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Asymptotic expansion of solutions for the Robin-Dirichlet problem of Kirchhoff-Carrier type with Balakrishnan-Taylor damping

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Abstract. In this paper, we consider the Robin-Dirichlet problem for a nonlinear wave equation of Kirchhoff-Carrier type with Balakrishnan-Taylor damping. First, under suitable conditions on the initial data, the local existence and uniqueness of a weak solution are proved. Next, an asymptotic expansion of solutions in a small parameter with high order is established. The used main tools are the linearization method for nonlinear terms together with the Faedo-Galerkin method, and the key lemmas of the expansion of high-order polynomials and the Taylor expansion for multi-variable functions.

1. Introduction

In this paper, we consider the following Robin-Dirichlet problem for a nonlinear wave equation with Balakrishnan-Taylor damping and nonlinear sources

$$\begin{cases} u_{tt} - \lambda u_{xxt} - \mu\left(t, a\left(u(t), u'(t)\right), \|u(t)\|^2, \|u(t)\|_a^2\right) u_{xx} \\ \quad = f\left(x, t, u, u_t, u_x, a\left(u(t), u'(t)\right), \|u(t)\|^2, \|u(t)\|_a^2\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - hu(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\lambda > 0$, $h \geq 0$ are given constants, $a(\cdot,\cdot)$ is the symmetric bilinear form on $H^1 \times H^1$ defined by

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + hu(0)v(0),\ \forall u,\ v \in H^1,$$

and $\|v\|_a = \sqrt{a(v,v)}$, $\forall v \in H^1$.

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Eq. $(1.1)_1$ is a model with Balakrishnan-Taylor damping, because of having $a(u(t), u'(t))$ in variables of the function μ, and $a(u(t), u'(t)) = \langle u_x(t), u_x'(t)\rangle$ when $h = 0$. This model was initially proposed by Balakrishnan and Taylor [1], and Bass and Zes [2], it was related to the panel flutter equation and to the spillover problem. Since then, there have been many stability results for the problem having Balakrishnan-Taylor damping, see [4], [6]-[9], [10], [11], [12], [22]-[23], [26] and references therein. For instance, in [26], Zarai and Tatar studied a viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping as follows

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u(t)\|_2^2 + \sigma\langle\nabla u(t), \nabla u_t(t)\rangle\right)\Delta u(t) \\ \qquad + \int_0^t h(t-s)\Delta u(s)ds = |u|^p u, \text{ in } \Omega \times [0, +\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \text{ in } \Omega, \\ u(x,t) = 0, \text{ in } \Gamma \times [0, +\infty), \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary. By using integral inequalities and multiplier techniques, the authors established polynomial decay estimates for the energy of the problem. In [10], Kang et. al. proved a general stability result for the viscoelastic problem with Balakrishnan-Taylor damping and time-varying delay of the form

$$\begin{cases} u_{tt} - \left(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds \\ \quad + \mu_1 f_1(u_t(x,t)) + \mu_2 f_2(u_t(x, t-\tau(t))) = 0, \text{ in } \Omega \times (0,\infty), \\ u = 0, \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \text{ in } \Omega, \\ u_t(x,t) = f_0(x,t), \text{ in } \Omega \times [-\tau(0), 0), \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary, a, b, σ are positive constants, $\mu_1 > 0$, $\mu_2 \neq 0$ is a real number, $\tau(t) > 0$ represents time-varying delay, and g, f_1, f_2 are given functions. In [4], Emmrich and Thalhammer considered a class of integro-differential equations with applications in nonlinear elastodynamics. They proposed a general model for description of nonlinear extensible beams incorporating weak, viscous, strong and Balakrishnan-Taylor damping as follows

$$\begin{aligned} & u_{tt} + \alpha\Delta^2 u + \xi u + \kappa u_t - \lambda\Delta u_t + \mu\Delta^2 u_t \\ & - \left[\beta + \gamma\int_\Omega |\nabla u|^2 dx + \delta\left|\int_\Omega \nabla u \nabla u_t dx\right|^{q-2}\int_\Omega \nabla u\nabla u_t dx\right]\Delta u = h, \end{aligned} \tag{1.4}$$

in $\Omega\times(0,\infty)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. The constants have the physical meaning: $\alpha > 0$ is the elasticity coefficient, $\gamma > 0$ is the extensibility coefficient, $\lambda \geq 0$ is the viscous damping coefficient, $\mu \geq 0$ is the strong damping coefficient, $\delta \geq 0$ is the Balakrishnan-Taylor damping coefficient, $\beta \in \mathbb{R}$ is the axial force coefficient ($\beta > 0$ traction or $\beta < 0$ compression), $\kappa \in \mathbb{R}$ is the weak damping coefficient (although without sign condition), $\xi \in \mathbb{R}$ is a source coefficient and the exponent q belongs to $[2,\infty)$. We note more that the Balakrishnan-Taylor damping $\int_\Omega u_x(x,t)u_{xt}(x,t)dx$ can be considered as a special case of nonlinear Balakrishnan-Taylor damping, $\left|\int_\Omega \nabla u\nabla u_t dx\right|^{q-2}\int_\Omega \nabla u\nabla u_t dx$, in (1.4). Recently, a generalization of (1.4) has been considered by Tavares et. al. in [24], with an alternative expression of the following Balakrishnan-Taylor term

$$\Phi(u, u_t) = \int_\Omega \nabla u\nabla u_t dx = -\int_\Omega (\Delta u)u_t dx.$$

The authors studied well-posedness and long-time dynamics to the following class of extensible beams with Balakrishnan-Taylor and frictional damping as follows

$$\begin{aligned} u_{tt} + \Delta^2 u - \left[\beta + \gamma\int_\Omega |\nabla u|^2 dx + \delta|\Phi(u,u_t)|^{q-2}\Phi(u,u_t)\right]\Delta u \\ + \kappa u_t + f(u) = h, \text{ in } \Omega \times \mathbb{R}_+, \end{aligned} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$.

Motivated by the above works, we consider the problem (1.1) with Balakrishnan-Taylor damping, where $(1.1)_1$ is a nonlinear equation of $a(u(t), u'(t))$, and then of $\langle u_x(t), u_x'(t)\rangle$. Obviously, there are some certain available difficulties for researchers to find the explicit solution of nonlinear initial boundary value problems, such as that of the problem (1.1). Therefore, in one way or another, they want to know more and more the informations of solutions, for example, they find the behavior of solutions. In this paper, in order to study the behavior of solutions of the problem (1.1), we introduce a method named the asymptotic expansion method, in which the solution is approximated by a polynomial in a small parameter and satisfied a high-order estimation. This method was used successfully in our published works, see [14]-[21], [25]. However, the asymptotic expansion method used here is different from our previous papers because of appearing the perturbed parameter h in the elements of nonlinear terms. We shall discuss briefly about the asymptotic expansion in a small parameter h of a weak solution of (1.1) as follows.

Suppose that the weak solution of (1.1) is a function of three variables $u = u(h, x, t)$, $(x, t) \in [0, 1] \times [0, T]$, $|h|$ is small enough. With a fixed pair (x, t), we assume that the function $h \longmapsto u(h, x, t)$ has the Maclaurin expansion given by

$$u(h, x, t) = \sum_{k=0}^{N} \frac{1}{k!} \frac{\partial^k u}{\partial h^k}(0, x, t)\, h^k + h^{N+1} R_N[u, h, x, t], \tag{1.6}$$

where

$$R_N[u, h, x, t] = \frac{1}{(N+1)!} \frac{\partial^{N+1} u}{\partial h^{N+1}}(\theta h, x, t),\ 0 < \theta < 1,$$

or

$$R_N[u, h, x, t] = \frac{1}{N!} \int_0^1 (1-\theta)^N \frac{\partial^{N+1} u}{\partial h^{N+1}}(\theta h, x, t)\, d\theta.$$

In a certain space X, suppose that $R_N[u, h, x, t]$ satisfies the following estimation

$$\|R_N[u, h, \cdot, \cdot]\|_X \le \tilde{C}_N,$$

where $\tilde{C}_N$ is a constant which is independent of h, x, t, $|h|$ is small enough. Then, it implies from (1.6) that

$$\left\| u(h, \cdot, \cdot) - \sum_{k=0}^{N} \frac{1}{k!} \frac{\partial^k u}{\partial h^k}(0, \cdot, \cdot)\, h^k \right\|_X \le \tilde{C}_N |h|^{N+1},\ \text{for } |h| \text{ small enough}, \tag{1.7}$$

we then obtain the following approximation

$$u(h, x, t) \approx \sum_{k=0}^{N} \frac{1}{k!} \frac{\partial^k u}{\partial h^k}(0, x, t)\, h^k,\ \text{for } |h| \text{ small enough},$$

in the sense of (1.7). By the fact that the explicit solution $u(h, x, t)$ is not known, we can not compute all derivatives $\frac{\partial^k u}{\partial h^k}(0, x, t)$, $k = 0, 1, \cdots, N$. To pass this difficulty, we find the functions $u_0(x, t), \cdots, u_N(x, t)$ (independent of h) such that the derivatives $\frac{\partial^k u}{\partial h^k}(0, x, t)$ in (1.7) can be replaced by these functions. Then, we have the following estimation

$$\left\| u(h, \cdot, \cdot) - \sum_{k=0}^{N} u_k(\cdot, \cdot)\, h^k \right\|_X \le \tilde{C}_N |h|^{N+1},\ \text{for } |h| \text{ small enough}. \tag{1.8}$$

Our plan in this paper is as follows. In Section 2, we give some notations and lemmas. In Section 3, we prove the local existence by applying the linearization method together with the Galerkin method.

Specially in Section 4, with the additional assumptions $\mu \in C^{N+1}([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$, $\mu(t,z_1,z_2,z_3) \geq \mu_* > 0$, for all $(t,z_1,z_2,z_3) \in [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$ and $f \in C^{N+1}([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^2)$, we establish an asymptotic expansion of the weak solution $u = u_h$ in a small parameter h with order $N+1$ in the sense of (1.8), via significant techniques with complicated computations. The results obtained here can be considered as a relative generalization of [14], [15], [19], [25].

2. Preliminaries

First, we put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot,\cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess \sup_{0<t<T} \|u(t)\|_X \text{ for } p = \infty.$$

Denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x,t)$, $u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t) = \frac{\partial u}{\partial x}(x,t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x,t)$.

With $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^2)$, $f = f(x,t,y_1,\cdots,y_6)$, we put $D_1 f = \dfrac{\partial f}{\partial x}$, $D_2 f = \dfrac{\partial f}{\partial t}$, $D_{i+2} f = \dfrac{\partial f}{\partial y_i}$, with $i = 1,\cdots,6$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_8^{\alpha_8} f$, $\alpha = (\alpha_1,\cdots,\alpha_8) \in \mathbb{Z}_+^8$, $|\alpha| = \alpha_1 + \cdots + \alpha_8 \leq k$, $D^{(0,\cdots,0)} f = f$.

Similarly, with $\mu \in C^k([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$, $\mu = \mu(t,y_1,\cdots,y_3)$, we put $D_1 \mu = \dfrac{\partial \mu}{\partial t}$, $D_{i+1} \mu = \dfrac{\partial \mu}{\partial y_i}$, with $i = 1,\cdots,3$ and $D^\beta \mu = D_1^{\beta_1} \cdots D_4^{\beta_4} \mu$, $\beta = (\beta_1,\cdots,\beta_4) \in \mathbb{Z}_+^4$, $|\beta| = \beta_1 + \cdots + \beta_4 \leq k$, $D^{(0,\cdots,0)} \mu = \mu$.

On $H^1 \equiv H^1(\Omega)$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{\frac{1}{2}}. \tag{2.1}$$

We set

$$V = \left\{ v \in H^1 : v(1) = 0 \right\}, \tag{2.2}$$

and

$$a(u,v) = \langle u_x, v_x \rangle + h u(0) v(0), \text{ for all } u, v \in V. \tag{2.3}$$

Then, V is a closed subspace of H^1 and three norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent on V.

V is continuously and it is densely embedded in L^2. Identifying L^2 with $\left(L^2\right)'$ (the dual of L^2), we have $V \hookrightarrow L^2 = \left(L^2\right)' \hookrightarrow V'$. We remark that the notation $\langle \cdot,\cdot \rangle$ is also used for the pairing between V and V'.

Then we have the following lemmas.

Lemma 2.1. *The embedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|v\|_{H^1} \text{ for all } v \in H^1. \tag{2.4}$$

Lemma 2.2. *Let $h \geq 0$. Then the embedding $V \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \leq \|v\|_a & \text{for all} \quad v \in V, \\ \frac{1}{\sqrt{2}}\|v\|_{H^1} \leq \|v_x\| \leq \|v\|_a \leq \sqrt{1+h}\,\|v\|_{H^1} & \text{for all} \quad v \in V. \end{cases} \tag{2.5}$$

Lemma 2.3. *Let $h \geq 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.3) *is continuous on $V \times V$ and coercive on V,* i.e.,

$$\begin{array}{lll} \text{(i)} & |a(u,v)| \leq (1+h)\,\|u_x\|\,\|v_x\|, & \text{for all} \quad u,\ v \in V, \\ \text{(ii)} & a(v,v) \geq \|v_x\|^2, & \text{for all} \quad v \in V. \end{array} \tag{2.6}$$

Lemma 2.4. *Let $h \geq 0$. Then there exists the Hilbert orthonormal base $\left\{w_j\right\}$ of the space L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

$$\begin{array}{ll} \text{(i)} & 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots,\ \lim_{j\to+\infty} \lambda_j = +\infty, \\ \text{(ii)} & a\left(w_j, v\right) = \lambda_j \left\langle w_j, v\right\rangle \text{ for all } v \in V,\ j = 1,2,\cdots. \end{array} \tag{2.7}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j, \text{ in } \Omega, \\ w_{jx}(0) - h w_j(0) = w_j(1) = 0,\ w_j \in V \cap C^\infty\left(\overline{\Omega}\right). \end{cases} \tag{2.8}$$

3. Local existence

In this section, we prove the local existence of solutions to the problem (1.1). For this purpose, we consider $T^* > 0$ fixed and make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in V \cap H^2,\ \tilde{u}_{0x}(0) - h\tilde{u}_0(0) = 0;$

(H_2) $\mu \in C^1([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$ and there exists a constant $\mu_* > 0$ such that
$$\mu(t, y_1, \cdots, y_3) \geq \mu_*,\ \forall (t, y_1, \cdots, y_3) \in [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2;$$

(H_3) $f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^2)$ such that
$f(1, t, 0, 0, y_3, \cdots, y_6) = 0,\ \forall t \in [0,T^*],\ \forall (y_3, \cdots, y_6) \in \mathbb{R} \times \mathbb{R}_+^2 \times \mathbb{R}.$

Let $\lambda > 0$, $h \geq 0$. For every $T \in (0, T^*]$, we say that u is a weak solution of the problem (1.1) if

$$\begin{aligned} u \quad \in \quad W_T = \{ & v \in L^\infty(0,T; V \cap H^2) : v' \in L^\infty(0,T; V \cap H^2), \\ & v'' \in L^2(0,T;V) \cap L^\infty(0,T;L^2)\}, \end{aligned}$$

and u satisfies the following variational equation

$$\langle u''(t), v\rangle + \lambda a(u'(t), v) + \mu[u](t) a(u(t), v) = \langle f[u](t), v\rangle, \tag{3.1}$$

for all $v \in V$, and a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$\begin{aligned} \mu[u](t) &= \mu\left(t, a(u(t), u'(t)), \|u(t)\|^2, \|u(t)\|_a^2\right), \\ f[u](x,t) &= f\left(x, t, u, u_t, u_x, a(u(t), u'(t)), \|u(t)\|^2, \|u(t)\|_a^2\right). \end{aligned} \tag{3.3}$$

For each $M > 0$ given, we set the constants $\tilde{K}_M(h,\mu)$, $K_M(h,f)$ as follows

$$\begin{cases} \tilde{K}_M(\mu) = \tilde{K}_M(h,\mu) = \left\|\mu\right\|_{C^1(\tilde{A}_M)} = \left\|\mu\right\|_{C^0(\tilde{A}_M)} + \sum_{i=1}^{4} \left\|D_i\mu\right\|_{C^0(\tilde{A}_M)}, \\ K_M(f) = K_M(h,f) = \left\|f\right\|_{C^1(A_M)} = \left\|f\right\|_{C^0(A_M)} + \sum_{i=1}^{8} \left\|D_i f\right\|_{C^0(A_M)}, \end{cases}$$

where

$$\begin{cases} \left\|\mu\right\|_{C^0(\tilde{A}_M)} = \sup_{(t,y_1,y_2,y_3)\in\tilde{A}_M} \left|\mu(t,y_1,y_2,y_3)\right|, \\ \left\|f\right\|_{C^0(A_M)} = \sup_{(x,t,y_1,\cdots,y_6)\in A_M} \left|f(x,t,y_1,\cdots,y_6)\right|, \\ \tilde{A}_M = [0,T^*]\times[-(1+h)M^2,(1+h)M^2]\times[0,M^2]\times[0,(1+h)M^2], \\ A_M = [0,1]\times[0,T^*]\times[-M,M]^3\times[-(1+h)M^2,(1+h)M^2] \\ \qquad\qquad \times[0,M^2]\times[0,(1+h)M^2]. \end{cases}$$

For each $T \in (0,T^*]$, we denote

$$V_T = \{v \in L^\infty(0,T;V\cap H^2) : v' \in L^\infty(0,T;V\cap H^2),\ v'' \in L^2(0,T;V)\},$$

is a Banach space with respect to the norm

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;V\cap H^2)};\ \|v'\|_{L^\infty(0,T;V\cap H^2)};\ \|v''\|_{L^2(0,T;V)}\}.$$

For every $M > 0$, we put

$$\begin{aligned} W(M,T) &= \{v \in V_T : \|v\|_{V_T} \le M\}, \\ W_1(M,T) &= \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{aligned}$$

Consider the recurrent sequence $\{u_m\}$ with $u_0 \equiv 0$, and suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.4}$$

we will find $u_m \in W_1(M,T)$, $m \ge 1$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t),v\rangle + \lambda a(u_m'(t),v) + \mu_m(t)a(u_m(t),v) = \langle F_m(t),v\rangle,\ \forall v \in V, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

where

$$\begin{cases} \mu_m(t) = \mu[u_{m-1}](t) = \mu\left(t, a\left(u_{m-1}(t), u_{m-1}'(t)\right), \|u_{m-1}(t)\|^2, \|u_{m-1}(t)\|_a^2\right), \\ F_m(t) = f[u_{m-1}](x,t) \\ \qquad = f\left(x,t,u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}(x,t),a\left(u_{m-1}(t),u_{m-1}'(t)\right),\right. \\ \qquad\qquad\qquad \left.\|u_{m-1}(t)\|^2, \|u_{m-1}(t)\|_a^2\right). \end{cases} \tag{3.6}$$

Theorem 3.1. *Let f, μ satisfy the conditions (H_2), (H_3) respectively and if $\tilde{u}_0$, $\tilde{u}_1 \in V\cap H^2$, then there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.5)-(3.6).

Proof of Theorem 3.1. Let $\{w_j\}$ be a completely orthonormal in L^2 as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)w_j,$$

where $c_{mj}^{(k)}$ are determined by the ordinary differential equations as below

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle + \lambda a\left(\dot{u}_m^{(k)}(t),w_j\right) + \mu_m(t)a\left(u_m^{(k)}(t),w_j\right) \\ \qquad\qquad = \left\langle F_m(t), w_j\right\rangle,\ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.7}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \text{ strongly in } V \cap H^2. \end{cases} \tag{3.8}$$

Note that (3.7) leads to an equivalent form of the system (3.9) as follows

$$\begin{cases} \ddot{c}_{mi}^{(k)}(t) + \lambda\lambda_i \dot{c}_{mi}^{(k)}(t) + \lambda_i \mu_m(t) c_{mi}^{(k)}(t) = f_{mi}(t), \\ c_{mi}^{(k)}(0) = \alpha_i^{(k)},\ \dot{c}_{mi}^{(k)}(0) = \beta_i^{(k)},\ 1 \le i \le k, \end{cases} \tag{3.9}$$

where

$$f_{mi}(t) = \langle F_m(t), w_i \rangle,\ 1 \le i \le k. \tag{3.10}$$

Using the Banach's contraction principle, it is not difficult to prove that the system (3.9) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0, T]$. The following priori estimates show the bounds of approximate solution $u_m^{(k)}(t)$.

First, we put

$$\begin{aligned} S_m^{(k)}(t) &= \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \lambda \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2 \\ &\quad + \mu_m(t) \left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 \right) \\ &\quad + 2\lambda \int_0^t \left(\left\|\dot{u}_m^{(k)}(s)\right\|_a^2 + \left\|\Delta \dot{u}_m^{(k)}(s)\right\|^2 \right) ds + 2 \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_a^2 ds. \end{aligned} \tag{3.11}$$

By computing directly to (3.7)-(3.8), we obtain

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\mu_m(0) \langle \Delta \tilde{u}_{0k}, \Delta \tilde{u}_{1k} \rangle \\ &\quad + 2 \int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds + 2 \int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds \\ &\quad + 2 \int_0^t a\left(F_m(s), \ddot{u}_m^{(k)}(s)\right) ds + 2 \int_0^t \mu_m(s) \left\|\Delta \dot{u}_m^{(k)}(s)\right\|^2 ds \\ &\quad + \int_0^t \mu_m'(s) \left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + 2\langle \Delta u_m^{(k)}(s), \Delta \dot{u}_m^{(k)}(s) \rangle \right) ds \\ &\quad - 2\mu_m(t) \langle \Delta u_m^{(k)}(t), \Delta \dot{u}_m^{(k)}(t) \rangle. \end{aligned} \tag{3.12}$$

Using the inequality

$$2ab \le \beta a^2 + \frac{1}{\beta} b^2,\ \forall a, b \ge 0,\ \forall \beta > 0,$$

and the following estimates

$$\begin{aligned} \mu_m(t) &\le \tilde{K}_M(\mu), \\ \|F_m(t)\| &\le K_M(f), \\ \|F_{mx}(t)\| &\le (1 + 3M) K_M(f), \\ \|F_m(t)\|_a &\le \sqrt{1+h}\, \|F_{mx}(t)\| \le \sqrt{1+h}\, (1 + 3M) K_M(f) \equiv \bar{K}_M(f), \end{aligned}$$

we deduce

$$\begin{aligned} 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds &\leq T K_M^2(f) + \int_0^t S_m^{(k)}(s)\,ds; \\ 2\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds &\leq T\bar{K}_M^2(f) + \int_0^t S_m^{(k)}(s)\,ds; \\ 2\int_0^t a\left(F_m(s), \ddot{u}_m^{(k)}(s)\right) ds &\leq 2T\bar{K}_M^2(f) + \frac{1}{4}S_m^{(k)}(t); \\ 2\int_0^t \mu_m(s)\left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 ds &\leq \frac{2}{\lambda}\tilde{K}_M(\mu)\int_0^t S_m^{(k)}(s)\,ds. \end{aligned} \tag{3.13}$$

By (H_2) and the inequality

$$\begin{aligned} S_m^{(k)}(t) &\geq \mu_m(t)\left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2\right) + \lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \\ &\geq \mu_*\left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2\right) \\ &\geq \mu_*\left\|\Delta u_m^{(k)}(t)\right\|^2 + \lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \\ &\geq 2\sqrt{\lambda\mu_*}\left\|\Delta u_m^{(k)}(t)\right\|\left\|\Delta\dot{u}_m^{(k)}(t)\right\|, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^t \mu_m'(s)\left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + 2\langle\Delta u_m^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\rangle\right) ds \\ &\leq \int_0^t \left|\mu_m'(s)\right|\left(\frac{S_m^{(k)}(s)}{\mu_*} + \frac{S_m^{(k)}(s)}{\sqrt{\lambda\mu_*}}\right) ds \\ &= \left(\frac{1}{\mu_*} + \frac{1}{\sqrt{\lambda\mu_*}}\right)\int_0^t \left|\mu_m'(s)\right| S_m^{(k)}(s)\,ds. \end{aligned} \tag{3.14}$$

We note that

$$\begin{aligned} \left\|\frac{\partial}{\partial s}\left(\mu_m(s)\Delta u_m^{(k)}(s)\right)\right\|^2 &= \left\|\mu_m(s)\Delta\dot{u}_m^{(k)}(s) + \mu_m'(s)\Delta u_m^{(k)}(s)\right\|^2 \\ &\leq 2\tilde{K}_M^2(\mu)\left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 + 2\left|\mu_m'(s)\right|^2\left\|\Delta u_m^{(k)}(s)\right\|^2 \\ &\leq 2\left(\frac{\tilde{K}_M^2(\mu)}{\lambda} + \frac{1}{\mu_*}\left|\mu_m'(s)\right|^2\right) S_m^{(k)}(s), \end{aligned}$$

then

$$\begin{aligned} \left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2 &= \left\|\mu_m(0)\Delta\tilde{u}_{0k} + \int_0^t \frac{\partial}{\partial s}\left(\mu_m(s)\Delta u_m^{(k)}(s)\right) ds\right\|^2 \\ &\leq 2\left\|\mu_m(0)\Delta\tilde{u}_{0k}\right\|^2 + 2T\int_0^t \left\|\frac{\partial}{\partial s}\left(\mu_m(s)\Delta u_m^{(k)}(s)\right)\right\|^2 ds \\ &\leq 2\left\|\mu_m(0)\Delta\tilde{u}_{0k}\right\|^2 \\ &\quad + 4T\int_0^t \left(\frac{\tilde{K}_M^2(\mu)}{\lambda} + \frac{1}{\mu_*}\left|\mu_m'(s)\right|^2\right) S_m^{(k)}(s)\,ds. \end{aligned}$$

Hence, $-2\mu_m(t)\langle \Delta u_m^{(k)}(t), \Delta \dot{u}_m^{(k)}(t)\rangle$ is estimated as follows

$$\begin{aligned}
&-2\mu_m(t)\langle \Delta u_m^{(k)}(t), \Delta \dot{u}_m^{(k)}(t)\rangle \\
&\le \frac{4}{\lambda}\left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2 + \frac{\lambda}{4}\left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2 \\
&\le \frac{4}{\lambda}\left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2 + \frac{1}{4}S_m^{(k)}(t) \\
&\le \frac{8}{\lambda}\left\|\mu_m(0)\Delta\tilde{u}_{0k}\right\|^2 \\
&+ \frac{16}{\lambda}T\int_0^t \left(\frac{\tilde{K}_M^2(\mu)}{\lambda} + \frac{1}{\mu_*}\left|\mu_m'(s)\right|^2\right) S_m^{(k)}(s)\,ds + \frac{1}{4}S_m^{(k)}(t).
\end{aligned} \tag{3.15}$$

By the inequalities (3.13), (3.14) and (3.15), we obtain

$$S_m^{(k)}(t) \le \bar{S}_{m,k} + 2T\left(K_M^2(f) + 3\bar{K}_M^2(f)\right) + \int_0^t \bar{\gamma}_m(s) S_m^{(k)}(s)\,ds, \tag{3.16}$$

where

$$\begin{cases}
\bar{S}_{m,k} = 2S_m^{(k)}(0) + 4\mu_m(0)\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle + \dfrac{16}{\lambda}\mu_m^2(0)\left\|\Delta\tilde{u}_{0k}\right\|^2, \\
\bar{\gamma}_m(s) = 4 + \dfrac{4}{\lambda}\tilde{K}_M(\mu) + \dfrac{32T\tilde{K}_M^2(\mu)}{\lambda^2} \\
\qquad +2\left(\dfrac{1}{\mu_*} + \dfrac{1}{\sqrt{\lambda\mu_*}}\right)\left|\mu_m'(s)\right| + \dfrac{32T}{\lambda\mu_*}\left|\mu_m'(s)\right|^2.
\end{cases} \tag{3.17}$$

Note that the real value $\mu_m(0) = \mu\left(0, a(\tilde{u}_{0x}, \tilde{u}_{1x}), \|\tilde{u}_0\|^2, \|\tilde{u}_0\|_a^2\right)$ is independent of m, so $\bar{S}_{m,k} = 2S_m^{(k)}(0) + 4\mu_m(0)\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle + \frac{16}{\lambda}\mu_m^2(0)\|\Delta\tilde{u}_{0k}\|^2$ is also independent of m. Then, by the convergences of $\tilde{u}_{0k}$, $\tilde{u}_{1k}$ given in (3.8), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$\bar{S}_{m,k} \le \frac{1}{4}M^2 \text{ for all } m,\ k \in \mathbb{N}. \tag{3.18}$$

Now, we need the lemmas below, their proofs are easy so we omit.

Lemma 3.2. *The following estimates are valid.*

$$\begin{aligned}
&\text{(i)}\ \left\|\mu_m'\right\|_{L^2(0,T)} \le \tilde{K}_M(\mu)\left[\left(1 + (5+2h)M^2\right)\sqrt{T} + (1+h)M^2\right] \equiv \sigma_T(M); \\
&\text{(ii)}\ \left\|\mu_m'\right\|_{L^1(0,T)} \le \sqrt{T}\left\|\mu_m'\right\|_{L^2(0,T)} \le \sqrt{T}\sigma_T(M); \\
&\text{(iii)}\ \left\|\bar{\gamma}_m\right\|_{L^1(0,T)} \le T\left(4 + \frac{4}{\lambda}\tilde{K}_M(\mu) + \frac{32T}{\lambda^2}\tilde{K}_M^2(\mu)\right) \\
&\qquad +2\left(\frac{1}{\mu_*} + \frac{1}{\sqrt{\lambda\mu_*}}\right)\sqrt{T}\sigma_T(M) + \frac{32T}{\lambda\mu_*}\sigma_T^2(M) \equiv \bar{\gamma}_M(T). \ \square
\end{aligned} \tag{3.19}$$

Lemma 3.3. *For every $T \in (0, T^*]$ and $\beta > 0$, we put*

$$\begin{aligned}
k_{\beta,T} &= \left(1 + \frac{1}{\sqrt{\mu_*}} + \frac{1}{\sqrt{2\lambda}}\right)\sqrt{2\beta(1+T)\bar{D}_M(f,\mu)} \\
&\quad \times \exp\left[\frac{T}{\beta} + \frac{1}{2\mu_*}\sqrt{T}\sigma_T(M)\right],
\end{aligned} \tag{3.20}$$

where

$$\bar{D}_M(f,\mu) = (2 + 5M + hM)^2 K_M^2(f) + M^4(5+h)\tilde{K}_M^2(\mu). \tag{3.21}$$

Let $\beta > 0$ such that

$$\left(1 + \frac{1}{\sqrt{\mu_*}} + \frac{1}{\sqrt{2\lambda}}\right)\sqrt{2\beta \bar{D}_M(f,\mu)} < 1.$$

Then, we can choose $T \in (0, T^*]$, such that

$$\begin{cases} \left(\frac{1}{4}M^2 + 2T\left(K_M^2(f) + 3\bar{K}_M^2(f)\right)\right)\exp\left(\bar{\gamma}_M(T)\right) \le M^2, \\ k_{\beta,T} < 1. \ \square \end{cases}$$

By (3.18), $(3.19)_3$, we have from (3.16) that

$$S_m^{(k)}(t) \le M^2 \exp\left(-\bar{\gamma}_M(T)\right) + \int_0^t \bar{\gamma}_m(s)\, S_m^{(k)}(s)\, ds. \tag{3.22}$$

Applying Gronwall's Lemma, it follows that

$$\begin{aligned} S_m^{(k)}(t) &\le M^2 \exp\left(-\bar{\gamma}_M(T)\right)\exp\left(\int_0^t \bar{\gamma}_m(s)\, ds\right) \\ &\le M^2 \exp\left(-\bar{\gamma}_M(T)\right)\exp\left(\left\|\bar{\gamma}_m\right\|_{L^1(0,T)}\right) \le M^2, \end{aligned}$$

for all $t \in [0, T]$, for all m and k. Therefore, we get

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k. \tag{3.23}$$

From (3.23), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty\left(0,T; V \cap H^2\right) \quad \text{weak*}, \\ \dot{u}_m^{(k)} \to u'_m & \text{in} \quad L^\infty\left(0,T; V \cap H^2\right) \quad \text{weak*}, \\ \ddot{u}_m^{(k)} \to u''_m & \text{in} \quad L^2(Q_T) \quad \text{weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.24}$$

Letting $k \to \infty$ in (3.7), we obtain u_m satisfying (3.5)-(3.6) in $L^2(0,T)$. By $(3.24)_4$, it is not difficult to check that $u''_m = \lambda \Delta u'_m + \mu_m(t)\Delta u_m + F_m \in L^\infty\left(0,T; L^2\right)$. Thus $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is proved completely. □

By using Theorem 3.1 and the compact imbedding theorems, we shall prove the existence and uniqueness of the weak local solution in time to the problem (1.1). First, we introduce the following space

$$W_1(T) = \left\{v \in L^\infty(0,T;V) : v' \in L^\infty\left(0,T;L^2\right) \cap L^2(0,T;V)\right\},$$

which is a Banach space with respect to the norm (see Lions [13])

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)} + \|v'\|_{L^2(0,T;V)}.$$

Theorem 3.3. *Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied. Then, the recurrent sequence $\{u_m\}$ defining by* (3.5)-(3.6) *converges strongly to a function u in $W_1(T)$ and u is the unique weak solution of* (1.1). *Moreover, we have the following estimate*

$$\|u_m - u\|_{W_1(T)} \le C_T\left(k_{\beta,T}\right)^m, \text{ for all } m \in \mathbb{N}, \tag{3.25}$$

where $k_{\beta,T} \in [0,1)$ is defined as in (3.20) *and C_T is a constant depending only on T, f, μ, $\tilde{u}_0$, $\tilde{u}_1$, λ, h and $k_{\beta,T}$.*

Proof of Theorem 3.3. First, we prove the local existence in time of (1.1). It is necessary to prove that $\{u_m\}$ (in Theorem 3.1) is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + \lambda a(v_m'(t), w) + \mu_{m+1}(t)\, a(v_m(t), w) \\ \quad = \langle F_{m+1}(t) - F_m(t), w\rangle + [\mu_{m+1}(t) - \mu_m(t)]\langle \Delta u_m(t), w\rangle, \ \forall w \in V, \\ v_m(0) = v_m'(0) = 0, \end{cases} \tag{3.26}$$

where

$$\begin{aligned} & F_{m+1}(t) - F_m(t) \\ = \ & f\left(x, t, u_m(x,t), u_m'(x,t), \nabla u_m(x,t), a(u_m(t), u_m'(t)), \|u_m(t)\|^2, \|u_m(t)\|_a^2\right) \\ & - f\left(x, t, u_{m-1}(x,t), u_{m-1}'(x,t), \nabla u_{m-1}(x,t), a\left(u_{m-1}(t), u_{m-1}'(t)\right),\right. \\ & \qquad \left. \|u_{m-1}(t)\|^2, \|u_{m-1}(t)\|_a^2\right), \\ & \mu_{m+1}(t) - \mu_m(t) \\ = \ & \mu\left(t, a(u_m(t), u_m'(t)), \|u_m(t)\|^2, \|u_m(t)\|_a^2\right) \\ & - \mu\left(t, a\left(u_{m-1}(t), u_{m-1}'(t)\right), \|u_{m-1}(t)\|^2, \|u_{m-1}(t)\|_a^2\right). \end{aligned}$$

Taking $w = v_m'$ in $(3.26)_1$ and then integrating in t, we get

$$\begin{aligned} S_m(t) = & \int_0^t \mu_{m+1}'(s)\, \|v_m(s)\|_a^2\, ds + 2\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle ds \\ & + 2\int_0^t [\mu_m(s) - \mu_{m+1}(s)]\langle \Delta u_m(s), v_m'(s)\rangle ds, \end{aligned} \tag{3.27}$$

where

$$S_m(t) = \left\|v_m'(t)\right\|^2 + 2\lambda \int_0^t \left\|v_m'(s)\right\|_a^2 ds + \mu_{m+1}(t)\, \|v_m(t)\|_a^2. \tag{3.28}$$

Next, we have to estimate the integrals on right-hand side of (3.27).
Since

$$S_m(t) \ge \mu_{m+1}(t)\, \|v_m(t)\|_a^2 \ge \mu_*\, \|v_m(t)\|_a^2,$$

it follows that

$$\int_0^t \mu_{m+1}'(s)\, \|v_m(s)\|_a^2\, ds \le \frac{1}{\mu_*}\int_0^t \left|\mu_{m+1}'(s)\right| S_m(s) ds. \tag{3.29}$$

By (H_3), it is clear to see that

$$\|F_{m+1}(t) - F_m(t)\| \le K_M(f)\,[2 + (5+h)M]\left(\|v_{m-1}(t)\|_a + \left\|v_{m-1}'(t)\right\|_a\right).$$

Hence

$$2\left|\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle ds\right| \tag{3.30}$$
$$\le 2K_M(f)\left[2 + (5+h)M\right]\int_0^t \left(\|v_{m-1}(s)\|_a + \left\|v_{m-1}'(s)\right\|_a\right)\left\|v_m'(s)\right\| ds$$
$$\le 2\beta K_M^2(f)\left[2 + (5+h)M\right]^2\left(T\|v_{m-1}\|^2_{L^\infty(0,T;V)} + \left\|v_{m-1}'(t)\right\|^2_{L^2(0,T;V)}\right)$$
$$+ \frac{1}{\beta}\int_0^t S_m(s)ds$$
$$\le 2\beta K_M^2(f)\left[2 + (5+h)M\right]^2(1+T)\|v_{m-1}\|^2_{W_1(T)} + \frac{1}{\beta}\int_0^t S_m(s)ds.$$

By the fact that

$$\begin{aligned} & \left|\mu_{m+1}(t) - \mu_m(t)\right| \\ \le\ & \tilde{K}_M(\mu)\left|a(u_m(t), u_m'(t)) - a\left(u_{m-1}(t), u_{m-1}'(t)\right)\right| \\ & + \tilde{K}_M(\mu)\left(\left|\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right| + \left|\|u_m(t)\|_a^2 - \|u_{m-1}(t)\|_a^2\right|\right) \\ \le\ & \tilde{K}_M(\mu)\left[(1+h)M\left(\|v_{m-1}(t)\|_a + \left\|v_{m-1}'(t)\right\|_a\right) + 4M\|v_{m-1}(t)\|_a\right] \\ \le\ & (5+h)M\tilde{K}_M(\mu)\left(\|v_{m-1}(t)\|_a + \left\|v_{m-1}'(t)\right\|_a\right), \end{aligned}$$

we obtain

$$2\left|\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right]\langle \Delta u_m(s), v_m'(s)\rangle ds\right| \tag{3.31}$$
$$\le 2M^2(5+h)\tilde{K}_M(\mu)\int_0^t \left(\|v_{m-1}(s)\|_a + \left\|v_{m-1}'(s)\right\|_a\right)\left\|v_m'(s)\right\| ds$$
$$\le 2\beta M^4(5+h)^2\tilde{K}_M^2(\mu)\left(T\|v_{m-1}\|^2_{L^\infty(0,T;V)} + \left\|v_{m-1}'\right\|^2_{L^2(0,T;V)}\right)$$
$$+ \frac{1}{\beta}\int_0^t S_m(s)ds$$
$$\le 2\beta M^4(5+h)^2\tilde{K}_M^2(\mu)(1+T)\|v_{m-1}\|^2_{W_1(T)} + \frac{1}{\beta}\int_0^t S_m(s)ds.$$

Combining (3.29), (3.30) and (3.31), we deduce from (3.27) that

$$\begin{aligned} S_m(t) \le\ & 2\beta(1+T)\bar{D}_M(f,\mu)\|v_{m-1}\|^2_{W_1(T)} \\ & + \int_0^t \left(\frac{2}{\beta} + \frac{1}{\mu_*}\left|\mu_{m+1}'(s)\right|\right)S_m(s)ds, \end{aligned}$$

where $\bar{D}_M(f,\mu)$ as in (3.21).

By using Gronwall's Lemma, we have

$$\begin{aligned} S_m(t) &\le 2\beta(1+T)\bar{D}_M(f,\mu)\|v_{m-1}\|^2_{W_1(T)}\exp\left[\int_0^t \left(\frac{2}{\beta} + \frac{1}{\mu_*}\left|\mu_{m+1}'(s)\right|\right)ds\right] \\ &\le 2\beta(1+T)\bar{D}_M(f,\mu)\|v_{m-1}\|^2_{W_1(T)}\exp\left(\frac{2T}{\beta} + \frac{1}{\mu_*}\left\|\mu_{m+1}'\right\|_{L^1(0,T)}\right) \\ &\le 2\beta(1+T)\bar{D}_M(f,\mu)\|v_{m-1}\|^2_{W_1(T)}\exp\left(\frac{2T}{\beta} + \frac{1}{\mu_*}\sqrt{T}\sigma_T(M)\right). \end{aligned} \tag{3.32}$$

Hence, it follows from (3.28) and (3.32) that

$$\|v_m\|_{W_1(T)} \le k_{\beta,T} \|v_{m-1}\|_{W_1(T)}, \ \forall m \in \mathbb{N},$$

where $k_{\beta,T} \in [0,1)$ is defined as in (3.20), which implies that

$$\left\|u_{m+p} - u_m\right\|_{W_1(T)} \le \frac{\left(k_{\beta,T}\right)^m}{1-k_{\beta,T}} \|u_1 - u_0\|_{W_1(T)} \le \frac{M}{1-k_{\beta,T}} \left(k_{\beta,T}\right)^m, \ \forall m, p \in \mathbb{N}.$$

The above inequality ensures that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \text{ strongly in } W_1(T). \tag{3.33}$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\left\{u_{m_j}\right\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty\left(0,T; V \cap H^2\right) \ \text{weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty\left(0,T; V \cap H^2\right) \ \text{weak*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \ \text{weak}, \\ u \in W(M,T). \end{cases} \tag{3.34}$$

We note that

$$\left\|F_m - f[u]\right\|_{L^2(Q_T)} \le \left(1+\sqrt{T}\right)[2+(5+h)M]\, K_M(f) \|u_{m-1} - u\|_{W_1(T)}.$$

Hence, since (3.33) we deduce

$$F_m \longrightarrow f[u] \ \text{strongly in } L^2(Q_T). \tag{3.35}$$

We also note that

$$\left\|\mu_m - \mu[u]\right\|_{L^2(0,T)} \le \left(1+\sqrt{T}\right) M^2 (5+h)\, \tilde{K}_M(\mu) \|u_{m-1} - u\|_{W_1(T)}. \tag{3.36}$$

On the other hand, for all $v \in V$, we have

$$\begin{aligned} &\left|\mu_m(t)\, a(u_m(t), v) - \mu[u](t)\, a(u(t), v)\right| \\ &\le \left|\mu_m(t)\, a(u_m(t) - u(t), v)\right| + \left|\mu_m(t) - \mu[u](t)\right| |a(u(t), v)| \\ &\le (1+h)\, \tilde{K}_M(\mu) \|u_m(t) - u(t)\|_a \|v\|_a + (1+h) \left|\mu_m - \mu[u](t)\right| \|u(t)\|_a \|v\|_a \\ &\le (1+h)\, \tilde{K}_M(\mu) \|v\|_a \|u_m - u\|_{W_1(T)} \\ &+ (1+h) \|u\|_{W_1(T)} \|v\|_a \left|\mu_m(t) - \mu[u](t)\right|. \end{aligned} \tag{3.37}$$

Hence, since (3.33), (3.36) and (3.37) we obtain

$$\begin{aligned} &\left|\int_0^T \mu_m(t)\, a(u_m(t), v)\, \phi(t)\, dt - \int_0^T \mu[u](t)\, a(u(t), v)\, \phi(t)\, dt\right| \\ &\le (1+h)\, \tilde{K}_M(\mu) \|v\|_a \left\|\phi\right\|_{L^1(0,T)} \|u_m - u\|_{W_1(T)} \\ &+ (1+h) \|u\|_{W_1(T)} \|v\|_a \int_0^T \left|\mu_m(t) - \mu[u](t)\right| \left|\phi(t)\right| dt \\ &\le (1+h)\, \tilde{K}_M(\mu) \|v\|_a \left\|\phi\right\|_{L^1(0,T)} \|u_m - u\|_{W_1(T)} \\ &+ (1+h) \|u\|_{W_1(T)} \|v\|_a \left\|\phi\right\|_{L^2(0,T)} \left\|\mu_m - \mu[u]\right\|_{L^2(0,T)} \ \to 0, \end{aligned} \tag{3.38}$$

$\forall v \in V, \forall \phi \in L^2(0,T)$.

Letting $m = m_j \to \infty$ in (3.5), (3.6) and using (3.34), (3.35) and (3.38), we get that there exists $u \in W(M,T)$ satisfying the equation

$$\langle u''(t), v\rangle + \lambda a(u'(t), v) + \mu[u](t) a(u(t), v) = \langle f[u](t), v\rangle, \ \forall v \in V, \tag{3.39}$$

and the initial conditions

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1.$$

Moreover, since the assumptions (H_2) and (H_3) we obtain from $(3.34)_4$ and (3.39) that

$$u'' = \lambda \Delta u' + \mu[u] \Delta u + f \in L^\infty\left(0,T;L^2\right),$$

thus we have $u \in W_1(M,T)$. The proof of existence is completed.

Finally, we need to prove the uniqueness. Let $u_1, u_2 \in W_1(M,T)$ be two weak solutions of the problem (1.1). Then $u = u_1 - u_2$ satisfies the variational proplem

$$\begin{cases} \langle u''(t), v\rangle + \lambda a(u'(t), v) + \bar{\mu}_1(t) a(u(t), v) \\ \qquad = \langle \bar{F}_1(t) - \bar{F}_2(t), v\rangle + \left[\bar{\mu}_1(t) - \bar{\mu}_2(t)\right]\langle \Delta u_2(t), v\rangle, \ \forall v \in V, \\ u(0) = u'(0) = 0, \end{cases}$$

where

$$\begin{cases} \bar{\mu}_i(t) = \mu[u_i](t) = \mu\left(t, a\left(u_i(t), u_i'(t)\right), \|u_i(t)\|^2, \|u_i(t)\|_a^2\right), \\ \bar{F}_i(t) = f[u_i](t) \\ \qquad = f\left(x,t,u_i,u_i',\nabla u_i, a\left(u_i(t), u_i'(t)\right), \|u_i(t)\|^2, \|u_i(t)\|_a^2\right), \ i = 1,2. \end{cases}$$

Taking $v = u'(t)$ and integrating in time from 0 to t, we get

$$\begin{aligned} Z(t) = &\int_0^t \bar{\mu}_1'(s) \|u(s)\|_a^2 ds + 2\int_0^t \langle \bar{F}_1(s) - \bar{F}_2(s), u'(s)\rangle ds \\ &+ 2\int_0^t \left(\bar{\mu}_1(s) - \bar{\mu}_2(s)\right)\langle \Delta u_2(s), u'(s)\rangle ds, \end{aligned} \tag{3.40}$$

where

$$Z(t) = \|u'(t)\|^2 + \bar{\mu}_1(t)\|u(t)\|_a^2 + 2\lambda\int_0^t \|u'(s)\|_a^2 ds. \tag{3.41}$$

Put $\bar{q}(s) = \bar{D}_M + \frac{2}{\mu_*}\left|\bar{\mu}_1'(s)\right|$, where $\bar{D}_M$ is a constant as follows

$$\begin{aligned} \bar{D}_M \ = \ & 2\left(\frac{1}{\sqrt{\mu_*}} + \frac{(2+5M+hM) K_M(f)}{\lambda}\right)(2+5M+hM) K_M(f) \\ & +2\left(\frac{1}{\sqrt{\mu_*}} + \frac{(5+h) M^2 \tilde{K}_M(\mu)}{\lambda}\right)(5+h) M^2 \tilde{K}_M(\mu). \end{aligned}$$

Then, by simple calculations, it follows from (3.40) that

$$Z(t) \le \int_0^t \bar{q}(s) Z(s) ds. \tag{3.42}$$

We note that

$$\left|\bar{\mu}_1'(s)\right| \le \tilde{K}_M(\mu)\left[1 + (5+2h) M^2 + (1+h) M \left\|u_1''(s)\right\|_a\right],$$

and since $u_1'' \in L^2(0,T;V)$, we obtain $\bar{\mu}_1' \in L^2(0,T)$, so $\bar{q} \in L^2(0,T)$.

Therefore, by (3.42) we have that

$$Z^2(t) \le \|\bar{q}\|^2_{L^2(0,T)} \int_0^t Z^2(s)\,ds.$$

By Gronwall lemma, it follows that $Z(t) \equiv 0$, i.e., $u_1 \equiv u_2$. The uniqueness is proved. Consequently, this completes the proof of Theorem 3.3. □

4. Asymptotic expansion of the solution

In this section, we consider the following perturbed problem, where h is a small parameter, with $0 \le h \le 1$:

$$(P_h)\begin{cases} u_{tt} - \lambda u_{xxt} - \mu\left(t, a_h(u(t),u'(t)), \|u(t)\|^2, \|u(t)\|^2_{a_h}\right) u_{xx} \\ \quad = f\left(x,t,u,u_t,u_x,a_h(u(t),u'(t)), \|u(t)\|^2, \|u(t)\|^2_{a_h}\right),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) - hu(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\lambda > 0$ is a given constant, $a_h(\cdot,\cdot)$ is the symmetric bilinear form on $H^1 \times H^1$ (depends on h) defined by

$$a_h(u,v) = \int_0^1 u_x(x)v_x(x)dx + hu(0)v(0),\ \forall u,\ v \in H^1,$$

and $\|v\|_{a_h} = \sqrt{a_h(v,v)}$, $\forall v \in H^1$.

First, we note that if the functions $(\tilde{u}_0, \tilde{u}_1)$, f, μ satisfy $(H_1)-(H_3)$ respectively, then a priori estimates of the Galerkin approximation sequence $\{u_m^{(k)}\}$ in the proof of Theorem 3.1 for Prob. (1.1), corresponding to $0 \le h \le 1$, lead to

$$u_m^{(k)} \in W_1(M,T),\ \text{for all } m,k \in \mathbb{N},$$

where M, T are constants which are independent of h.

We also note that the positive constants M and T are chosen as in (3.19), Lemma 3.2, with $K_M(f) \equiv K_M(h,f)$, $\tilde{K}_M(\mu) \equiv \tilde{K}_M(h,\mu)$, $a(\cdot,\cdot) \equiv a_h(\cdot,\cdot)$ and $\sigma_T(T)$, standing for $K_M(1,f)$, $\tilde{K}_M(1,\mu)$, $a_1(\cdot,\cdot)$, $\tilde{K}_M(1,\mu)\left[(1+7M^2)\sqrt{T}+2M^2\right]$, respectively.

Hence, the limit u_h in suitable function spaces of the sequence $\{u_m^{(k)}\}$ as $k \to +\infty$, after $m \to +\infty$, is a unique weak solution of the problem (P_h) satisfying

$$u_h \in W_1(M,T).$$

When $h = 0$, (P_h) is denoted by (P_0). We shall study the asymptotic expansion of the solution u_h of (P_h) with respect to a small parameter h.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_N,\ \alpha! = \alpha_1! \cdots \alpha_N!, \\ x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \\ \alpha,\ \beta \in \mathbb{Z}_+^N,\ \alpha \le \beta \Longleftrightarrow \alpha_i \le \beta_i\ \ \forall i = 1, \cdots, N. \end{cases}$$

First, we shall need the following lemmas.

Lemma 4.1. *Let $m, N \in \mathbb{N}$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, and $h \in \mathbb{R}$. Then*

$$\left(\sum_{k=1}^N x_k h^k\right)^m = \sum_{k=m}^{mN} P_k^{[m]}(N,x)h^k, \tag{4.1}$$

where the coefficients $P_k^{[m]}(N,x)$, $m \le k \le mN$ *depending on* $x = (x_1, \cdots, x_N)$ *are defined by the formula*

$$P_k^{[m]}(N,x) = \begin{cases} x_k, & 1 \le k \le N,\ m = 1, \\ \sum\limits_{\alpha \in A_k^{[m]}(N)} \dfrac{m!}{\alpha!} x^\alpha, & m \le k \le mN,\ m \ge 2, \end{cases} \tag{4.2}$$

with $A_k^{[m]}(N) = \left\{ \alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^N i\alpha_i = k \right\}$.

Lemma 4.2. *Let* $N \in \mathbb{N}$, $(x_0, \cdots, x_N)$, $(y_0, \cdots, y_N) \in \mathbb{R}^{N+1}$, *and* $h \in \mathbb{R}$. *Then*

$$\left(\sum_{k=0}^N x_k h^k \right) \left(\sum_{k=0}^N y_k h^k \right) = \sum_{k=0}^{2N} \left(\sum_{i=0}^k x_i y_{k-i} \right) h^k. \tag{4.3}$$

In the case of $x_0 = y_0 = 0$, *we have*

$$\left(\sum_{k=1}^N x_k h^k \right) \left(\sum_{k=1}^N y_k h^k \right) = \sum_{k=2}^{2N} \left(\sum_{i=1}^{k-1} x_i y_{k-i} \right) h^k. \tag{4.4}$$

The proof of these lemmas is easy, and we omit the details. □

Now, we assume that

$(H_2^{(N)})$ $\mu \in C^{N+1}([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$ and there exists a constant $\mu_* > 0$ such that $\mu(t, y_1, \cdots, y_3) \ge \mu_*$, $\forall (t, y_1, \cdots, y_3) \in [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$;

$(H_3^{(N)})$ $f \in C^{N+1}([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^2)$ such that $f(1, t, 0, 0, y_3, \cdots, y_6) = 0$, $\forall t \in [0,T^*]$, $\forall (y_3, \cdots, y_6) \in \mathbb{R} \times \mathbb{R}_+^2 \times \mathbb{R}$.

We also use the notations

$$\begin{aligned}
f_h[u](x,t) &= f\left(x, t, u, u_t, u_x, a_h\left(u(t), u'(t)\right), \|u(t)\|^2, \|u(t)\|_{a_h}^2\right) \\
f[u](x,t) &= f(x, t, u, u_t, u_x, \langle u_x(t), u_x'(t) \rangle, \|u(t)\|^2, \|u_x(t)\|^2), \\
\mu_h[u](t) &= \mu\left(t, a_h\left(u(t), u'(t)\right), \|u(t)\|^2, \|u(t)\|_{a_h}^2\right), \\
\mu[u](t) &= \mu\left(t, \langle u_x(t), u_x'(t) \rangle, \|u(t)\|^2, \|u_x(t)\|^2\right), \\
D^m f &= D_3^{m_1} D_4^{m_2} D_5^{m_3} D_6^{m_4} D_7^{m_5} D_8^{m_6} f,\ m = (m_1, \cdots, m_6) \in \mathbb{Z}_+^6, \\
D^n \mu &= D_2^{n_1} D_3^{n_2} D_4^{n_3} \mu,\ n = (n_1, n_2, n_3) \in \mathbb{Z}_+^3.
\end{aligned}$$

According to the above, $u_0 \in W_1(M,T)$ is the weak solution of the problem

$$(P_0) \begin{cases} u_0'' - \lambda \Delta u_0' - \mu[u_0](t) \Delta u_0 = f[u_0] \equiv \tilde{F}_0,\ 0 < x < 1,\ 0 < t < T, \\ u_{0x}(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x),\ u_0'(x,0) = \tilde{u}_1(x). \end{cases}$$

Considering the sequence of weak solutions $u_k \in W_1(M,T)$, $1 \le k \le N$, of the following problems:

$$(\bar{P}_k) \begin{cases} u_k'' - \lambda \Delta u_k' - \mu[u_0](t) \Delta u_k = \tilde{F}_k,\ 0 < x < 1,\ 0 < t < T, \\ u_{kx}(0,t) = u_{k-1}(0,t),\ u_k(1,t) = 0, \\ u_k(x,0) = u_k'(x,0) = 0, \end{cases}$$

where $\tilde{F}_k$, $1 \le k \le N$, are defined by the recurrent formulas

$$\tilde{F}_k = \begin{cases} \sum\limits_{\substack{|m|=1 \\ m \in \mathbb{Z}_+^6}} \frac{1}{m!} D^m f[u_0] \Phi_1[m, \vec{u}_*] + \Delta u_0 \sum\limits_{\substack{|n|=1 \\ n \in \mathbb{Z}_+^3}} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_1[n, \vec{u}_*], & k = 1, \\ \sum\limits_{\substack{1 \le |m| \le k \\ m \in \mathbb{Z}_+^6}} \frac{1}{m!} D^m f[u_0] \Phi_k[m, \vec{u}_*] + \Delta u_0 \sum\limits_{\substack{1 \le |n| \le k \\ n \in \mathbb{Z}_+^3}} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_k[n, \vec{u}_*] & \\ \qquad + \sum\limits_{i=1}^{k-1} \sum\limits_{\substack{1 \le |n| \le i \\ n \in \mathbb{Z}_+^3}} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_i[n, \vec{u}_*] \Delta u_{k-i}, & 2 \le k \le N, \end{cases} \tag{4.5}$$

with $\vec{u}_* = (u_0, u_1, \cdots, u_N)$ and $\Phi_k[m, \vec{u}_*]$, $\hat{\Phi}_k[n, \vec{u}_*]$, $1 \le k \le N$, defined by the formulas:

(a) *Formula* $\Phi_k[m, \vec{u}_*]$:

$$\Phi_k[m, \vec{u}_*] = \sum_{\vec{k} \in \bar{A}_k(m)} P_{k_1}^{[m_1]}(N, \vec{u}) P_{k_2}^{[m_2]}(N, \vec{u}') P_{k_3}^{[m_3]}(N, \vec{u}_x) P_{k_4}^{[m_4]}(2N+1, \vec{\xi}_1) \times P_{k_5}^{[m_5]}(2N, \vec{\xi}_2) P_{k_6}^{[m_6]}(2N+1, \vec{\xi}_3), \tag{4.6}$$

in which

$$\begin{aligned} \bar{A}_k(m) = \{ &\vec{k} = (k_1, \cdots, k_6) \in \mathbb{Z}_+^6 : \left|\vec{k}\right| = k_1 + \cdots + k_6 = k, \\ &m_i \le k_i \le N m_i, \ i = 1, 2, 3; \\ &m_j \le k_j \le (2N+1) m_j, \ j = 4, 6; \ m_5 \le k_5 \le 2N m_5 \}, \\ &m = (m_1, \cdots, m_6) \in \mathbb{Z}_+^6, \\ &|m| = m_1 + \cdots + m_6, \\ &m! = m_1! \cdots m_6!, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\vec{u} = (u_1, \cdots, u_N), \ \vec{u}' = (u_1', \cdots, u_N'), \\ &\vec{u}_x = (u_{1x}, \cdots, u_{Nx}) = (\nabla u_1, \cdots, \nabla u_N), \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\vec{\xi}_1 = (\xi_{11}, \cdots, \xi_{1,2N+1}) \in \mathbb{R}^{2N+1}, \\ &\vec{\xi}_2 = (\xi_{21}, \cdots, \xi_{2,2N}) \in \mathbb{R}^{2N}, \\ &\vec{\xi}_3 = (\xi_{31}, \cdots, \xi_{3,2N+1}) \in \mathbb{R}^{2N+1}, \end{aligned} \tag{4.9}$$

are defined by

$$\begin{aligned} &\xi_{1k} = \begin{cases} \sum_{i=0}^{k} \langle \nabla u_i, \nabla u_{k-i}' \rangle + \sum_{i=0}^{k-1} u_i(0,t) u_{k-1-i}'(0,t), & 1 \le k \le 2N, \\ \sum_{i=0}^{2N} u_i(0,t) u_{2N-i}'(0,t), & k = 2N+1, \end{cases} \\ &\xi_{2k} = \sum_{i=0}^{k} \langle u_i, u_{k-i} \rangle, \ 1 \le k \le 2N, \\ &\xi_{3k} = \begin{cases} \sum_{i=0}^{k} \langle \nabla u_i, \nabla u_{k-i} \rangle + \sum_{i=0}^{k-1} u_i(0,t) u_{k-1-i}(0,t), & 1 \le k \le 2N, \\ \sum_{i=0}^{2N} u_i(0,t) u_{2N-i}(0,t), & k = 2N+1. \end{cases} \end{aligned} \tag{4.10}$$

(b) *Formula* $\hat{\Phi}_k[n, \vec{u}_*]$:

$$\hat{\Phi}_k[n, \vec{u}_*] = \sum_{\vec{k} \in \hat{A}_k(n)} P_{k_1}^{[n_1]}(2N+1, \vec{\xi}_1) P_{k_2}^{[n_2]}(2N, \vec{\xi}_2) P_{k_3}^{[n_3]}(2N+1, \vec{\xi}_3), \tag{4.11}$$

where

$$\begin{aligned} \hat{A}_k(n) = \{ &\vec{k} = (k_1, k_2, k_3) \in \mathbb{Z}_+^3 : \left|\vec{k}\right| = k_1 + k_2 + k_3 = k, \\ &n_j \le k_j \le (2N+1) n_j, \ j = 1, 3; \ n_2 \le k_2 \le 2N n_2 \}, \\ &n = (n_1, n_2, n_3) \in \mathbb{Z}_+^3, \\ &|n| = n_1 + n_2 + n_3, \ n! = n_1! n_2! n_3!. \end{aligned} \tag{4.12}$$

Now, we need the lemma below, its proof is easy so we omit.

Lemma 4.3. *Let* $H = \sum_{k=0}^{N} u_k h^k$, *then*

$$\begin{cases} \xi_1 = a_h(H, H') - \langle u_{0x}, u'_{0x} \rangle = \sum_{k=1}^{2N+1} \xi_{1k} h^k, \\ \xi_2 = \|H\|^2 - \|u_0\|^2 = \sum_{k=1}^{2N} \xi_{2k} h^k, \\ \xi_3 = \|H\|_{a_h}^2 - \|u_{0x}\|^2 = \sum_{k=1}^{2N+1} \xi_{3k} h^k, \end{cases} \tag{4.13}$$

where $\xi_{1k}, \xi_{3k}, (k = \overline{1, 2N+1}), \xi_{2k}, (k = \overline{1, 2N})$ *are defined by the formulas* (4.10).

Therefore, we can prove the following key lemmas (Lemma 4.4 and Lemma 4.5).

Lemma 4.4. *Let* $\Phi_k[m, \vec{u}_*]$, $\hat{\Phi}_k[n, \vec{u}_*]$, $0 \le k \le N$, *be the functions defined by formulas* (4.6) *and* (4.11). *Let* $H = \sum_{r=0}^{N} u_r \varepsilon^r$. *Then we have*

$$f_h[H] = f[u_0] + \sum_{k=1}^{N} \left(\sum_{\substack{1 \le |m| \le k \\ m \in \mathbb{Z}_+^6}} \frac{1}{m!} D^m f[u_0] \Phi_k[m, \vec{u}_*] \right) h^k + h^{N+1} R_N[f, \vec{u}_*, h], \tag{4.14}$$

and

$$\mu_h[H] = \mu[u_0] + \sum_{k=1}^{N} \left(\sum_{\substack{1 \le |n| \le k \\ n \in \mathbb{Z}_+^3}} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_k[n, \vec{u}_*] \right) h^k + h^{N+1} \hat{R}_N[\mu, \vec{u}_*, h], \tag{4.15}$$

with $\left\| R_N[f, \vec{u}_*, h] \right\|_{L^\infty(0,T;L^2)} + \left\| \hat{R}_N[\mu, \vec{u}_*, h] \right\|_{L^\infty(0,T)} \le C$, *where* C *is a constant depending only on* N, T, f, μ, u_k, $k = \overline{0, N}$.

Proof of Lemma 4.4. (i) In the case of $N = 1$, the proof of (4.14) is easy, hence we omit the details. We only prove the case of $N \ge 2$. Let $H = u_0 + \sum_{k=1}^{N} u_k h^k \equiv u_0 + H_1$. We rewrite as below

$$\begin{aligned} f_h[H] &= f\left(x, t, H, H', H_x, a_h(H, H'), \|H\|^2, \|H\|_{a_h}^2\right) \\ &= f\left(x, t, u_0 + H_1, u'_0 + H'_1, u_{0x} + H_{1x}, \langle u_{0x}, u'_{0x} \rangle + \xi_1, \|u_0\|^2 + \xi_2, \|u_{0x}\|^2 + \xi_3\right), \end{aligned} \tag{4.16}$$

where $\xi_1 = a_h(H, H') - \langle u_{0x}, u'_{0x} \rangle$, $\xi_2 = \|H\|^2 - \|u_0\|^2$, $\xi_3 = \|H\|_{a_h}^2 - \|u_{0x}\|^2$.

By using Taylor's expansion of the function $f_h[H] = f_h[u_0 + h_1]$ around the point

$$[u_0] = \left(x, t, u_0, u'_0, u_{0x}, \langle u_{0x}, u'_{0x} \rangle, \|u_0\|^2, \|u_{0x}\|^2\right)$$

up to order $N + 1$, we obtain

$$\begin{aligned} f_h[H] = f[u_0] + &\sum_{\substack{1 \le |m| \le N \\ m = (m_1, \cdots, m_6) \in \mathbb{Z}_+^6}} \frac{1}{m!} D^m f[u_0] H_1^{m_1} (H'_1)^{m_2} (H_{1x})^{m_3} \xi_1^{m_4} \xi_2^{m_5} \xi_3^{m_6} \\ &+ R_N^{(1)}[f, \vec{u}_*, H_1, \xi_1, \xi_2, \xi_3], \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &R_N^{(1)}[f,\vec{u}_*,H_1,\xi_1,\xi_2,\xi_3] \\ &= \sum_{\substack{|m|=N+1 \\ m\in\mathbb{Z}_+^6}} \frac{N+1}{m!}\left(\int_0^1 (1-\theta)^N D^m f(\theta)d\theta\right) H_1^{m_1}(H_1')^{m_2}\,(H_{1x})^{m_3}\,\xi_1^{m_4}\xi_2^{m_5}\xi_3^{m_6} \\ &= h^{N+1}R_N^{(2)}[f,\vec{u}_*,h], \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} D^m f(\theta) = D^m f\Big(&x,t,u_0+\theta H_1,u_0'+\theta H_1',u_{0x}+\theta H_{1x},\langle u_{0x},u_{0x}'\rangle+\theta\xi_1, \\ &\|u_0\|^2+\theta\xi_2,\|u_{0x}\|^2+\theta\xi_3\Big). \end{aligned} \tag{4.19}$$

By the formulas (4.1), (4.3), (4.4), it follows that

$$\begin{aligned} H_1^{m_1} &= \left(\sum_{k=1}^{N} u_k h^k\right)^{m_1} = \sum_{k=m_1}^{Nm_1} P_k^{[m_1]}(N,\vec{u})h^k, \\ (H_1')^{m_2} &= \left(\sum_{k=1}^{N} u_k' h^k\right)^{m_2} = \sum_{k=m_2}^{Nm_2} P_k^{[m_2]}(N,\vec{u}')h^k, \\ (H_x)^{m_3} &= \left(\sum_{k=1}^{N} u_{kx} h^k\right)^{m_3} = \sum_{k=m_3}^{Nm_3} P_k^{[m_1]}(N,\vec{u}_x)h^k, \\ \xi_1^{m_4} &= \left(\sum_{k=1}^{2N+1} \xi_{1k} h^k\right)^{m_4} = \sum_{k=m_4}^{(2N+1)m_4} P_k^{[m_4]}(2N+1,\vec{\xi}_1)h^k, \\ \xi_2^{m_5} &= \left(\sum_{k=1}^{2N} \xi_{2k} h^k\right)^{m_5} = \sum_{k=m_5}^{2Nm_5} P_k^{[m_5]}(2N,\vec{\xi}_2)h^k, \\ \xi_3^{m_6} &= \left(\sum_{k=1}^{2N+1} \xi_{3k} h^k\right)^{m_6} = \sum_{k=m_6}^{(2N+1)m_6} P_k^{[m_6]}(2N+1,\vec{\xi}_3)h^k. \end{aligned} \tag{4.20}$$

Therefore, it follows from (4.20), that

$$H_1^{m_1}(H_1')^{m_2}\,(H_{1x})^{m_3}\,\xi_1^{m_4}\xi_2^{m_5}\xi_3^{m_6} = \sum_{k=|m|}^{N} \Phi_k[m,\vec{u}_*]h^k + \sum_{k=N+1}^{\bar{N}_m} \Phi_k[m,\vec{u}_*]h^k, \tag{4.21}$$

where

$$\begin{aligned} \Phi_k[m,\vec{u}_*] = \sum_{\vec{k}\in\bar{A}_k(m)} &P_{k_1}^{[m_1]}(N,\vec{u})P_{k_2}^{[m_2]}(N,\vec{u}')P_{k_3}^{[m_3]}(N,\vec{u}_x) \\ &\times P_{k_4}^{[m_4]}(2N+1,\vec{\xi}_1)P_{k_5}^{[m_5]}(2N,\vec{\xi}_2)P_{k_6}^{[m_6]}(2N+1,\vec{\xi}_3), \\ \bar{N}_m = N\,(|m|&+m_4+m_5+m_6)+m_4+m_6, \end{aligned} \tag{4.22}$$

with $\bar{A}_k(m)$ as in (4.7).

Hence, we deduce from (4.17), (4.18) and (4.21) that

$$\begin{aligned} f_h[H] &= f[u_0] + \sum_{1\le|m|\le N} \frac{1}{m!} D^m f[u_0] \sum_{k=|m|}^{N} \Phi_k[m,\vec{u}_*]h^k \\ &\quad + \sum_{1\le|m|\le N} \frac{1}{m!} D^m f[u_0] \sum_{k=N+1}^{\bar{N}_m} \Phi_k[m,\vec{u}_*]h^k + h^{N+1} R_N^{(2)}[f,\vec{u}_*,h] \\ &= f[u_0] + \sum_{k=1}^{N} \left(\sum_{1\le|m|\le k} \frac{1}{m!} D^m f[u_0] \Phi_k[m,\vec{u}_*] \right) h^k + h^{N+1} R_N[f,\vec{u}_*,h], \end{aligned} \tag{4.23}$$

where

$$h^{N+1} R_N[f,\vec{u}_*,h] = \sum_{1\le|m|\le N} \frac{1}{m!} D^m f[u_0] \sum_{k=N+1}^{\bar{N}_m} \Phi_k[m,\vec{u}_*]h^k + h^{N+1} R_N^{(2)}[f,\vec{u}_*,h], \tag{4.24}$$

with $\Phi_k[m,\vec{u}_*]$, $1 \le k \le \bar{N}_m$ are defined by (4.22), and $\left\| R_N[f,\vec{u}_*,h] \right\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on $N, T, f, u_k, k = \overline{0,N}$.

Hence, the formula (4.14) is proved.

(ii) In the case of $\mu_h[H] = \mu\left(t, a_h(H,H'), \|H\|^2, \|H\|_{a_h}^2\right)$. Applying the formulas (4.6) - (4.10) and (4.14) with $f = f(t,y_1,y_2,y_3) = \mu(t,y_1,y_2,y_3)$, $D_j f = 0$, $j = 1,3,4,5$; $D_2 f = D_1\mu = \frac{\partial\mu}{\partial t}$, $D_6 f = D_2\mu = \frac{\partial\mu}{\partial y_1}$, $D_7 f = D_3\mu = \frac{\partial\mu}{\partial y_2}$, $D_8 f = D_4\mu = \frac{\partial\mu}{\partial y_3}$, we obtain formulas (4.11), (4.12) and the formula (4.15) is proved.

This completes the proof of Lemma 4.4. □

Let $u = u_h \in W_1(M,T)$ be a unique weak solution of the problem (P_h). Then $v = u - \sum_{k=0}^{N} u_k h^k \equiv u - H = u - u_0 - H_1$ satisfies the problem

$$\begin{cases} v'' - \lambda\Delta v' - \mu_h[v+H]\Delta v = f_h[v+H] - f_h[H] + (\mu_h[v+H] - \mu_h[H])\Delta H \\ \qquad\qquad + E_h(x,t),\ 0 < x < 1,\ 0 < t < T, \\ v_x(0,t) - hv(0,t) = h^{N+1} u_N(0,t),\ v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \end{cases} \tag{4.25}$$

where

$$E_h(x,t) = f_h[H] - f[u_0] + (\mu_h[H] - \mu[u_0])\Delta H - \sum_{k=1}^{N} \tilde{F}_k h^k. \tag{4.26}$$

Lemma 4.5. *Under the assumptions* $(H_1) - (H_3)$, $(H_2^{(N)})$ *and* $(H_3^{(N)})$, *we have*

$$\|E_h\|_{L^\infty(0,T;L^2)} \le \tilde{C}_* h^{N+1}, \tag{4.27}$$

where $\tilde{C}_*$ *is a constant depending only on* $N, T, f, \mu, u_k, 0 \le k \le N$.

Proof of Lemma 4.5. In the case of $N = 1$, the proof of Lemma 4.5 is easy. The details are omitted. We only consider $N \ge 2$.

By using formulas (4.4), (4.15), we obtain

$$
\begin{aligned}
&(\mu_h[H] - \mu[u_0])\,\Delta H \qquad (4.28)\\
&= \sum_{k=1}^{N}\left(\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\right)h^k\Delta H + h^{N+1}\hat{R}_N[\mu,\vec{u}_*,h]]\Delta H\\
&= \sum_{k=1}^{N}\left(\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\right)h^k\left(\Delta u_0 + \sum_{k=1}^{N}\Delta u_k h^k\right) + h^{N+1}\hat{R}_N[\mu,\vec{u}_*,h]]\Delta H\\
&= \Delta u_0\sum_{k=1}^{N}\left(\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\right)h^k\\
&+ \sum_{k=1}^{N}\left(\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\right)h^k\sum_{k=1}^{N}\Delta u_k h^k + h^{N+1}\hat{R}_N[\mu,\vec{u}_*,h]]\Delta H
\end{aligned}
$$

$$
\begin{aligned}
=\;& \Delta u_0\sum_{k=1}^{N}\left(\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\right)h^k\\
&+\sum_{k=2}^{2N}\sum_{i=1}^{k-1}\left(\sum_{1\le|n|\le i}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_i[n,\vec{u}_*]\right)\Delta u_{k-i}h^k + h^{N+1}\hat{R}_N[\mu,\vec{u}_*,h]]\Delta H\\
=\;& \Delta u_0\left(\sum_{|n|=1}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_1[n,\vec{u}_*]\right)h\\
&+\sum_{k=2}^{N}\left[\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\Delta u_0 + \sum_{i=1}^{k-1}\left(\sum_{1\le|n|\le i}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_i[n,\vec{u}_*]\right)\Delta u_{k-i}\right]h^k\\
&+h^{N+1}\left[\sum_{k=N+1}^{2N}\sum_{i=1}^{k-1}\left(\sum_{1\le|n|\le i}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_i[n,\vec{u}_*]\right)\Delta u_{k-i}h^{k-N-1} + \hat{R}_N[\mu,\vec{u}_*,h]]\Delta H\right]\\
=\;& \Delta u_0\left(\sum_{|n|=1}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_1[n,\vec{u}_*]\right)h\\
&+\sum_{k=2}^{N}\left[\sum_{1\le|n|\le k}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_k[n,\vec{u}_*]\Delta u_0 + \sum_{i=1}^{k-1}\left(\sum_{1\le|n|\le i}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_i[n,\vec{u}_*]\right)\Delta u_{k-i}\right]h^k\\
&+h^{N+1}\hat{R}_N^{(1)}[\mu,\vec{u}_*,h],
\end{aligned}
$$

where

$$
\hat{R}_N^{(1)}[\mu,\vec{u}_*,h] = \left[\sum_{k=N+1}^{2N}\sum_{i=1}^{k-1}\left(\sum_{1\le|n|\le i}\frac{1}{n!}D^n\mu[u_0]\hat{\Phi}_i[n,\vec{u}_*]\right)\Delta u_{k-i}h^{k-N-1} + \hat{R}_N[\mu,\vec{u}_*,h]]\Delta H\right]. \qquad (4.29)
$$

By using formulas (4.14), (4.26), (4.29), we deduce from (4.5) that

$$\begin{aligned} E_h(x,t) &= f_h[H] - f[u_0] + (\mu_h[H] - \mu[u_0])\,\Delta H - \sum_{k=1}^{N} \tilde{F}_k h^k \\ &= \left[\sum_{|m|=1} \frac{1}{m!} D^m f[u_0] \Phi_1[m,\vec{u}_*] + \Delta u_0 \left(\sum_{|n|=1} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_1[n,\vec{u}_*] \right) - \tilde{F}_1 \right] h \\ &+ \sum_{k=2}^{N} \left[\sum_{1\le|m|\le k} \frac{1}{m!} D^m f[u_0] \Phi_k[m,\vec{u}_*] + \sum_{1\le|n|\le k} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_k[n,\vec{u}_*] \Delta u_0 \right. \\ &\qquad \left. + \sum_{i=1}^{k-1} \left(\sum_{1\le|n|\le i} \frac{1}{n!} D^n \mu[u_0] \hat{\Phi}_i[n,\vec{u}_*] \right) \Delta u_{k-i} - \tilde{F}_k \right] h^k \\ &+ h^{N+1} \left(R_N[f,\vec{u}_*,h] + \hat{R}_N^{(1)}[\mu,\vec{u}_*,h] \right) \\ &= h^{N+1} \left(R_N[f,\vec{u}_*,h] + \hat{R}_N^{(1)}[\mu,\vec{u}_*,h] \right). \end{aligned} \tag{4.30}$$

By the functions $u_k \in W_1(M,T)$, $0 \le k \le N$, we obtain from (4.24), (4.29) and Lemma 4.4 that $R_N[f,\vec{u}_*,h] + \hat{R}_N^{(1)}[\mu,\vec{u}_*,h]$ is bounded in $L^\infty(0,T;L^2)$ by a constant $\tilde{C}_*$ depending only on N, T, f, μ, u_k, $0 \le k \le N$, i.e.,

$$\|E_h\|_{L^\infty(0,T;L^2)} \le \tilde{C}_* h^{N+1}. \tag{4.31}$$

This completes the proof of Lemma 4.5. □

Now, we estimate $v = u - \sum_{k=0}^{N} u_k h^k$.

By multiplying the two sides of (4.25) by v', we verify without difficulty that

$$\begin{aligned} S(t) &= 2\int_0^t \langle E_h(s), v'(s)\rangle ds - 2h^{N+1} \int_0^t \tilde{\mu}_h(s) u_N(0,s) v'(0,s) ds \\ &- 2\lambda h^{N+1} \int_0^t u_N'(0,s) v'(0,s) ds \\ &+ \int_0^t \tilde{\mu}_h'(s) \|v(s)\|_{a_h}^2 ds + 2\int_0^t \langle f_h[v+H] - f_h[H], v'(s)\rangle ds \\ &+ 2\int_0^t (\mu_h[v+H] - \mu_h[H]) \langle \Delta H, v'(s)\rangle ds \\ &\equiv \sum_{i=0}^{6} S_i, \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} \tilde{\mu}_h(s) &= \mu_h[v+H] = \mu\left(t, a_h(v+H, v'+H'), \|v+H\|^2, \|v+H\|_{a_h}^2\right) \equiv \mu[v+H], \\ S(t) &= \|v'(t)\|^2 + \tilde{\mu}_h(t) \|v(t)\|_{a_h}^2 + 2\lambda \int_0^t \|v'(s)\|_{a_h}^2 ds \\ &\ge \|v'(t)\|^2 + \mu_* \|v(t)\|_{a_h}^2 + 2\lambda \int_0^t \|v'(s)\|_{a_h}^2 ds. \end{aligned} \tag{4.33}$$

Put $M_1 = (N+2)M$, it is not difficult to prove that the following inequalities hold

$$\begin{aligned}
S_1 &= 2\int_0^t \langle E_h(s), v'(s)\rangle ds \le T\tilde{C}_*^2 h^{2N+2} + \int_0^t S(s)ds; \qquad (4.34)\\
S_2 &= -2h^{N+1}\int_0^t \tilde{\mu}_h(s) u_N(0,s) v'(0,s) ds \\
&\le 2h^{N+1} M \tilde{K}_{M_1}(\mu) \int_0^t \left\| v_x'(s)\right\| ds \\
&\le \frac{T}{2\beta\lambda} h^{2N+2} M^2 \tilde{K}_{M_1}^2(\mu) + 2\beta\lambda \int_0^t \left\| v_x'(s)\right\|^2 ds \\
&\le \frac{T}{2\beta\lambda} h^{2N+2} M^2 \tilde{K}_{M_1}^2(\mu) + \beta S(t); \\
S_3 &= -2\lambda h^{N+1}\int_0^t u_N'(0,s) v'(0,s) ds \\
&\le 2\lambda M h^{N+1} \int_0^t \left\| v_x'(s)\right\| ds \\
&\le \frac{T}{2\beta} \lambda M^2 h^{2N+2} + 2\beta\lambda \int_0^t \left\| v_x'(s)\right\|^2 ds \\
&\le \frac{T}{2\beta} \lambda M^2 h^{2N+2} + \beta S(t).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\tilde{\mu}_h'(s) &= D_1\mu[v+H] + D_2\mu[v+H]\left[\|v'+H'\|_{a_h}^2 + a_h(v+H, v''+H'')\right] \qquad (4.35)\\
&\quad + 2D_3\mu[v+H]\langle v+H, v'+H'\rangle + 2D_4\mu[v+H] a_h(v+H, v'+H').
\end{aligned}$$

By $u_k, u \in W_1(M,T)$, $k = 0, 1, \cdots, N$, we can deduce the existence of a constant $D_M > 0$ independent on h and $s \in [0,T]$ such that

$$\left|\tilde{\mu}_h'(s)\right| \le D_M\left(1 + \left\| v_x''(s) + H_x''(s)\right\|\right) = \Psi_M(s). \qquad (4.36)$$

Moreover

$$\begin{aligned}
\int_0^T \Psi_M^2(s) ds &\le D_M^2 \int_0^T \left(1 + \left\| v_x''(s) + H_x''(s)\right\|\right)^2 ds \qquad (4.37)\\
&\le 2D_M^2\left[T + \int_0^T \left\| u_x''(s)\right\|^2 ds\right] \\
&= 2D_M^2\left(T + \|u''\|_{L^2(0,T;V)}^2\right) \le 2D_M^2\left(T + M^2\right).
\end{aligned}$$

It follows from (4.36) that

$$S_4 = \int_0^t \tilde{\mu}_h'(s)\, \|v(s)\|_{a_h}^2\, ds \le \frac{1}{\mu_*}\int_0^t \Psi_M(s) S(s) ds. \qquad (4.38)$$

By the fact that

$$\left\| f_h[v+H] - f_h[H]\right\| \le (2 + 6M_1) K_{M_1}(f)\left(\|v(s)\|_{a_h} + \|v'(s)\|_{a_h}\right), \qquad (4.39)$$

we get

$$S_5 = 2\int_0^t \langle f_h[v+H] - f_h[H], v'(s)\rangle ds \tag{4.40}$$
$$\leq 2(2+6M_1)K_{M_1}(f)\int_0^t \left(\|v(s)\|_{a_h} + \|v'(s)\|_{a_h}\right)\|v'(s)\|\,ds$$
$$\leq 2\sigma_1\int_0^t \left(\frac{1}{\sqrt{\mu_*}}\sqrt{S(s)} + \|v'(s)\|_{a_h}\right)\sqrt{S(s)}ds$$
$$\leq 2\frac{\sigma_1}{\sqrt{\mu_*}}\int_0^t S(s)ds + \frac{1}{\beta}\sigma_1^2\int_0^t S(s)ds + \beta\int_0^t \|v'(s)\|_{a_h}^2\,ds$$
$$\leq \left(\frac{2\sigma_1}{\sqrt{\mu_*}} + \frac{1}{\beta}\sigma_1^2\right)\int_0^t S(s)ds + \frac{\beta}{2\lambda}S(t).$$

Similarly, by the following estimate

$$\left|\mu_h[v+H] - \mu_h[H]\right| \leq 6M_1\tilde{K}_{M_1}(\mu)\left(\|v(s)\|_{a_h} + \|v'(s)\|_{a_h}\right), \tag{4.41}$$

we obtain

$$S_6 = 2\int_0^t \left(\mu_h[v+H] - \mu_h[H]\right)\langle \Delta H, v'(s)\rangle ds \tag{4.42}$$
$$\leq 12M_1^2\tilde{K}_{M_1}(\mu)\int_0^t \left(\|v(s)\|_{a_h} + \|v'(s)\|_{a_h}\right)\|v'(s)\|\,ds$$
$$\leq 2\sigma_2\int_0^t \left(\frac{1}{\sqrt{\mu_*}}\sqrt{S(s)} + \|v'(s)\|_{a_h}\right)\sqrt{S(s)}ds$$
$$\leq \left(\frac{2\sigma_2}{\sqrt{\mu_*}} + \frac{1}{\beta}\sigma_2^2\right)\int_0^t S(s)ds + \frac{\beta}{2\lambda}S(t).$$

We can choose $\beta > 0$ such that $\beta\left(2+\frac{1}{\lambda}\right) \leq 1/2$, combining (4.32), (4.34), (4.38), (4.40) and (4.42), we then obtain

$$S(t) \leq TD_M^{(1)}h^{2N+2} + 2\int_0^t \left[D_M^{(2)} + \frac{1}{\mu_*}\Psi_M(s)\right]S(s)ds, \tag{4.43}$$

where

$$D_M^{(1)} = 2\left[\tilde{C}_*^2 + \frac{M^2}{2\beta\lambda}\left(\lambda^2 + \tilde{K}_{M_1}^2(\mu)\right)\right], \tag{4.44}$$
$$D_M^{(2)} = 1 + \frac{2\sigma_1 + 2\sigma_2}{\sqrt{\mu_*}} + \frac{\sigma_1^2 + \sigma_2^2}{\beta}.$$

By Gronwall's lemma, we obtain from (4.37), (4.43) that

$$S(t) \leq TD_M^{(1)}h^{2N+2}\exp\left[2\int_0^T \left(D_M^{(2)} + \frac{1}{\mu_*}\Psi_M(s)\right)ds\right] \tag{4.45}$$
$$\leq TD_M^{(1)}h^{2N+2}\exp\left[2TD_M^{(2)} + \frac{2\sqrt{T}}{\mu_*}\|\Psi_M\|_{L^2(0,T)}\right]$$
$$\leq TD_M^{(1)}h^{2N+2}\exp\left[2\left(TD_M^{(2)} + \frac{\sqrt{2T(T+M^2)}D_M}{\mu_*}\right)\right]$$
$$\equiv D_M^{(3)}(T)h^{2N+2}.$$

Hence

$$\|v\|_{W_1(T)} \leq \left(1 + \frac{1}{\sqrt{\mu_*}} + \frac{1}{\sqrt{2\lambda}}\right) \sqrt{D_M^{(3)}(T)} h^{N+1},$$

or

$$\left\| u_h - \sum\nolimits_{k=0}^{N} u_k h^k \right\|_{W_1(T)} \leq C_T h^{N+1}. \tag{4.46}$$

Consequently, we obtain Theorem 4.6 below.

Theorem 4.6. *Let* $(H_1) - (H_3)$, $(H_2^{(N)})$ *and* $(H_3^{(N)})$ *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* h, *with* $0 \leq h \leq 1$, *the problem* (P_h) *has a unique weak solution* $u_h \in W_1(M,T)$ *satisfying an asymptotic estimation up to order* $N+1$ *as in* (4.46), *where the functions* u_k, $k = 0, 1, \cdots, N$ *are weak solutions of* (P_0), $(\bar{P}_k)$, $k = 1, \cdots, N$, *respectively.* □

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