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Hyperspaces and function graphs in digital topology

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Abstract. We adapt the study of hyperspaces and function spaces from classical topology to digital topology. We define digital hyperspaces and digital function graphs, and study some of their relationships and graphical properties.

1. Introduction

Classical topology has a large literature devoted to the study of hyperspaces, in which a topology is induced on some set of subsets of a given topological space. By the time of the publication of [27], hundreds of papers had been published on hyperspaces, and many more have appeared subsequently. Typically, the topology of a hyperspace is induced by using the Hausdorff metric, which essentially measures how two objects approximate each other with respect to position. The Hausdorff metric can be computed efficiently [14, 29] and has been used by some students of digital image processing as a crude measure of whether two images might represent the same real-world object. Other metrics have been developed in order to compare objects with respect to topological or geometric properties [2, 4, 7, 9, 16, 17]. Variations on the Hausdorff metric were introduced in [11, 13, 18, 32]

Classical topology also has a large literature on *function spaces*, in which the set of functions

$$Y^X = \{f : X \to Y \mid f \text{ is continuous}\}$$

between topological spaces, or some interesting subset of Y^X, is considered as a topological space whose topology is determined from those of X and Y; see, e.g., [3, 5, 6, 8, 33].

In the current paper, we develop notions of hyperspaces and function graphs (the latter, an analog of function spaces) for digital topology. The paper is organized as follows.

- Section 2 reviews basics of digital topology.
- In section 3, we introduce the adjacency that we use to form a hyperspace of digital images.
- Section 4 has elementary observations on the cardinalities of digital hyperspaces.

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- In section 5 we discuss certain digitally continuous functions on hyperspaces. In section 5.3, we introduce the concept of a function graph as a digital analog of a function space. Classical topology studies relations between hyperspaces and function spaces, e.g., [5, 6, 8]; in section 5.3 and later in the paper, we study relations between digital hyperspaces and function graphs.
- In section 6 we study connectedness properties of digital hyperspaces.
- In section 7 we consider various notions of continuous multivalued functions in digital topology and their relations with digital hyperspaces.
- In section 8 we obtain results concerning cycles and Girth in digital hyperspaces.
- In sections 9 and 10, we study, respectively, dominating sets and diameters of digital hyperspaces.
- We give some concluding remarks in section 11.

2. Preliminaries

Much of this section is quoted or paraphrased from [12, 13].

We use $\mathbb{N}$ to indicate the set of natural numbers, $\mathbb{Z}$ for the set of integers, and $\mathbb{R}$ for the set of real numbers. We use $\#X$ for the number of points in a set X.

2.1. Adjacencies

A digital image is a graph (X, κ), where X is a nonempty subset of $\mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation for the points of X. The c_u-adjacencies are commonly used. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

Let $u \in \mathbb{N}$, $1 \leq u \leq n$. We say x and y are *c_u-adjacent* if

- There are at most u indices i for which $|x_i - y_i| = 1$.
- For all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Often, a c_u-adjacency is denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

We write $x \leftrightarrow_\kappa x'$, or $x \leftrightarrow x'$ when κ is understood, to indicate that x and x' are κ-adjacent. Similarly, we write $x \leftrightarroweq_\kappa x'$, or $x \leftrightarroweq x'$ when κ is understood, to indicate that x and x' are κ-adjacent or equal.

A sequence $P = \{y_i\}_{i=0}^m$ in a digital image (X, κ) is a *κ-path from $a \in X$ to $b \in X$* if $a = y_0$, $b = y_m$, and $y_i \leftrightarroweq_\kappa y_{i+1}$ for $0 \leq i < m$.

$Y \subset X$ is *κ-connected* [28], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a κ-path in Y from a to b.

Let $N(X, x, \kappa)$ be the set

$$N(X, x, \kappa) = \{y \in X \mid x \leftrightarrow_\kappa y\}.$$

2.2. Digitally continuous functions

In a metric space, the continuity of $f : X \to Y$ is defined to preserve the intuition that if x_0 and x_1 are sufficiently close, then $f(x_0)$ and $f(x_1)$ are close; i.e., "closeness," and therefore connectivity, are preserved by a continuous function. Digital continuity is defined to preserve connectedness, as at Definition 2.1 below. By using adjacency as our standard of "closeness," we get Theorem 2.2 below.

Definition 2.1. [12] (generalizing a definition of [28]) *Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y.*

If $Y \subset X$, we use the abbreviation κ-continuous for (κ, κ)-continuous.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

Theorem 2.2. [12, 28] *A function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarroweq f(x')$.*

See also [19, 20], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

Proposition 2.3. [12] *If $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (W, \mu)$ are continuous maps between digital images, then $g \circ f : X \to W$ is (κ, μ)-continuous.*

Remark 2.4. *Notice P is a κ-path if and only if there is a (c_1, κ)-continuous function $p : [0, n]_{\mathbb{Z}} \to X$ such that $p([0, n]_{\mathbb{Z}}) = P$. It is therefore common to call such a function a κ-*path.

To express the idea of following one path and then another, the *product* or *concatenation* of paths is defined as follows.

Definition 2.5. [24] *Let $p_1 : [0, m]_{\mathbb{Z}} \to X$ and $p_2 : [0, n]_{\mathbb{Z}} \to X$ be κ-paths such that $p_1(m) = p_2(0)$. The* product *or* concatenation *of these paths is the function $p_1 \cdot p_2 : [0, m + n]_{\mathbb{Z}} \to X$ given by*

$$(p_1 \cdot p_2)(t) = \begin{cases} p_1(t) & \text{if } 0 \le t \le m; \\ p_2(t - m) & \text{if } m \le t \le m + n. \end{cases}$$

Lemma 2.6. [10] *The concatenation of paths is associative, i.e.,*

$$(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3).$$

Let $Y \subset X$. A κ-continuous function $r : X \to Y$ is a *retraction*, and Y is a *κ-retract of* X, if $r|_Y = \mathrm{id}_Y$.

A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

Definition 2.7. ([12]; see also [24]) *Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, λ)-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that*

- *for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by*

  $$F_x(t) = F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

  is $(2, \lambda)$−continuous. That is, $F_x(t)$ is a path in Y.
- *for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by*

  $$F_t(x) = F(x, t) \text{ for all } x \in X$$

  is (κ, λ)−continuous.

Then F is a digital $(\kappa, \lambda)-$homotopy between *f and g, and f and g are* digitally $(\kappa, \lambda)-$homotopic in Y, *denoted* $f \sim_{\kappa,\lambda} g$.

If for some $x_0 \in X$ and $y_0 \in Y$ we have $F(x_0, t) = F(x_0, 0) = y_0 \in Y$ for all $t \in [0, m]_{\mathbb{Z}}$, we say F holds x fixed, *F is a* pointed homotopy, *and x_0 and y_0 are* basepoints *of the homotopy.* □

A different notion of digital homotopy has been introduced by [26, 30]. The latter paper calls this *strong homotopy*. It is defined as follows.

Definition 2.8. *Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, λ)-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that*

- *for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$; and*
- *if $x \leftrightarrows_\kappa y$ in X and $t_0 \leftrightarrows_{c_1} t_1$ in $[0, m]_{\mathbb{Z}}$, then $F(x, t_0) \leftrightarrows_\lambda F(y, t_1)$ in Y.*

Then F is a strong homotopy *between f and g, and f and g are strongly (κ, λ)-homotopic in Y.*

If for some $x_0 \in X$ and $y_0 \in Y$ we have $F(x_0, t) = F(x_0, 0) = y_0 \in Y$ for all $t \in [0, m]_{\mathbb{Z}}$, we say F holds x_0 fixed, *F is a* strong pointed homotopy, *and x_0 and y_0 are* basepoints *of the homotopy.*

If there is a (strong) (pointed) (κ, κ)-homotopy $F : X \times [0, m]_{\mathbb{Z}} \to X$ between the identity function 1_X and a constant function, we say F is a (digital) *(strong) (pointed) κ-contraction* and X is *(strongly) (pointed) κ-contractible*.

If there are continuous $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (X, \kappa)$ such that $g \circ f$ is (strongly) (pointed) homotopic to id_X and $f \circ g$ is (strongly) (pointed) homotopic to id_Y, then (X, κ) and (Y, λ) are *(strongly) (pointed) homotopy equivalent* or *have the same (strong) (pointed) homotopy type*.

If $r : X \to X$ is a κ-retraction of X to $Y \subset X$ that is (strongly) homotopic to id_X, then r is a (*strong* in the sense of digital homotopy) *deformation retraction*. If a (*strong* in the sense of digital homotopy) deformation retraction of X to $Y \subset X$ holds fixed every point of Y, then r is a *strong* (in the sense of deformation theory) (*strong* in the sense of digital homotopy) *deformation retraction*.

If $f : (X, \kappa) \to (Y, \lambda)$ is a continuous bijection such that $f^{-1} : (Y, \lambda) \to (X, \kappa)$ is continuous, then f is an *isomorphism* (called *homeomorphism* in [10]) and (X, κ) and (Y, λ) are *isomorphic*.

3. Hyperspaces

The book [27] is a good source for much of the material discussed in this section that is taken from classical topology.

In classical topology, given a topological space X, we denote by 2^X the set or *hyperspace* of nonempty compact subsets of X. If X is a metric space with metric d, 2^X becomes a metric space with the Hausdorff metric, or some other metric, based on d.

Given a digital image (X, κ), we seek a somewhat parallel construction of a graph based on finite subsets of X. We let

$$2^X = \{Y \mid \emptyset \neq Y \subset X,\ \#Y < \infty\}.$$

We define the κ' adjacency for 2^X as follows.

Definition 3.1. *Let $\{A, B\} \subset 2^X$, $A \neq B$. Then $A \leftrightarrow_{\kappa'} B$ if and only if given $a \in A$ and $b \in B$, there exist $a_0 \in A$ and $b_0 \in B$ such that $a \leftrightarrows_\kappa b_0$ and $b \leftrightarrows_\kappa a_0$.*

The pair $(2^X, \kappa')$ is a graph or *tolerance space* [34], the *hyperspace of* (X, κ). Note we do not call this hyperspace a digital image, since 2^X is not a subset of $\mathbb{Z}^n$. However, since digital topology's notions of continuous functions are defined in terms of graph adjacency, or, alternately, graph connectedness, they are naturally applied to this construction.

In classical topology, it is common to denote by $C(X)$ the subset of 2^X consisting of connected members of 2^X. Since the notation $C(X,\kappa)$ is established in the literature of digital topology as the set of κ-continuous self maps on X, we use the notation

$$K(X,\kappa') = (\{A \in 2^X \mid A \text{ is } \kappa\text{-connected}\}, \kappa').$$

We will use the abbreviation $K(X)$ when κ is understood.

Example 3.2. *$K([a,b]_{\mathbb{Z}}, c_1')$ and $(\{(x,y) \in \mathbb{Z}^2 \mid a \le x \le y \le b\}, c_2)$ are isomorphic graphs.*

Proof. Let $X = K([a,b]_{\mathbb{Z}}, c_1')$, $Y = \{(x,y) \in \mathbb{Z}^2 \mid a \le x \le y \le b\}$. Consider the function $F : X \to Y$ given by $F([m,n]_{\mathbb{Z}}) = (m,n)$. It is elementary to show that F is a (c_1', c_2)-isomorphism. □

4. Cardinality

Remark 4.1. *Let (X,κ) be a digital image such that $\#X = n$. Then $\#2^X = 2^n - 1$. This is because for each $x \in X$ and $A \in 2^X$, either $x \in A$ or $x \notin A$. This yields 2^n possible combinations of pixels, but we exclude the empty set.*

However, the following example shows that $K(X)$ may be considerably smaller than 2^X.

Example 4.2. $\#K([1,n]_{\mathbb{Z}}, c_1') = n(n+1)/2$.

Proof. For $i \in [1,n]_{\mathbb{Z}}$, the members of $K([1,n]_{\mathbb{Z}}, c_1')$ that have i as their largest member are those of $\{[j,i]_{\mathbb{Z}}\}_{j=1}^{i}\}$. Since there are i digital intervals with largest member i in $K([1,n]_{\mathbb{Z}}, c_1')$,

$$\#K([1,n]_{\mathbb{Z}}, c_1') = \sum_{i=1}^{n} i = n(n+1)/2.$$

□

5. Maps on digital hyperspaces

In this section, we study maps induced on hyperspaces by continuous maps between digital images.

5.1. Induced maps

Given a continuous map $f : (X,\kappa) \to (Y,\lambda)$, we show below that f induces a (κ',λ')-continuous map $f_* : 2^X \to 2^Y$ such that $f_*|_{K(X)} : K(X) \to K(Y)$ is also (κ',λ')-continuous. In the following, we will use the notation f_* to abbreviate $f_*|_{K(X)}$.

Theorem 5.1. *Let (X,κ) and (Y,λ) be digital images and let $f : X \to Y$. Then f is (κ,λ)-continuous if and only if the induced functions $f_* : 2^X \to 2^Y$ and $f_* : K(X) \to K(Y)$ defined by $f_*(A) = f(A)$ are (κ',λ')-continuous.*

Proof. Since a digitally continuous function preserves adjacency and connectivity, the same argument works for both of the induced functions.

Suppose f is continuous. Let $A, B \in 2^X$ such that $A \leftrightarrow_{\kappa'} B$. Let $x \in A$. There exists $y \in B$ such that $x \leftrightarroweq_{\kappa} y$. By the continuity of f we have $f(x) \leftrightarroweq_{\lambda} f(y)$. Similarly, for $b \in B$, there exists $a \in A$ such that $a \leftrightarroweq_{\kappa} b$ and $f(a) \leftrightarroweq_{\lambda} f(b)$. Therefore, $f_*(A) = f(A) \leftrightarroweq_{\lambda'} f(B) = f_*(B)$. Thus f_* is (κ',λ')-continuous.

Suppose f_* is (κ',λ')-continuous. Let $x, y \in X$ such that $x \leftrightarrow_{\kappa} y$. Then $\{x\} \leftrightarrow_{\kappa'} \{y\}$, so

$$\{f(x)\} = f_*(\{x\}) \leftrightarroweq_{\lambda'} f_*(\{y\}) = \{f(y)\}.$$

Therefore, $f(x) \leftrightarroweq_{\lambda} f(y)$. Thus, f is (κ,λ)-continuous. □

We have the following as an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let (X,κ) and (Y,λ) be digital images and let $f : X \to Y$. Then the following are equivalent.*

- *f is a (κ,λ)-isomorphism;*
- *the induced function $f_* : 2^X \to 2^Y$ is a (κ',λ')-isomorphism; and*
- *the induced function $f_* : K(X) \to K(Y)$ is a (κ',λ')-isomorphism*

Proposition 5.3. *Given continuous functions $f : (X,\kappa) \to (Y,\lambda)$ and $g : (Y,\lambda) \to (W,\mu)$, we have $(g \circ f)_* = g_* \circ f_*$.*

Proof. The assertion follows from the observation that $A \in 2^X$ implies

$$(g \circ f)_*(A) = (g \circ f)(A) = g(f(A)) = g_*(f_*(A)) = (g_* \circ f_*)(A).$$

□

The following is elementary.

Lemma 5.4. *Let (X,κ) be a digital image. Then $(\mathrm{id}_X)_* = \mathrm{id}_{(2^X,\kappa')}$ and $(\mathrm{id}_X)_* = \mathrm{id}_{(K(X),\kappa')}$.*

Theorem 5.5. *The hyperspace construction yields covariant functors F, F' from the category of digital images and continuous functions to the category of graphs and continuous functions (respectively, to the category of connected graphs and continuous functions), in which $F(X,\kappa) = (2^X,\kappa')$, $F'(X,\kappa) = K(X,\kappa')$ and for $f : (X,\kappa) \to (Y,\lambda)$ we have $F(f) = f_*$, $F'(f) = f_*$.*

Proof. This follows from Proposition 5.3 and Lemma 5.4. □

Not every continuous function on the hyperspace of a digital image is induced by a continuous map between digital images, as shown by the following.

Example 5.6. *Let $X = [0,1]_{\mathbb{Z}}$. Let $F : K(X,c_1') \to K(X,c_1')$ be the function given by $F(A) = X$ for all $A \in K(X)$. F is constant, hence continuous, and is not induced by any $f : X \to X$ since for each such function, e.g., $f(0) \in X = \{0,1\} = f_*(0)$, hence $f_*(\{0\}) \neq F(0)$.*

5.2. Retraction and homotopy

Theorem 5.7. *Let (X,κ) and (Y,κ) be digital images and let $r : X \to Y$ be a κ-retraction. Then the induced maps $r_* : 2^X \to 2^Y$ and $r_* : K(X) \to K(Y)$ are κ'-retractions.*

Proof. It follows from Theorem 5.1 that each version of r_* is κ'-continuous. It is clear that $r_*|_{2^Y} = \mathrm{id}_{2^Y}$, $r_*(2^X) = 2^Y$, $r_*(K(X)) = K(Y)$, $r_*|_{K(Y)} = \mathrm{id}_{K(Y)}$. The assertion follows. □

Theorem 5.8. *Let f and g be (strongly) (pointed) homotopic maps from (X,κ) to (Y,λ). Then f_* and g_* are (strongly) (pointed) homotopic maps from $(2^X,\kappa')$ to $(2^Y,\lambda')$, and from $K(X)$ to $K(Y)$. In the case of (strongly) pointed homotopy, if $x_0 \in X$ is held fixed by the (strong) pointed homotopy from f to g, then $\{x_0\}$ is held fixed by the (strong) pointed homotopy from f_* to g_*.*

Proof. We give a proof for homotopy using 2^X and 2^Y. The proofs for strong or pointed homotopies and for $K(X)$ and $K(Y)$ are similar.

By hypothesis, there is a function $H : X \times [0,n]_{\mathbb{Z}} \to Y$ for some $n \in \mathbb{N}$ such that

- $H(x,0) = f(x)$ and $H(x,n) = g(x)$ for all $x \in X$.
- For all $x \in X$, the induced function $H_x : [0,n]_{\mathbb{Z}} \to Y$ defined by $H_x(t) = H(x,t)$ is (c_1,λ)-continuous.
- For all $t \in [0,n]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ defined by $H_t(x) = H(x,t)$ is (κ,λ)-continuous.

Let $H_* : 2^X \times [0,n]_{\mathbb{Z}} \to 2^Y$ be the function $H_*(A,t) = H_t(A)$.

- We have

  $$H_*(A,0) = H_0(A) = f(A) = f_*(A).$$

  Similarly, $H_*(A,n) = g_*(A)$.

- For all $A \in 2^X$, consider the induced function $H_{*,A} : [0,n]_{\mathbb{Z}} \to 2^Y$ defined by

  $$H_{*,A}(t) = H(A,t) = \bigcup_{x \in A} \{H_x(t)\}.$$

  Since $0 \le t < n$ implies $H_x(t) \leftrightarrows_{\lambda} H_x(t+1)$, it follows that $H_{*,A}(t) \leftrightarrows_{\lambda'} H_{*,A}(t+1)$. Thus the induced function $H_{*,A}$ is (c_1, λ')-continuous.

- For all $t \in [0,n]_{\mathbb{Z}}$, consider the induced function $H_{*,t} : 2^X \to 2^Y$ given by

  $$H_{*,t}(A) = H(A,t) = \bigcup_{x \in A} \{H_t(x)\}.$$

  If $A \leftrightarrow_{\kappa'} B$ then for each $a \in A$ there exists $b \in B$ such that $a \leftrightarrows_{\kappa} b$, and for each $\beta \in B$ there exists $\alpha \in A$ such that $\alpha \leftrightarrows_{\kappa} \beta$. Therefore, $H_t(a) \leftrightarrows_{\lambda} H_t(b)$ and $H_t(\alpha) \leftrightarrows_{\lambda} H_t(\beta)$. It follows that $H_{*,t}(A) \leftrightarrows_{\lambda'} H_{*,t}(B)$. Thus $H_{*,t}$ is (κ', λ')-continuous.

The above shows that H_* is a homotopy from f_* to g_*. □

Theorem 5.9. *Let (X,κ) and (Y,λ) be digital images.*

1. *If (X,κ) and (Y,λ) have the same (pointed) homotopy type, then $(2^X,\kappa')$ and $(2^Y,\lambda')$ have the same (pointed) homotopy type; as do $K(X,\kappa')$ and $K(Y,\lambda')$.*
2. *If (X,κ) and (Y,λ) have the same strong (pointed) homotopy type, then $(2^X,\kappa')$ and $(2^Y,\lambda')$ have the same strong (pointed) homotopy type; as do $K(X,\kappa')$ and $K(Y,\lambda')$.*
3. *Let (X,κ) be (pointed) contractible (respectively, (pointed) strongly contractible). Then $(2^X,\kappa')$ is contractible (respectively, (pointed) strongly contractible); as is $K(X,\kappa')$.*

Proof. We give proofs for the full hyperspaces 2^X and 2^Y; the proofs for $K(X,\kappa')$ and $K(Y,\lambda')$ are similar.

1. By hypothesis, there are continuous (pointed) maps $f : (X,\kappa) \to (Y,\lambda)$ and $g : (Y,\lambda) \to (X,\kappa)$ (with basepoints $x_0 \in X$, $y_0 \in Y$) such that $g \circ f \sim_{(\kappa,\kappa)} \mathrm{id}_X$ (holding x_0 fixed) and $f \circ g \sim_{(\lambda,\lambda)} \mathrm{id}_Y$ (holding y_0 fixed). By Proposition 5.3 and Theorem 5.8,

   $$g_* \circ f_* = (g \circ f)_* \sim_{(\kappa',\kappa')} (\mathrm{id}_X)_* = \mathrm{id}_{2^X} \text{ (holding } \{x_0\} \text{ fixed)}$$

   and

   $$f_* \circ g_* = (f \circ g)_* \sim_{(\lambda',\lambda')} (\mathrm{id}_Y)_* = \mathrm{id}_{2^Y} \text{ (holding } \{y_0\} \text{ fixed).}$$

   Therefore, $(2^X,\kappa')$ and $(2^Y,\lambda')$ have the same homotopy type.
2. The proof for strong homotopy type is similar.
3. Since (pointed) contractible (respectively, (pointed) strongly contractible) means having the same (pointed) homotopy type (respectively, (pointed) strong homotopy type) as a digital image of a single point, the assertions concerning (pointed) contractibility (respectively, (pointed) strong contractibility) follow from the above.

□

Theorem 5.10. *Let $H : (X,\kappa) \times [0,n]_{\mathbb{Z}} \to (X,\kappa)$ be a (strong, in the sense of strong homotopy) (strong, in the sense of retraction theory) deformation retraction of X to a subset Y, i.e., a (strong) homotopy between the induced maps $H_0, H_n : X \to X$ such that $H_0 = \mathrm{id}_X$ and H_n is a retraction (that holds fixed every point of Y). Then $H_* : (2^X,\kappa') \times [0,n]_{\mathbb{Z}} \to (2^X,\kappa')$ (respectively, $H_* : (K(X),\kappa') \times [0,n]_{\mathbb{Z}} \to (K(X),\kappa')$) is a (strong, in the sense of strong homotopy) (strong, in the sense of retraction theory) deformation retraction of $(2^X,\kappa')$ to $(2^Y,\kappa')$ (respectively, of $(K(X),\kappa')$ to $(K(X),\kappa')$).*

Proof. These assertions follow from Theorems 5.9 and 5.7. □

5.3. Function graphs

In this section, we explore an analog of function spaces for digital images.

Definition 5.11. *Let (X, κ) and (Y, λ) be digital images. Consider the set $(Y, \lambda)^{(X,\kappa)}$, or Y^X when κ and λ can be assumed, defined by*

$$(Y, \lambda)^{(X,\kappa)} = \{f : X \to Y \mid f \text{ is } (\kappa, \lambda)\text{-continuous}\}.$$

We say $f, g \in Y^X$ are $\Phi(\kappa, \lambda)$-adjacent*, or* Φ-adjacent *when κ and λ can be assumed, if for all $x \in X$ we have $f(x) \eqsim_\lambda g(x)$.*

A more restrictive adjacency for Y^X, which we denote as $\Psi(\kappa, \lambda)$, is proposed in [26]. We have the following.

Definition 5.12. [26] *Let $f, g \in Y^X$. Then $f \leftrightarrow_{\Psi(\kappa,\lambda)} g$ if given $x_0 \eqsim_\kappa x_1$ in X, $f(x_0) \eqsim_\lambda g(x_1)$ in Y.*

Remark 5.13. *It is clear that $f \leftrightarrow_{\Psi(\kappa,\lambda)} g$ implies $f \leftrightarrow_{\Phi(\kappa,\lambda)} g$. The converse is not generally valid. For example, consider the functions $f, g : [0, 2]_\mathbb{Z} \to [0, 2]_\mathbb{Z}$ given by $f(x) = x$, $g(x) = \min\{2, x + 1\}$. It is easily seen that $f, g \in C([0, 2]_\mathbb{Z}, c_1)$ and $f \leftrightarrow_{\Phi(c_1,c_1)} g$. However, since $0 \leftrightarrow_{c_1} 1$ and $f(0) = 0 \not\leftrightarrow_{c_1} 2 = g(1)$, f and g are not $\Psi(c_1, c_1)$-adjacent.*

We show below, at Example 5.17, an important difference between $\Phi(c_1, c_1)$ and $\Psi(c_1, c_1)$.

Lemma 5.14. *Let (X, κ) and (Y, λ) be digital images. Let $f, g \in Y^X$. Then f and g are homotopic in one step if and only if $f \eqsim_\Phi g$.*

Proof. Suppose f and g are homotopic in one step. Then there exists $H : X \times [0, 1]_\mathbb{Z} \to Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and the induced function $H_x : [0, 1]_\mathbb{Z} \to Y$ given by $H_x(t) = H(x, t)$ is (c_1, λ) continuous. The latter implies

$$f(x) = H(x, 0) \eqsim_\lambda H(x, 1) = g(x)$$

for all $x \in X$, so $f \eqsim_\Phi g$.

Suppose $f \eqsim_\Phi g$. Then one sees easily that the function $H : X \times [0, 1]_\mathbb{Z} \to Y$ defined by

$$H(x, 0) = f(x), \quad H(x, 1) = g(x),$$

is a homotopy in one step from f to g. □

The following was suggested by an anonymous reviewer.

Theorem 5.15. *Let (X, κ) and (Y, λ) be digital images. Let $f, g \in Y^X$. Then f and g are homotopic if and only if f and g belong to the same component of (Y^X, Φ).*

Proof. Since both homotopy between functions and being connected by a path are transitive relations, the assertion follows from Lemma 5.14. □

Let (S_n, κ) be any cyclic graph of $n > 4$ points, with point set $S_n = \{x_i\}_{i=0}^{n-1}$ such that $x_i \leftrightarrow_\kappa x_j$ if and only if $|i - j| \in \{1, n - 1\}$. Let $r_j \in C(S_n, \kappa)$ be the rotation $r_j(x_i) = x_{(i+j) \bmod n}$. We have the following.

Theorem 5.16. [15] *If $f \in C(S_n, \kappa)$ such that f and id_{S_n} are κ-homotopic, then $f = r_j$ for some j, $0 \le j < n$.*

We do not get a similar outcome if we substitute Ψ for Φ in Theorem 5.15, as shown in the following.

Example 5.17. *Let (S_n, κ) be any cyclic graph of $n > 4$ points. Then all the rotations r_j are homotopic. However, no distinct r_j and r_k belong to the same component of $(S_n^{S_n}, \Psi)$.*

Proof. Without loss of generality, $k = j + m < n$ for some m, $0 < m < n - j$. Then $H : S_n \times [0, m]_{\mathbb{Z}} \to S_n$, defined by $H(x_i, t) = r_{j+t}(x_i)$, is a homotopy from r_j to r_k.

It follows from Theorem 5.16 that every induced map H_t of H for $t \in [0, m]_{\mathbb{Z}}$, and in particular, H_1, is a rotation.

- For $1 \leq m < n - 2$, $r_j(x_0) = x_j$ and $r_k(x_1) = x_{(j+m+1) \bmod n}$ are not κ-adjacent.
- For $m = n - 2$, we cannot follow the pattern used above, since

  $$r_k(x_1) = r_{j+n-2}(x_1) = r_{j-2 \bmod n}(x_1) = x_{j-1 \bmod n}$$

  is adjacent to $r_j(x_0)$. However,

  $$r_k(x_0) = r_{j+n-2}(x_0) = r_{j-2 \bmod n}(x_0) = x_{j-2 \bmod n}$$

  is neither adjacent nor equal to $r_j(x_0)$.
- For $m = n - 1$ we must have $j = 0$. Therefore, $r_j(x_1) = x_1$ and $r_k(x_0) = x_{n-1}$ are not κ-adjacent.

In every case, r_j and r_k are not Ψ-adjacent. This completes the proof. □

Theorem 5.18. *Let (X, κ) and (Y, λ) be digital images. Let W be a λ-retract of Y. Then $(W^X, \Phi(\kappa, \lambda))$ is a retract of $(Y^X, \Phi(\kappa, \lambda))$.*

Proof. Let $r : Y \to W$ be a λ-retraction. Then for every (κ, λ)-continuous $f : X \to Y$, $r \circ f : X \to W$ is continuous by Proposition 2.3. Further, if $f(X) \subset W$ then $r \circ f = f$. The assertion follows. □

We present results that link the topics of hyperspaces and function graphs.

Theorem 5.19. *Let $f \leftrightarrow_{\Phi(\kappa,\lambda)} g$ in Y^X. Then for $A \in 2^X$, $f(A) \leftrightarroweq_{\lambda'} g(A)$.*

Proof. Let $y_f \in f(A)$. Let $x_f \in f^{-1}(y_f)$. Then $y_f = f(x_f) \leftrightarroweq_{\lambda} g(x_f)$. Similarly, given $y_g \in g(A)$, there exists $x_g \in g^{-1}(y_g)$ such that $f(x_g) \leftrightarroweq_{\lambda} g(x_g) = y_g$. It follows that $f(A) \leftrightarroweq_{\lambda'} g(A)$. □

Theorem 5.20. *Let (W, κ), (X, λ), and (Y, μ) be digital images. Suppose $f, g \in Y^X$ are (λ, μ)-continuous. If f and g are*

- *(strongly) (λ, μ)-homotopic;*
- *(strongly) pointed (λ, μ)-homotopic with x_0 held fixed,*

then the induced maps $f_*, g_* : (X^W, \Phi(\kappa, \lambda)) \to (Y^W, \Phi(\kappa, \mu))$, defined for all $F \in X^W$ by

$$f_*(F) = f \circ F, \quad g_*(F) = g \circ F,$$

are $(\Phi(\kappa, \lambda), \Phi(\kappa, \mu))$-continuous and, respectively, f_* and g_* are,

- *(strongly) $(\Phi(\kappa, \lambda), \Phi(\kappa, \mu))$-homotopic;*
- *(strongly) pointed $(\Phi(\kappa, \lambda), \Phi(\kappa, \mu))$-homotopic with the constant function $\hat{x}_0$ held fixed.*

Proof. Let $F, G \in X^W$ be (κ, λ)-continuous with $F \leftrightarrow_{\Phi(\kappa,\lambda)} G$ and let $w \in W$. Then

$$F(w) \leftrightarroweq_{\lambda} G(w), \text{ so } f_*(F)(w) = f \circ F(w) \leftrightarroweq_{\mu} f \circ G(w) = f_*(G)(w),$$

so $f_*(F) \leftrightarroweq_{\Phi(\kappa,\mu)} f_*(G)$, hence f_* is continuous. Similarly, g_* is continuous.

We proceed with a proof for homotopic maps; the other assertions are proven similarly.

Let $H : X \times [0, n]_{\mathbb{Z}} \to Y$ be a (λ, μ)-homotopy from f to g. Let $H_* : (X^W, \Phi(\kappa, \lambda)) \times [0, n]_{\mathbb{Z}} \to (Y^W, \Phi(\kappa, \mu))$ be given by $H_*(F, t)(x) = H(F(x), t)$. We have the following.

- $H_*(F,0)(x) = H(F(x),0) = f(F(x)) = f_*(F)(x)$, so $H_*|_{t=0} = f_*$; and $H_*(F,n)(x) = H(F(x),n) = g(F(x)) = g_*(F)(x)$, so $H_*|_{t=n} = g_*$.
- Given $F \in X^W$, the induced function $H_{*,F} : [0,n]_{\mathbb{Z}} \to Y^W$ given by $H_{*,F}(t)(w) = H(F(w),t)$ satisfies, for $t_0 \leftrightarrow_{c_1} t_1$ in $[0,n]_{\mathbb{Z}}$,

$$H_{*,F}(t_0)(w) = H(F(w),t_0) \leftrightarroweq_\mu H(F(w),t_1)(w) = H_{*,F}(t_1)(w),$$

  so $H_{*,F}$ is $(c_1, \Phi(\lambda,\mu))$-continuous.
- Given $t \in [0,n]_{\mathbb{Z}}$, the induced function $H_{*,t} : X^W \to Y^W$ given by $H_{*,t}(F)(w) = H(F(w),t)$ satisfies, for $F_0 \leftrightarrow_{\Phi(\kappa,\lambda)} F_1$ in X^W,

$$H_{*,t}(F_0)(w) = H(F_0(w),t) \leftrightarroweq_\mu H(F_1(w),t) = H_{*,t}(F_1)(w).$$

  Therefore, $H_{*,t}$ is $(\Phi(\kappa,\lambda), \Phi(\kappa,\mu))$-continuous.

Therefore, H_* is a homotopy from f_* to g_*. □

Proposition 5.21. *Let (V,κ), (W,λ), (X,μ), (Y,ν) be digital images. Let $f : (W,\lambda) \to (X,\mu)$ and $g : (X,\mu) \to (Y,\nu)$ be continuous. Consider the induced maps $f_* : W^V \to X^V$ and $g_* : X^W \to Y^W$. We have $(g \circ f)_* = g_* \circ f_* : W^V \to Y^V$.*

Proof. Given $F : V \to W$, we have

$$g_* \circ f_*(F) = g_*(f \circ F) = g \circ (f \circ F) = (g \circ f) \circ F = (g \circ f)_*(F).$$

The assertion follows. □

Corollary 5.22. *Let (X,κ) and (Y,λ) be digital images. Suppose (X,κ) and (Y,λ) have the same (strong) (pointed) homotopy type. Then $(X^X, \Phi(\kappa,\kappa))$ and $(Y^Y, \Phi(\lambda,\lambda))$ have the same (strong) (pointed) homotopy type, respectively.*

Proof. We give a proof for "same homotopy type"; the other assertions are established similarly (in the pointed cases, if $x_0 \in X$ and $y_0 \in Y$ are the basepoints of the assumption, then the constant maps $\hat{x_0} \in X^X$ and $\hat{y_0} \in Y^Y$ are the basepoints of the conclusion).

If (X,κ) and (Y,λ) have the same homotopy type, then there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim_{\kappa,\kappa} \mathrm{id}_X$ and $f \circ g \sim_{\lambda,\lambda} \mathrm{id}_Y$. By Theorem 5.20 and Proposition 5.21, we have

$$(g \circ f)_* \sim_{\Phi(\kappa,\kappa)} (\mathrm{id}_X)_* = \mathrm{id}_{X^X}$$

and similarly,

$$(f \circ g)_* \sim_{\Phi(\lambda,\lambda)} (\mathrm{id}_Y)_* = \mathrm{id}_{Y^Y}.$$

Thus, $(X^X, \Phi(\kappa,\kappa))$ and $(Y^Y, \Phi(\lambda,\lambda))$ have the same homotopy type. □

Corollary 5.23. *Let (X,κ) be a digital image. Suppose (X,κ) is*

- *(strongly) contractible;*
- *(strongly) pointed contractible with basepoint x_0.*

Then, respectively, $(X^X, \Phi(\kappa,\kappa))$ is

- *(strongly) contractible;*
- *(strongly) pointed contractible with the constant function $\hat{x_0}$ as basepoint.*

Proof. We give a proof for "contractible"; the other assertions follow similarly.

Since "contractible" means homotopy equivalent to a digital image with a single point, the assertion follows from Corollary 5.22. □

6. Connectedness in digital hyperspaces

We have the following.

Proposition 6.1. *Let (X, κ) be a digital image. Let W be a nonempty κ'-connected subset of $K(X)$. Then $W' = \bigcup_{Y \in W} Y$ is a κ-connected subset of X.*

Proof. Let $x_0, x_1 \in W'$. There exist $Y_i \in W$ such that $x_i \in Y_i \in K(X, \kappa')$. Since W is κ'-connected, there exists a κ'-path $\{W_i\}_{i=0}^n \subset W$ from Y_0 to Y_1, i.e., $W_0 = Y_0$, $W_i \leftrightarrow_{\kappa'} W_{i+1}$, and $W_n = Y_1$.

By Definition 3.1, there exist $p_i \in W_i$, $q_{i+1} \in W_{i+i}$ such that $p_i \leftrightarrows_\kappa q_{i+1}$. As each W_i is κ-connected, there exist κ-paths $P_0 \subset Y_0 = W_0$ from x_0 to p_0; $P_i \subset W_i$ from q_i to p_{i+1}, $1 \leq i < n$; and $P_n \subset W_n = Y_1$ from p_n to x_1.

Then $\bigcup_{i=0}^n P_i$ is a κ-path in W' from x_0 to x_1. It follows that W' is κ-connected. □

Proposition 6.2. *Let C_0 and C_1 be distinct components of (X, κ). Let*

$$A_i \in K(C_i, \kappa') \text{ for } i \in \{0, 1\}. \tag{1}$$

Then A_0 and A_1 are points of distinct components of $K(X, \kappa')$.

Proof. Were A_0 and A_1 in the same component of $K(X, \kappa')$, then there would exist a path $\{B_j\}_{j=0}^n \subset K(X, \kappa')$ such that $A_0 = B_0$,

$$B_j \leftrightarrow_{\kappa'} B_{j+1} \text{ for } 1 \leq j < n, \tag{2}$$

and $B_n = A_1$. By (1), there is a smallest $k \in \mathbb{N}$ such that $0 \leq k < n$, $B_k \subset C_0$, and $B_{k+1} \not\subset C_0$. But by (2), $B_k \cup B_{k+1}$ is κ-connected and therefore must be a subset of C_0, contrary to our choice of k. It follows that A_0 and A_1 are points of distinct components of $K(X, \kappa')$. □

Proposition 6.3. *Let (X, κ) be a finite connected digital image. Then $K(X, \kappa')$ is connected.*

Proof. Let $A \in K(X, \kappa')$. We show there is a path in $K(X, \kappa')$ from A to X. If $A = X$, we are done. Otherwise, since X is connected, there are sequences $\{x_i\}_{i=1}^m \subset X \setminus A$ and $\{A_j\}_{j=0}^m$ such that $A = A_0$, $A_{j+1} = A_j \cup \{x_{j+1}\}$, A_{j+1} is connected, and $A_m = X$. Therefore, $A_j \leftrightarrow_{\kappa'} A_{j+1}$. Thus $\{A_j\}_{j=0}^m$ is a κ'-path in $K(X, \kappa')$ from A to X.

Since A was arbitrarily chosen, it follows that $K(X, \kappa')$ is connected. □

Proposition 6.4. *Let D be a component of (X, κ). Then $K(D, \kappa')$ is a component of $K(X, \kappa')$.*

Proof. By Proposition 6.3, $K(D, \kappa')$ is connected. The conclusion follows from Proposition 6.2. □

Theorem 6.5. *Let (X, κ) be a digital image. Then X is κ-connected if and only if $K(X, \kappa')$ is κ'-connected.*

Proof. Suppose (X, κ) is connected. By Proposition 6.4, $K(X, \kappa')$ is κ'-connected.

Conversely, suppose $K(X, \kappa')$ is κ'-connected. By Proposition 6.2, (X, κ) must be connected. □

Lemma 6.6. *Let (X, κ) be a digital image. Let A be a finite member of $K(X, \kappa')$. Then there is a path $\mathcal{P}$ in $K(A, \kappa')$ from a singleton to A.*

Proof. Let $x_0 \in A$. By Proposition 6.3, there is a path in $K(A, \kappa')$ from $\{ x_0 \}$ to A. □

Suppose (X, κ) is a connected digital image. We say $Y \subset X$ *disconnects* (X, κ) if $X \setminus Y$ is not κ-connected.

Theorem 6.7. *Let (X, κ) be a connected digital image. Let $Y \subset X$. Let*

$$\mathcal{Y} = \{B \in K(X, \kappa') \mid B \cap Y \neq \emptyset\}. \tag{3}$$

If Y disconnects (X, κ) then $\mathcal{Y}$ disconnects $K(X, \kappa')$.

Proof. Suppose Y disconnects (X,κ). Then there are x_0, x_1 that are in distinct components of $X \setminus Y$.

Suppose $\mathcal{Y}$ fails to disconnect $K(X,\kappa')$. Then there exists a κ'-path

$$\mathcal{P} = \{B_j\}_{j=0}^{n} \subset K(X,\kappa') \setminus \mathcal{Y} \tag{4}$$

from $\{x_0\}$ to $\{x_1\}$. By Definition 3.1, there exist $y_j, z_j \in B_j$ such that $y_j \leftrightarrows_\kappa z_{j+1}$ for $j < n$. Since B_j is connected, there are κ-paths $P_0 \subset B_0$ from x_0 to y_0, $P_j \subset B_j$ from z_j to y_j, and $P_n \subset B_n$ from z_n to x_1. Then $P = \bigcup_{j=0}^{n} P_j$ is a κ-path in $\bigcup_{j=0}^{n} B_j \subset X$ from x_0 to x_1. Since Y disconnects X, we must have $P \cap Y \neq \emptyset$. Hence for some k, $B_k \cap Y \neq \emptyset$, contrary to (4). The contradiction establishes that $\mathcal{Y}$ disconnects $K(X,\kappa')$. □

7. Multivalued functions and hyperspaces

In this section, we examine relations between various notions of continuous multivalued functions between digital images, and hyperspaces of digital images.

Definition 7.1. *A multivalued function $F : (X,\kappa) \multimap (Y,\lambda)$*

- *has* strong continuity [31] *if for each pair of adjacent $x, y \in X$, every point of $F(x)$ is adjacent or equal to some point of $F(y)$ and every point of $F(y)$ is adjacent or equal to some point of $F(x)$;*
- *has* weak continuity [31] *if for each pair of adjacent $x, y \in X$, $F(x)$ and $F(y)$ are adjacent sets in Y, i.e., there exist $a \in F(x)$, $b \in F(y)$ such that $a \leftrightarrows_\lambda b$;*
- *is* connectivity preserving [25] *if $F(A) \subset Y$ is connected whenever $A \subset X$ is connected;*
- *is* continuous *[21, 22] if $X \subset \mathbb{Z}^n$, $\kappa = c_u$ for $1 \leq u \leq n$, and F is* generated *by a continuous function $f : S(X,r) \to Y$ for some positive integer r; where $S(X,r) = \bigcup_{x \in X} S(\{x\}, r)$, where for $x = (x_1, \ldots, x_n)$, $S(\{x\}, r)$ is the set of all points $(y_1, \ldots, y_n)$ such that for each index i we have $y_i = x_i + k_i/r$ for some integer k_i such that $0 \leq k_i < r$; $S(X,r)$ inherits c_u in the sense that $(y_1, \ldots, y_n) \leftrightarrows_{c_u} (a_1, \ldots, a_n)$ in $S(X,r)$ if for at most u indices i, $|y_i - a_i| = 1/r$ and for all other indices j, $y_j = a_j$; and "F is* generated *by f" means for all $x \in X$, $F(x) = \bigcup_{y \in S(\{x\},r)} \{f(y)\}$.*

We have the following.

Theorem 7.2. *Let $F : (X,\kappa) \multimap (Y,\lambda)$ be a strongly continuous multifunction between digital images. Then the function $F_* : (2^X, \kappa') \to (2^Y, \lambda')$ defined by $F_*(A) = F(A)$ is continuous.*

Proof. Let $A_0 \leftrightarrow_{\kappa'} A_1$ in 2^X. We must show that $F(A_0) \leftrightarrows_{\lambda'} F(A_1)$ in 2^Y.

Let $x \in A_0$, $y \in A_1$ such that $x \leftrightarrows_\kappa y$. By Definition 7.1, for every $p \in F(x)$ there exists $q \in F(y)$ such that $p \leftrightarrows_\lambda q$. Similarly, given $u \in A_1$, $v \in A_0$ such that $u \leftrightarrow_\kappa v$, for every $r \in F(u)$ there exists $s \in F(v)$ such that $r \leftrightarrows_\lambda s$. The assertion follows. □

The following shows that in substituting weak continuity, continuity, or connectivity-preserving for strong continuity, we fail to obtain a result analogous to Theorem 7.2.

Example 7.3. *Let $F : ([0,1]_\mathbb{Z}, c_1) \multimap ([0,2]_\mathbb{Z}, c_1)$ be defined by $F(0) = \{0\}$, $F(1) = \{1,2\}$. Then F has weak c_1-continuity, is c_1-continuous, and is c_1-connectivity-preserving, but since $2 \in F(1)$ has no c_1-neighbor in $F(0)$, the induced function $F_* : (2^X, c_1') \to (2^Y, c_1')$ is not (c_1', c_1')-continuous.*

8. Cycles and Girth

The reader is reminded that:

- a point in 2^X is a nonempty subset of X;
- a cycle in X is a closed path of at least 3 distinct points in which no node repeats, but in which a point x can be adjacent to points distinct from the predecessor and successor of x in the path (the cycle does not need to be chordless).

Proposition 8.1. *Let (X, κ) be a digital image. Then $K(X, \kappa')$ has a 3-cycle if and only if (X, κ) has a non-isolated point.*

Proof. It is elementary that if the points of (X, κ) are all isolated, then $K(X, \kappa')$ has no cycle.

Suppose $x \in X$ is not isolated in (X, κ). Then there exists $y \in X$ such that $x \leftrightarrow_\kappa y$. Then $\{\{x\}, \{x, y\}, \{y\}\}$ is a 3-cycle in $K(X, \kappa')$. □

The *girth* of a graph (X, κ) is variously described in the literature as the length of a shortest or of a longest [1] cycle in (X, κ). We may distinguish these concepts as *girth* and *Girth*, respectively. In light of Proposition 8.1, the Girth is more interesting, so in the following we focus on Girth.

Example 8.2. *If (X, κ) is a digital image and $x \in X$ such that $N(X, x, \kappa)$ has distinct points u and v that are not κ-adjacent, then $K(X, \kappa')$ has Girth of at least 6.*

Proof. By hypothesis, there exist distinct $u, v \in N(X, x, \kappa)$. Then by Definition 3.1, $K(X, \kappa')$ has a 6-cycle

$$\{u\}, \{u, x\}, \{u, x, v\}, \{x, v\}, \{v\}, \{x\}.$$

□

We have the following.

Example 8.3. *The Girth of $(2^{[1,4]_\mathbb{Z}}, c_1')$ is 15, which is equal to $\#(2^{[1,4]_\mathbb{Z}}, c_1')$. I.e., $(2^{[1,4]_\mathbb{Z}}, c_1')$ has a cycle containing all members of $(2^{[1,4]_\mathbb{Z}}, c_1')$.*

Proof. It is easy to see that the following sequence of the 15 distinct members of $(2^{[1,4]_\mathbb{Z}}, c_1')$ is a c_1'-cycle.

$$\{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{2, 3, 4\}, \{2, 3\},$$

$$\{2, 4\}, \{3, 4\}, \{4\}, \{3\}, \{2\}, \{1\}$$

□

9. Dominating set

A subset D of a graph (X, κ) is a *dominating set* for, or *dominates*, (X, κ), if given $x \in X$ there exists $d \in D$ such that $d \leftrightarroweq_\kappa x$.

Theorem 9.1. *Let (X, κ) be a digital image and let $D \subset X$. Let*

$$\mathcal{D} = \{A \in 2^X \mid A \cap D \neq \emptyset\}$$

Then D dominates (X, κ) if and only if $\mathcal{D}$ dominates $(2^X, \kappa')$.

Proof. Suppose D dominates (X, κ). Let $x \in A \in 2^X$. There exists $y \in D$ such that $x \leftrightarroweq_\kappa y$. It follows from Definition 3.1 that

$$A' = A \cup \{y\} \leftrightarroweq_{\kappa'} A.$$

Since A is arbitrary and $A' \in \mathcal{D}$, it follows that $\mathcal{D}$ dominates $(2^X, \kappa')$.

Suppose $\mathcal{D}$ dominates $(2^X, \kappa')$. Let $x \in X$. Then there exists $A \in \mathcal{D}$ such that $A \leftrightarroweq_{\kappa'} \{x\}$. Therefore, for all $a \in A$ we have $a \leftrightarroweq_\kappa x$. Since there exists $d \in A \cap D$, $d \leftrightarroweq_\kappa x$. Thus, D dominates X. □

10. Diameter

Definition 10.1. [23] *Let* (X, κ) *be a connected graph. The* shortest path metric *for* (X, κ) *is*

$$d_\ell(x, y) = \min\{length(P) \mid P \text{ is a } \kappa\text{-path in } X \text{ from } x \text{ to } y\}, \text{ for } x, y \in X.$$

Definition 10.2. *The* diameter *of a finite connected graph* (X, κ) *is*

$$diam(X, \kappa) = \max\{d_\ell(x, y) \mid x, y \in X\}.$$

Definition 10.3. [1] *Let* (X, κ) *be a connected digital image. For* $x \in X$*, the* associated number $e(x)$ of x *is*

$$e(x) = \max\{d_\ell(x, y) \mid y \in X\}.$$

A center *of* (X, κ) *is a point* $x_0 \in X$ *such that*

$$e(x_0) = \min\{e(x) \mid x \in X\}.$$

The associated number of the center is the radius *of* (X, κ)*.*

We have the following.

Theorem 10.4. *Let* (X, κ) *be a finite connected digital image with radius* r*. Let* $\#X = n$*. Then* $diam(K(X, \kappa')) < 2(n + r - 1)$*.*

Proof. Let x_0 be a center of (X, κ). Let $A_0, A_1 \in K(X, \kappa')$. Let $y_0 \in A_0$, $y_1 \in A_1$. By assumption, there are paths P_i of length at most r from x_0 to y_i. Thus, $P_0 \cup P_1$ is a κ-path in X of length at most $2r$ from y_0 to y_1. It follows from Definition 3.1 that $\mathcal{P} = \{\{p\} \mid p \in P_0 \cup P_1\}$ is a κ'-path in $K(X)$ of length at most $2r$ from $\{y_0\}$ to $\{y_1\}$.

Let $Q_0 = \{y_0\}$. We argue inductively as follows. Suppose we have $Q_k \in K(X, \kappa')$ such that $Q_k \subset A_0$. If $Q_k \neq A_0$, then since A_0 is connected, there exists $q \in A_0 \setminus Q_k$ such that for some $q' \in Q_k$, $q \leftrightarrow_\kappa q'$. By Definition 3.1, we have

$$Q_{k+1} = Q_k \cup \{q'\} \leftrightarrow_{\kappa'} Q_k.$$

Since $Q_{\#A_0 - 1} = A_0$, the set $\mathcal{P}_0 = \{Q_j\}_{j=0}^{\#A_0 - 1}$ is a path in $K(X, \kappa')$ of length $\#A_0 - 1$ from $\{y_0\}$ to A_0; equivalently, from A_0 to $\{y_0\}$.

Similarly, we can construct a path $\mathcal{P}_1$ in $K(X, \kappa')$ of length $\#A_1 - 1$ from $\{y_1\}$ to A_1. Therefore, $\mathcal{P}_0 \cup \mathcal{P} \cup \mathcal{P}_1$ is a path in $K(X, \kappa')$ of length at most

$$\#A_0 - 1 + 2r + \#A_1 - 1 \leq 2(n + r - 1)$$

from A_0 to A_1. Further, we may assume $\min\{\#A_0, \#A_1\} < n$; since otherwise $A_0 = X = A_1$, so there is a path of length 0 from A_0 to A_1 in $K(X, \kappa')$. It follows that for any $A_0, A_1 \in K(X, \kappa')$ there is a path in $K(X, \kappa')$ from A_0 to A_1 of length less than $2(n + r - 1)$. The assertion follows. □

11. Further remarks

We have introduced into digital topology the study of hyperspaces of digital images, and have taken a somewhat different approach to function graphs than that introduced in [26]. We have studied some relations between digital hyperspaces and digital function graphs. We have examined a number of properties of digital hyperspaces concerning cardinality, continuous maps and homotopy, connectivity, cycles and Girth, dominating sets, and diameters.

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