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Hyperbolic Navier-Stokes equations in three space dimensions

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Abstract. We consider in this paper a hyperbolic quasilinear version of the Navier-Stokes equations in three space dimensions, obtained by using Cattaneo type law instead of a Fourier law. In our earlier work [2], we proved the global existence and uniqueness of solutions for initial data small enough in the space $H^4(\mathbb{R}^3)^3 \times H^3(\mathbb{R}^3)^3$. In this paper, we refine our previous result in [2], we establish the existence under a significantly lower regularity. We first prove the local existence and uniqueness of solution, for initial data in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, $\delta > 0$. Under weaker smallness assumptions on the initial data and the forcing term, we prove the global existence of solutions. Finally, we show that if ε is close to 0, then the solution of the perturbed equation is close to the solution of the classical Navier-Stokes equations.

1. Introduction

The classical Navier-Stokes equations describe the evolution of Newtonian incompressible viscous fluids

$$(NS)\begin{cases} v_t - \nu\Delta v + (v.\nabla)v = -\nabla p + f, \\ \operatorname{div} v = 0, \\ v(0,y) = v_0, \end{cases} \tag{1.1}$$

with $\nu > 0$ being the viscosity, for the velocity vector $v = v(t,y) : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ of a fluid, $p = p(t,y) : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ the pressure and f the forcing term.
This equation arise from the momentum equations

$$\partial_t v + v.\nabla v = divS, \tag{1.2}$$

and the Fourier law for the tensor S,

$$S(t) = -p(t)Id + \nu(\nabla u(t) + (\nabla u)^T(t)). \tag{1.3}$$

However, the system (NS) gives a velocity with infinite propagation speed, which is contradict with the physical. In order to avoid this, Cattaneo replace the Fourier law (1.3) by the retarded equation

$$S(t+\varepsilon) = -p(t)Id + \nu(\nabla u(t) + (\nabla u)^T(t)), \tag{1.4}$$

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Using Taylor approximation of $S(t+\varepsilon) \simeq S(t) + \varepsilon\partial_t S(t)$ and applying the operator $(\varepsilon\partial_t + Id)$ to the momentum equations (1.2), we obtain the following quasilinear hyperbolic version of Navier-Stokes equations:

$$(HNS)_\varepsilon \begin{cases} \varepsilon u^\varepsilon_{tt} - \nu\Delta u^\varepsilon + u^\varepsilon_t + \varepsilon(u^\varepsilon.\nabla)u^\varepsilon_t + ((\varepsilon u^\varepsilon_t + u).\nabla)u^\varepsilon = -\nabla p + f^\varepsilon, \\ \operatorname{div} u^\varepsilon = 0, \\ (u^\varepsilon, u^\varepsilon_t)(0,y) = (u^\varepsilon_0, u^\varepsilon_1). \end{cases} \tag{1.5}$$

This resulting system has finite propagation speed, so compatible with physical law.
In literature, there exist several examples of hyperbolic perturbation of Navier Stokes equations. We remark that, if we neglect $\varepsilon(u^\varepsilon.\nabla)u^\varepsilon_t + \varepsilon(u^\varepsilon_t.\nabla)u^\varepsilon$, we obtain the hyperbolic version of Navier-Stokes equations:

$$\begin{cases} \varepsilon u^\varepsilon_{tt} - \nu\Delta u^\varepsilon + u^\varepsilon_t + (u^\varepsilon.\nabla)u^\varepsilon = -\nabla p + f^\varepsilon, \\ \operatorname{div} u^\varepsilon = 0, \\ (u^\varepsilon, u^\varepsilon_t)(0,y) = (u^\varepsilon_0, u^\varepsilon_1). \end{cases} \tag{1.6}$$

Brenier, Natalini and Puel in [4] obtained also this system (1.6) after relaxation of the Euler equations and rescaling variables. Moreover, they proved the global existence and uniqueness of solution to the system (1.6), in a two-dimensional space under smallness condition of the initial data in $H^2(\mathbb{R}^2)^2 \times H^1(\mathbb{R}^2)^2$ and without a forcing term. Paicu and Raugel in [17, 18] improved this result. They stated the global existence and uniqueness under significantly improved regularity for the initial data by using the Strichartz estimate. In [17], they obtained the global existence under suitable smallness assumptions on the initial data in the space $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$. In [18], the authors proved global existence and uniqueness in three-dimensional space under a smallness condition of the initial data in $H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$, for $\delta > 0$. Hachicha [13] obtained the global existence and uniqueness under suitable assumptions on the initial data in the space $H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^\times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$, $n = 2, 3$ by using a modulated energy method. Also, she proves the convergence of solution of hyperbolic version of Navier-Stokes to a solution of the classical Navier Stokes equations.

The purpose of this paper is to study the quasilinear hyperbolic perturbation of Navier Stokes equations $(HNS)_\varepsilon$. This model has already been studied by Racke and Saal [20, 21]. They have shown the local existence and the uniqueness of solution, when the initial data belong to $H^{m+2}(\mathbb{R}^n)^n \times H^{m+1}(\mathbb{R}^n)^n$, for $m > \frac{n}{2}$ integer. Under smallness assumption on the initial data in the space $H^{m+2}(\mathbb{R}^n)^n \times H^{m+1}(\mathbb{R}^n)^n$, for $m \geq 12$ integer, they proved the global existence of solution. Schöwe in [23, 24] improved this result. More precisely, in three dimensional case, he obtained the global existence under smallness condition on the initial data in the space $H^{m+2}(\mathbb{R}^3)^3 \times H^{m+1}(\mathbb{R}^3)^3$, for $m \geq 4$ integer. In our previous work [2], we proved the global existence and uniqueness where the initial data are small enough in the space $H^4(\mathbb{R}^3)^3 \times H^3(\mathbb{R}^3)^3$ and we proved that if ε is close to 0, then the solution of the perturbed system $(HNS)_\varepsilon$ is close to the solution of the classical Navier stokes equations (NS). Recently, Couloud Hachicha and Raugel [10] investigated to this equation in two dimensional space. They proved the local existence in the space $H^{\eta+2}(\mathbb{R}^2)^2 \times H^{\eta+1}(\mathbb{R}^2)^2$, $0 < \eta < 1$, and under suitable smallness assumption on the initial data they proved the global existence.

The aim of this paper is to improve the results of [2, 20, 21, 23, 24]. First, we show the local existence and uniqueness of solution to the equation $(HNS)_\varepsilon$ in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, $\delta > 0$. In addition, under suitable smallness assumptions on the initial data and forcing term, we prove the global existence to these equations. Finally, we prove that if ε is close to 0 then the solution of $(HNS)_\varepsilon$ is close to the solution of the classical Navier-Stokes equations (NS).
The main interesting point in our result is the significantly improved regularity for the initial data. We claim that the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, $\delta > 0$ is the minimum regularity in the Sobolev space needed to establish the existence results.

As staded above, we first adapt the Friedrichs method to our equation and prove the local well-posedness result of the system $(HNS)_\varepsilon$ in $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, $\delta > 0$.

Theorem 1.1. *Let $\delta > 0$, $(u^\varepsilon_0, u^\varepsilon_1) \in H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$ and the forcing term $f^\varepsilon \in C^0((0,+\infty), H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$. There exist a positive time T and a unique local weak solution $u^\varepsilon \in C^1([0,T], H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3) \cap C([0,T], H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3)$ of the System $(HNS)_\varepsilon$.*

Our second goal in this paper is to refined our earlier work [2]. We prove the global existence and uniqueness of solution for the system $(HNS)_\varepsilon$ under suitable smallness assumptions on the initial data in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$.

Theorem 1.2. *Let $\delta > 0$, There exist positive constants ε_0, R such that for all $0 < \varepsilon \leq \varepsilon_0$, if we assume that the initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$ and the forcing term $f^\varepsilon \in L^1((0,+\infty), H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3) \cap L^2((0,+\infty), H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$ satisfy, for all $\theta \in [0,1]$*

$$\varepsilon^{\frac{1+\delta}{2}-\theta(\frac{3}{4}+\frac{\delta}{2})}\Big(\|u_0^\varepsilon\|_{L^2}^\theta |u_0^\varepsilon|_{\frac{3}{2}+\delta}^{1-\theta} + \varepsilon^{\frac{1}{2}}\|\nabla u_0^\varepsilon\|_{L^2}^\theta |\nabla u_0^\varepsilon|_{\frac{3}{2}+\delta}^{1-\theta} + \varepsilon\|u_1^\varepsilon\|_{L^2}^\theta |u_1^\varepsilon|_{\frac{3}{2}+\delta}^{1-\theta}\Big) \leq R, \tag{1.7}$$

$$\varepsilon^{\frac{1+\delta}{2}-\theta(\frac{3}{4}+\frac{\delta}{2})}\Big(\Big\| \|f^\varepsilon\|_{L^2}^\theta |f^\varepsilon|_{\frac{3}{2}+\delta}^{1-\theta}\Big\|_{L^1(0,+\infty)} + \varepsilon^{\frac{1}{2}}\Big\| \|f^\varepsilon\|_{L^2}^\theta |f^\varepsilon|_{\frac{3}{2}+\delta}^{1-\theta}\Big\|_{L^2(0,+\infty)}\Big) \leq R, \tag{1.8}$$

then there exists a unique global solution $u^\varepsilon \in C^1(\mathbb{R}^+, H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3) \cap C(\mathbb{R}^+, H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3)$ to the System $(HNS)_\varepsilon$.

Remark 1.3.
1. *With respect to our earlier work [2], the improvement in Theorem 1.2 lays in the lower regularity of the initial data. Indeed, the initial data in our previous work [2] belongs to the space $H^4(\mathbb{R}^3)^3 \times H^3(\mathbb{R}^3)^3$. However, in this paper, we do better, we obtained the global existence and uniqueness result for initial data in $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, $\delta > 0$.*
2. *The smallness assumption* (1.7)-(1.8) *on the initial data and forcing terms can be reduced to the choice of a sufficiently small ε.*

The third result in this paper is dedicated to the convergence of solution from the relaxed system to the classical Navier-Stokes equations.

Theorem 1.4. *Let $\delta > 0$, $v_0 \in H^3(\mathbb{R}^3)^3$ be a divergence-free vector field.*
Let $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$ the initial data and $f^\varepsilon \in L^1(\mathbb{R}^+, H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3) \cap L^2(\mathbb{R}^+, H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$ the forcing term verify (1.7), (1.8). *Assume, moreover, that there exist positive constants ε_1, M such that for all $0 < \varepsilon \leq \varepsilon_1$,*

$$\varepsilon^{-1}\|u_0^\varepsilon - v_0\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\nabla(u_0^\varepsilon - v_0)\|_{L^2} + \|u_1^\varepsilon - v_1\|_{L^2} \leq M, \tag{1.9}$$

$$\|f_t^\varepsilon\|_{L^2(\dot{H}^{-1})}^2 + \varepsilon\|f_t^\varepsilon\|_{L^2(L^2)}^2 \leq M, \tag{1.10}$$

where $v_1 := \nu\Delta v_0 - \mathbb{P}(v_0.\nabla)v_0 + \mathbb{P}f^\varepsilon(0,y)$.
Then the global solution u^ε of the system $(HNS)_\varepsilon$ obtained in Theorem 1.2 close to the solution of the incompressible Navier-Stokes equations with v_0 as initial data, when ε close to 0, in the space $L^\infty_{loc}(\mathbb{R}^+, H^1(\mathbb{R}^3)^3)$.
Moreover, for all positive time T, there exists a positive constant $K(T,v)$, depending only on T and v, such that for all $0 \leq t \leq T$

$$\|u^\varepsilon(t) - v(t)\|_{L^2} + \varepsilon^{\frac{1}{2}}\|\nabla(u^\varepsilon(t) - v(t))\|_{L^2} \leq \varepsilon K(T,v).$$

Remark 1.5.
1. *As a consequence of the assumptions in $(u_0^\varepsilon, u_1^\varepsilon)$,* (1.7)- (1.9), *we obtain the smallness condition of initial data v_0, which is a necessary condition to the existence of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$.*
2. *We will not give the proof of Theorem 1.4, because the convergence result is obtained by a simple modification of the proof of Theorem* 1.2 *in [2] page* 222 – 224.

The structure of this paper is as follows: in Section 2, we introduce notation and some preliminary results. The Section 3 is dedicated to the proof of the local existence Theorem 1.1. Finally, in section 4, we prove the global existence of the system $(HNS)_\varepsilon$, namely the proof of Theorem 1.2.

2. Notation and preliminary results

In the beginning of this section, We introduce several function space and recall some estimates.
We designate by $L^p(\mathbb{R}^3)$, $1 \leq p \leq \infty$, the Lebesgue space with norm $\|.\|_{L^p}$. For $p = 2$, we note $(.,.)$ the scalar

product.
Also, we introduce the inhomogeneous Sobolev spaces, for $s \in \mathbb{R}$

$$H^s(\mathbb{R}^3) = \{u \in \mathcal{S}'(\mathbb{R}^3);\ \hat{u} \in L^2_{loc}(\mathbb{R}^3) \text{ and } (1 + |\xi|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^3)\},$$

equipped with the norm

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{u}|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u, given by

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^3} u(x)e^{-ix\xi}dx.$$

Also, for $s \geq 0$, we introduce the operator D_s by

$$D_s u = \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)).$$

We remark that

$$\|u\|_s = \|D_s u\|_{L^2}.$$

We will need also to introduce the homogenous Sobolev spaces, for $s \geq 0$,

$$\dot{H}^s(\mathbb{R}^3) = \{u \in \mathcal{S}'(\mathbb{R}^3);\ \hat{u} \in L^2_{loc}(\mathbb{R}^3) \text{ and } |\xi|^s\widehat{u} \in L^2(\mathbb{R}^3)\},$$

with the seminorm

$$\|u\|^2_{\dot{H}^s} = |u|^2_s = \int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{u}|^2 d\xi.$$

For more details see [1].
In order to perform energy estimates, we need to estimate the product of two vectors. For the reader's convenience, we recall the following elementary laws

Lemma 2.1. 1. *Let $s \geq 1$, there exists a positive constant $C_0 = C_0(s)$, such that, for all $u \in H^{s+1}$ and $v \in H^{s-1}$, we have*

$$\|[D_s, u]v\|_{L^2} \leq C_0(s)\|D_s\nabla u\|_{L^2}\|D_{s-1}v\|_{L^2}. \tag{2.11}$$

2. *For any $s \geq 0$, there exists a positive constant $C_1 = C_1(s)$, such that, for all $f, g \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, the following holds*

$$\|fg\|_s \leq C_1(s)(\|f\|_{L^\infty}\|g\|_s + \|f\|_s\|g\|_{L^\infty}).$$

3. *For any $s \geq 0$, there exists a positive constant $C_2 = C_2(s)$, such that, for all $f, g \in H^{s+1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have*

$$\|f\nabla g\|_s \leq C_2(s)(\|f\|_{L^\infty}\|g\|_{s+1} + \|f\|_{s+1}\|g\|_{L^\infty}).$$

Remark 2.2. 1. *For the proof of item 1), we refer the reader to Lemma 2.4 in [26]. Items 2) and 3) are stated in [3, 8, 18]*

2. *Using Sobolev embedding $H^{\frac{3}{2}+\delta}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we write item 2) as follows*

$$\|fg\|_{\frac{3}{2}+\delta} \leq C\|f\|_{\frac{3}{2}+\delta}\|g\|_{\frac{3}{2}+\delta}, \quad \forall f, g \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3), \tag{2.12}$$

and

$$\|f\nabla g\|_{\frac{3}{2}+\delta} \leq C\|f\|_{\frac{3}{2}+\delta}\|\nabla g\|_{\frac{3}{2}+\delta}, \quad \forall f \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3), g \in H^{\frac{5}{2}+\delta}(\mathbb{R}^3). \tag{2.13}$$

In the study of equation dependent of ε with fixed initial data, it is convenient to perform a change of variables, which fix the equation and transform the ε-dependence into the initial data. For this end, we introduce the diffusive scaling:

$$u^\varepsilon(\tau,y)=\frac{1}{\sqrt{\varepsilon}}u(\frac{\tau}{\varepsilon},\frac{y}{\sqrt{\varepsilon}}),\ \ p^\varepsilon(\tau,y)=\frac{1}{\varepsilon}p(\frac{\tau}{\varepsilon},\frac{y}{\sqrt{\varepsilon}}),\ \ f^\varepsilon(\tau,y)=\frac{1}{\varepsilon\sqrt{\varepsilon}}f(\frac{\tau}{\varepsilon},\frac{y}{\sqrt{\varepsilon}}),$$

and we set $t=\dfrac{\tau}{\varepsilon}, x=\dfrac{y}{\sqrt{\varepsilon}}$.

This scaling transforms the System $(HNS)_\varepsilon$ into the following System of equations with initial data which depend on ε:

$$\begin{cases} u_{tt}-\nu\Delta u+u_t+(u.\nabla)u_t+(u_t.\nabla)u+(u.\nabla)u=-\nabla p+f,\\ \operatorname{div}u=0,\\ (u,u_t)(0,x)=(u_0,u_1)(x)=(\sqrt{\varepsilon}u_0^\varepsilon(\sqrt{\varepsilon}x),\varepsilon^{\frac{3}{2}}u_1^\varepsilon(\sqrt{\varepsilon}x)).\end{cases}\tag{2.14}$$

An elementary calcul shows that, by this scaling, we have the following equality of norms

$$|u|_s=\varepsilon^{\frac{s}{2}-\frac{1}{4}}|u^\varepsilon|_s,\quad |u_t|_s=\varepsilon^{\frac{s}{2}+\frac{3}{4}}|u_t^\varepsilon|_s,\ \forall s\in\mathbb{R},$$

$$\|u\|_{L^p}=\varepsilon^{-\frac{3}{2p}+\frac{1}{2}}\|u^\varepsilon\|_{L^p},\ \forall p>1,$$

and

$$\|f\|_{L^p(\dot H^s)}=\varepsilon^{\frac{3}{4}+\frac{s}{2}-\frac{1}{p}}\|f^\varepsilon\|_{L^p(\dot H^s)},\forall s\in\mathbb{R},\ p\geq 1.$$

We introduce $\mathbb{P}$ the Leray projector over divergence-free vector fields which maps $L^2(\mathbb{R}^3)^3$ into $L^2_\sigma(\mathbb{R}^3)^3=\{f\in L^2(\mathbb{R}^3)^3;\quad \operatorname{div}f=0\}$. Therefore, if we apply $\mathbb{P}$ to (HNS), we obtain the following equation

$$(HNS)\begin{cases} u_{tt}-\nu\Delta u+u_t=-\mathbb{P}(u.\nabla)u-\mathbb{P}(u.\nabla)u_t-\mathbb{P}(u_t.\nabla)u+\mathbb{P}f,\\ \operatorname{div}u=0,\\ (u,u_t)(0,y)=(u_0,u_1)\in H^{\frac{5}{2}+\delta}\times H^{\frac{3}{2}+\delta}.\end{cases}\tag{2.15}$$

As usual, the pressure p may be computed from the velocity field.

We now introduce the concept of weak solution of (HNS)

Definition 2.3. *Let $\delta>0$. A vector filed (u,u_t) in the space $C^0_w([0,T],H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3\times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$ is a weak solution of (HNS) if div $u=0$, $(u,u_t)_{t=0}=(u_0,u_1)$ and for any smooth function φ of compact support and divergence-free, we have*

$$\int_{\mathbb{R}^3}(u_{tt}-\nu\Delta u+u_t)(t,x)\varphi(x)dx\ =\ -\int_{\mathbb{R}^3}\mathbb{P}\Big((u.\nabla)u+(u.\nabla)u_t+(u_t.\nabla)u\Big)(t,x)\varphi(x)dx-\int_{\mathbb{R}^3}\mathbb{P}f(t,x)\varphi(x)dx.$$

3. Local existence in $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3\times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$

In this section, we show local existence and uniqueness of a weak solution to the system (HNS) in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3\times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$. More precisely, we prove Theorem 1.1. For this end, we apply the Friedrichs method, which relies on 3 parts:

- First, we introduce a regularised ordinary differential system $(HNS)_n$ dependent on a parameter n. By using the Cauchy-Lipschitz Theorem to this ordinary differential equation, we prove the existence of unique solution $(u_n,\partial_t u_n)$. The crucial interest of this approximation that we obtain a very regular solution.

- Second, we perform energy estimate on the solutions $(u_n, \partial_t u_n)$ of the regularised system, in order to prove that $(u_n, \partial_t u_n)$ is bounded in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, on $[0,T]$, which the time T is independent on n.

- Finally the conclusion is obtained by passing to the limit, we prove that the solution of the regularised system $(HNS)_n$ converges to a solution $(u, \partial_t u)$ of (HNS).

Note that, the proof is very close to that in Couloud, Hachicha and Raugel [10] in two space dimensions, except in our situation, three space dimensions, where we replace the space by $H^{\frac{5}{2}+\delta} \times H^{\frac{3}{2}+\delta}$. For this reason, we sketch briefly the proof, for more details, we refer the interested reader to section 3 in [10].

3.1. Approximate system

First, we approximate (HNS) by a sequence of ordinary differential equations, which depends on a parameter $n \in \mathbb{N}$:

$$(HNS)_n \begin{cases} \partial_t^2 u_n + \partial_t u_n - J_n \Delta \mathbf{J}_n u_n \\ \quad + \mathbb{P} J_n(J_n u_n.\nabla J_n u_n) + \mathbb{P} J_n(J_n u_n.\nabla J_n \partial_t u_n) + \mathbb{P}(J_n \partial_t u_n.\nabla J_n u_n) = \mathbb{P} J_n f \\ div u_n = 0 \\ (u_n(0), \partial_t u_n(0)) = (J_n u_0, J_n u_1), \end{cases}$$

where J_n is the Fourier cut-off operator defined by

$$J_n(u) = \mathcal{F}^{-1}(\chi_n \widehat{u}),$$

and χ_n is the cut-off function, given by

$$\chi_n(\xi) = \begin{cases} 1, & \text{if} \quad |\xi| \le n, \\ 0, & \text{if} \quad |\xi| > n. \end{cases}$$

The crucial role of this operator is its regularising effect. We have, for all $u \in H^s$,

$$\|J_n u\|_\sigma \le C n^{\sigma - s} \|J_n u\|_s \le C n^{\sigma - s} \|u\|_s, \quad \forall \sigma \ge s,$$

$$C n^{s-\sigma} \|(I - J_n)u\|_\sigma \le \|(I - J_n)u\|_s, \quad \forall \sigma \le s,$$

and

$$J_n(u) \to u \quad \text{strongly in } H^s, \text{ when } n \to \infty.$$

If we consider $v_n = (u_n, \partial_t u_n)$, then the equation $(HNS)_n$ is equivalent to the following first order ordinary differential equation

$$\partial_t v_n = F(t, v_n), \quad v_n(0) = (J_n u_0, J_n u_1). \tag{3.16}$$

Thanks to the continuity properties of J_n, we easily prove that $F : [0, +\infty[\times J_n H^{\frac{5}{2}+\delta} \times J_n H^{\frac{3}{2}+\delta} \to J_n H^{\frac{5}{2}+\delta} \times J_n H^{\frac{3}{2}+\delta}$ is continuous and locally lipschitz in v_n. For more details, we refer the interested reader to [10].
This leads to apply Cauchy-Lipschitz's Theorem, thus we obtain for every $(u_0, u_1) \in H^{\frac{5}{2}+\delta} \times H^{\frac{3}{2}+\delta}$ with a divergence free, there exists a positive maximal time T_n and a unique solution $(u_n, \partial_t u_n) \in C^1([0, T_n], H^{\frac{5}{2}+\delta} \times H^{\frac{3}{2}+\delta})$. We deduce by uniqueness of solution that

$$J_n u_n = u_n, \text{ and } \quad J_n \partial_t u_n = \partial_t u_n.$$

Thus, the solution $(u_n, \partial_t u_n)$ is as regular as wanted. This is the crucial tool in the next steps.

3.2. A priori estimate

Let us consider the energy

$$E_n(t) = \frac{1}{2}(\|u_n + \partial_t u_n\|^2_{\frac{3}{2}+\delta} + \|\partial_t u_n\|^2_{\frac{3}{2}+\delta}) + \|\nabla u_n\|^2_{\frac{3}{2}+\delta},$$

see e.g. [9, 22, 27]. In order to obtain a priori estimate of $(u_n, \partial_t u_n)$ in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, we apply the operator $D_{\frac{3}{2}+\delta}$ to the equation $(HNS)_n$ and we take the L^2-inner product with $D_{\frac{3}{2}+\delta}(u_n + 2\partial_t u_n)$, we obtain

$$\frac{dE_n(t)}{dt} + \|\partial_t u_n\|^2_{\frac{3}{2}+\delta} + \nu\|\nabla u_n\|^2_{\frac{3}{2}+\delta} \quad = \quad -\Big(D_{\frac{3}{2}+\delta}((u_n.\nabla)u_n + (u_n.\nabla)\partial_t u_n + (\partial_t u_n.\nabla)u_n + \mathbb{P}J_n f, D_{\frac{3}{2}+\delta}(u_n + 2\partial_t u_n)\Big).$$

Using several integration by parts, Hölder inequality and estimates (2.12), (2.13), we show that for all $t \in [0, T_n]$

$$\frac{dE_n(t)}{dt} \leq G(t)E_n^2(t), \tag{3.17}$$

where G is a locally integrable function depends only on f.

3.3. Pass to the limit

Thanks to estimate (3.17), we can choose a positive time T, independent of n, such that $(u_n(t), \partial_t u_n(t))$ is uniformly bounded in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, with respect to n, $\forall t \in [0, T]$.
Hence, there exist a subsequence $(u_n, \partial_t u_n) \in L^\infty([0,T], H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$ and $(u, \partial_t u) \in L^\infty([0,T], H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3)$ such that

$$(u_n(t), \partial_t u_n(t)) \rightharpoonup (u(t), \partial_t u(t)) \;\text{ weakly in }\; H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3, \;\text{ for all }\; t \in [0, T].$$

To establish strong convergence between $(u_n, \partial_t u_n)$ and $(u, \partial_t u)$, we use compactness embedding results of the Sobolev spaces and Arzela-Ascoli Theorem.
Consequently, we can pass to the limit, when n goes to 0, in the following equation

$$\begin{aligned}&\int_{\mathbb{R}^3}(\partial_t^2 u_n - \nu\Delta u_n + \partial_t u_n)(t,x)\varphi(x)dx \\ &= -\int_{\mathbb{R}^3}\mathbb{P}\Big((u_n.\nabla)u_n + (u_n.\nabla)\partial_t u_n + (\partial_t u_n.\nabla)u_n\Big)(t,x)\varphi(x)dx - \int_{\mathbb{R}^3}\mathbb{P}J_n f(t,x)\varphi(x)dx.\end{aligned}$$

We deduce that, $(u, \partial_t u)$ is a local weak solution of (HNS).
Finally, we prove the uniqueness and the time continuity of the obtained solution.

4. Global existence in $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$

This section is devoted to proof the global existence for $(HNS)_\varepsilon$ in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$, with a considerably relaxed assumptions on the regularity and with a ε-dependent assumption of initial data. One of the main tools in the proof, we use a specific energy $E(t)$ and an additional related quantity $Y(t)$.

4.1. Energy estimate

Let $E(t)$ the energy functional defined as follows

$$E(t) = \frac{1}{2}(\|u + u_t\|^2_{\frac{3}{2}+\delta} + \|u_t\|^2_{\frac{3}{2}+\delta}) + \nu\|\nabla u\|^2_{\frac{3}{2}+\delta},$$

and we introduce $Y(t)$ by

$$Y(t) = \|u_t\|^2_{\frac{3}{2}+\delta} + \nu\|\nabla u\|^2_{\frac{3}{2}+\delta}.$$

The energy estimate of (u, u_t) in $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$ is given by the following lemma

Lemma 4.1. *There exist positives constants C and K such that, for all $t \in [0,T]$, the following energy estimate holds.*

$$E(t) + \int_0^t \Big(\frac{1}{2} - C(\|u(s)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(s))\Big)Y(s)ds \quad \leq \quad E(0) + \frac{1}{2}\sup_{s\in[0,t]} E(s) + K\Big(\|f\|^2_{L^1([0,t],H^{\frac{3}{2}+\delta})} + \|f\|^2_{L^2([0,t],H^{\frac{3}{2}+\delta})}\Big).$$

Proof. Applying the operator $D_{\frac{3}{2}+\delta}$ to the equation (HNS) and taking the L^2-inner product of the resulting equation with $D_{\frac{3}{2}+\delta}(u+2u_t)$, we find

$$\frac{dE(t)}{dt} + Y(t) = \sum_{i=1}^{4}\Big(D_{\frac{3}{2}+\delta}A_i, D_{\frac{3}{2}+\delta}(u+2u_t)\Big), \tag{4.18}$$

where $A_1 = -\mathbb{P}(u.\nabla)u$, $A_2 = -\mathbb{P}(u.\nabla)u_t$, $A_3 = -\mathbb{P}(u_t.\nabla)u$, and $A_4 = \mathbb{P}f$.
For the first term, if we write $D_{\frac{3}{2}+\delta}(u.\nabla)u = [D_{\frac{3}{2}+\delta}, u]\nabla u + uD_{\frac{3}{2}+\delta}\nabla u$, then since $divu = 0$ we obtain by integration by parts

$$\Big(D_{\frac{3}{2}+\delta}A_1, D_{\frac{3}{2}+\delta}u\Big) = \Big([D_{\frac{3}{2}+\delta}, u]\nabla u, D_{\frac{3}{2}+\delta}u\Big).$$

Thanks to item 1) of Lemma 2.1 and Sobolev embedding, we see that

$$\Big(D_{\frac{3}{2}+\delta}A_1, D_{\frac{3}{2}+\delta}u\Big) \leq c\|u\|_{\frac{3}{2}+\delta}Y(t).$$

Besides, Hölder inequality and estimate (2.13) yields

$$\Big(D_{\frac{3}{2}+\delta}A_1, D_{\frac{3}{2}+\delta}u_t\Big) \leq c\|u\|_{\frac{3}{2}+\delta}\|\nabla u\|_{\frac{3}{2}+\delta}\|u_t\|_{\frac{3}{2}+\delta} \leq c\|u\|_{\frac{3}{2}+\delta}Y(t).$$

Consequently, we have

$$\Big(D_{\frac{3}{2}+\delta}A_1, D_{\frac{3}{2}+\delta}(u+2u_t)\Big) \leq c\|u\|_{\frac{3}{2}+\delta}Y(t), \tag{4.19}$$

For the second term, we remark that $D_{\frac{3}{2}+\delta}(u.\nabla u_t) = D_{\frac{3}{2}+\delta}\nabla(u\otimes u_t)$. Using integrate by part, we obtain

$$(D_{\frac{3}{2}+\delta}A_2, D_{\frac{3}{2}+\delta}u) = \Big(D_{\frac{3}{2}+\delta}(u\otimes u_t), D_{\frac{3}{2}+\delta}\nabla u\Big).$$

Applying Hölder inequality and the inequality (2.12), we get

$$(D_{\frac{3}{2}+\delta}A_2, D_{\frac{3}{2}+\delta}u) \leq c\|u\|_{\frac{3}{2}+\delta}\|u_t\|_{\frac{3}{2}+\delta}\|\nabla u\|_{\frac{3}{2}+\delta} \leq c\|u\|_{\frac{3}{2}+\delta}Y(t). \tag{4.20}$$

Next, we decompose A_2 as follows $-D_{\frac{3}{2}+\delta}A_2 = [D_{\frac{3}{2}+\delta}, u]\nabla u_t + uD_{\frac{3}{2}+\delta}\nabla u_t$. Besides, by integrations by parts and $divu = 0$, we obtain

$$-(D_{\frac{3}{2}+\delta}A_2, D_{\frac{3}{2}+\delta}u_t) = ([D_{\frac{3}{2}+\delta}, u]\nabla u_t, D_{\frac{3}{2}+\delta}u_t)$$

Now, we use Hölder inequality and estimate (2.11), we find that

$$(D_{\frac{3}{2}+\delta}A_2, D_{\frac{3}{2}+\delta}u_t) \leq \|[D_{\frac{3}{2}+\delta}, u]\nabla u_t\|_{L^2}\|u_t\|_{\frac{3}{2}+\delta} \leq c\|\nabla u\|_{\frac{3}{2}+\delta}\|u_t\|^2_{\frac{3}{2}+\delta} \leq cY^{\frac{3}{2}}(t). \tag{4.21}$$

Thus, we infer from the inequalities (4.20) and (4.21), that

$$\Big(D_{\frac{3}{2}+\delta}A_2, D_{\frac{3}{2}+\delta}(u+2u_t)\Big) \leq c(\|u\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t))Y(t) \tag{4.22}$$

Let us now estimate the term A_3. Also, applying Hölder inequality and estimate (2.12), we obtain

$$\begin{aligned}\Big(D_{\frac{3}{2}+\delta}A_3, D_{\frac{3}{2}+\delta}(u+2u_t)\Big) &\leq c\|u_t.\nabla u\|_{\frac{3}{2}+\delta}(\|u\|_{\frac{3}{2}+\delta} + \|u_t\|_{\frac{3}{2}+\delta})\\ &\leq c\|u_t\|_{\frac{3}{2}+\delta}\|\nabla u\|_{\frac{3}{2}+\delta}(\|u\|_{\frac{3}{2}+\delta} + \|u_t\|_{\frac{3}{2}+\delta})\\ &\leq c(\|u\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t))Y(t).\end{aligned} \tag{4.23}$$

For the last term, we deduce from the Hölder inequality and the Young inequality that

$$\begin{aligned}\Big(D_{\frac{3}{2}+\delta}A_4, D_{\frac{3}{2}+\delta}(u+2u_t)\Big) &= \Big(D_{\frac{3}{2}+\delta}f, D_{\frac{3}{2}+\delta}(u+u_t)\Big) + \Big(D_{\frac{3}{2}+\delta}f, D_{\frac{3}{2}+\delta}u_t\Big) \\ &\leq \|f\|_{\frac{3}{2}+\delta}E(t)^{\frac{1}{2}} + c\|f\|^2_{\frac{3}{2}+\delta} + \frac{1}{2}Y(t). \qquad (4.24)\end{aligned}$$

By summing up the above estimates (4.19)-(4.24), we get

$$\frac{d}{dt}E(t) + \Big(\frac{1}{2} - C\Big(\|u\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)\Big)\Big)Y(t) \leq \|f\|_{\frac{3}{2}+\delta}E(t)^{\frac{1}{2}} + c\|f\|^2_{\frac{3}{2}+\delta}.$$

Integrating the above inequality between 0 and t, we obtain

$$E(t) + \int_0^t \Big(\frac{1}{2} - C(\|u(s)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(s))\Big)Y(s)ds \leq E(0) + c\sup_{s\in[0,t]} E^{\frac{1}{2}}(s)\int_0^t \|f\|_{\frac{3}{2}+\delta} + c\|f\|^2_{L^2([0,t],H^{\frac{3}{2}+\delta})}.$$

Using Young inequality we obtain the desired energy estimate. This concludes the proof of Lemma 4.1. □

4.2. *Proof of Theorem 1.2*

Let T_{max} be the maximal time of existence of (u, u_t) in the space $H^{\frac{5}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3$. We will prove that, if we assume T_{max} is finite and the initial data and forcing small enough, then $\|u(t)\|_{\frac{5}{2}+\delta} + \|u_t(t)\|_{\frac{3}{2}+\delta}$ is uniformly bounded on the time, which contradicts the fact that T_{max} is finite.
For this end, we claim first that $\|u(t)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)$ is uniformly bounded on the time. First, we remark that

$$C(\|u(t)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)) \leq K(\nu)E^{\frac{1}{2}}(t). \qquad (4.25)$$

Thus, if we choose $K(\nu)E^{\frac{1}{2}}(0) \leq \frac{1}{4}$, and due to continuity of (u, u_t), we allow that there exists $T > 0$ such that for all $0 \leq t \leq T$

$$K(\nu)E^{\frac{1}{2}}(t) \leq \frac{1}{2},$$

and

$$C(\|u(t)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)) \leq \frac{1}{2}. \qquad (4.26)$$

Now, we introduce $\tilde{T}$ the maximal time such estimate (4.26) hold:

$$\tilde{T} = \sup\Big\{t,\ C(\|u(t)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)) \leq \frac{1}{2}\Big\}. \qquad (4.27)$$

If we assume that $\tilde{T} < T_{max}$, then by Lemma 4.1, we obtain

$$\sup_{t\in[0,\tilde{T}]} E(t) \leq 2E(0) + 2C_0(\|f\|^2_{L^1(H^{\frac{3}{2}+\delta})} + \|f\|^2_{L^2(H^{\frac{3}{2}+\delta})}). \qquad (4.28)$$

If we assume the initial data (u_0, u_1) and the forcing term f satisfy

$$\Big(E(0) + C_0(\|f\|^2_{L^1(H^{\frac{3}{2}+\delta})} + \|f\|^2_{L^2(H^{\frac{3}{2}+\delta})})\Big)^{\frac{1}{2}} \leq \frac{1}{4K(\nu)},$$

then by estimate (4.25), we find that for all $0 \leq t \leq \tilde{T}$

$$C(\|u(t)\|_{\frac{3}{2}+\delta} + Y^{\frac{1}{2}}(t)) < \frac{1}{2}.$$

we have a contradiction from (4.27).
We conclude that $\tilde{T} \geq T_{max}$ and $\|u(t)\|_{\frac{5}{2}+\delta} + \|u_t(t)\|_{\frac{3}{2}+\delta}$ is uniformly bounded on the interval time $[0, T_{max})$, which contradict the fact that T_{max} is finite. This finishes the proof of Theorem 1.2.

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