z}||_2^2} \geq \mathcal{A} \frac{1}{||\Phi^*||_2^2}. \end{aligned}$$

□

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