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Semilinear periodic parabolic problem with discontinuous coefficients: Mathematical analysis and numerical simulation

Hamza Alaa[a], Malika El Ghabi[a], Abderrahim Charkaoui[a,b]

[a] *Laboratory LAMAI, Faculty of Science and Technology, University Cadi Ayyad, B.P. 549, Street Abdelkarim Elkhattabi, Marrakesh-40000, Morocco*

[b] *Higher School of Education and Training Berrechid (ESEFB), Hassan First University, BP 539, Route de Casablanca Km 3.5, 26100, Berrechid, Morocco*

Abstract. We develop a new technique to mathematically analyze and numerically simulate the weak periodic solution to a class of semilinear periodic parabolic equations with discontinuous coefficients. We reformulate our problem into a minimization problem via a least-squares cost function. By using variational calculus theory, we establish the existence of an optimal solution and based on the Lagrangian method, we calculate the derivative of our cost function. To illustrate the validity and efficiency of our proposed method, we present some numerical examples with different periods of time and diverse choices of discontinuous coefficients.

1. Statement of the problem

In these last decades, a huge interest has been given to the studies of partial differential equations not only for the linear case but also they are involving nonlinear terms. Diverse methods have been investigated to examine the existence, uniqueness, regularity and stability of the considered solution of such problems. We mainly refer the interesting readers to see the works [1, 2, 4, 12, 13, 16, 20, 24].

The purpose of this work is to study a semilinear periodic parabolic equation with discontinuous coefficients whose model is

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x)\nabla u) + g(t,x,u) = f(t,x) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \Sigma_T, \end{cases} \tag{1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, f is a measurable function, periodic in time with period T and belonging to certain Lebesgue space, $A(t,x) = \left(a_{ij}(t,x)\right)_{1\le i,j\le N}$ is a periodic bounded matrix, and $g : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function periodic with respect to T and satisfying some assumptions.

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Email addresses: hamza.alaa@ced.uca.ma (Hamza Alaa), malika.elghabi@ced.uca.ma (Malika El Ghabi), abderrahim.charkaoui@edu.uca.ma (Abderrahim Charkaoui)

To enrich the discussion, we refer the readers to see the book by Hess [19] for a comprehensive introduction to periodic parabolic equations with regular data. In [3] Amann was also studied periodic parabolic equations with regular data. He proved the existence of classical periodic solutions by the mean of the sub and super solution technique. In [21] Lions established the existence and uniqueness of weak solutions to a class of periodic parabolic equations with Leray-Lions type operators. Their proof was based on the theory of maximal monotone operators in Banach spaces. The work [14] by Deuel and Hess was interested in the quasilinear case with an additional lower order term. The authors showed the existence and established the regularity property of weak periodic solutions via the techniques of sub- and super-solutions. Recently, Alaa et al generalized the work [14] by considering the same equation but with L^1 data. Based on the truncation method, they proved the existence of a weak periodic solution which is called SOLA solution (Solution Obtained as the Limit of Approximation). Let us mention that all the above cited works are dedicated to present a theoretical analysis of periodic partial differential equations. However, the consideration of numerical simulations of periodic parabolic problems is by far more limited in the literature. In the following, we propose to refer the readers to some interesting works on the same subject. We start with the work of Carasso [6] in which the author was based on the least squares approach to numerically simulate some periodic solutions. Lust et al [23] proposed an iterative algorithm to construct numerically the periodic solutions to an ordinary differential system when the period is unknown. In [26] another approach was investigated. The authors started by formulating a nonlinear heat conduction periodic problem to an evolutionary equation in a suitable Banach space. They proved the existence of a periodic solution by using semigroup theory and fixed point theorems. Thereafter, they based on Newton's method to present some numerical simulations of the periodic solutions of such problems. Let us remark that all the early mentioned works are focused on the numerical simulations of periodic parabolic equations with continuous coefficients.

In this work, we build a new iterative method to numerically construct the periodic solution of (1). We start by formulating the periodic problem (1) into an optimization problem via the introduction of a least squares criterion. Therefore, we establish the existence of an optimal solution to the optimization problem which leads to affirm that the considered problem is well-posed. Thereafter, we follow Lagrange's method to calculate the gradient of the considered cost function through the introduction of an auxiliary problem called the adjoint equation. Based on the derivative of the cost function, we develop an iterative algorithm to solve numerically the considered optimization problem.

We have organized the rest of this paper as follows: In Section 2, we state the necessary hypothesis and we define the notion of a weak periodic solution to (1). Section 3 is devoted to formulating questions about the existence of weak periodic solution of (1) into an equivalent minimization problem. We will introduce a least-squares cost function and therefore we show the existence of an optimal solution to the considered minimization problem. Thereafter, we compute the gradient of the cost function via the Lagrange method. We reserve Section 4 to discretize our problem into a finite element problem. Based on the derivative of the cost function, we will present a numerical algorithm to solve numerically the minimization problem. In Section 5, we validate our theoretical study by making some numerical examples.

2. Mathematical Background and Definitions

Let us start this section by imposing the necessary hypothesis to solve (1).

2.1. Assumptions

Throughout this work, we assume that $A = \left(a_{ij}\right)_{1\leq i,j\leq N}$ is an elliptic bounded matrix periodic with respect to the time, namely

($\mathcal{A}_1$) $\left(a_{ij}\right)_{1\leq i,j\leq N}$ are measurable functions periodic and belonging to $L^\infty(Q_T)^N$

($\mathcal{A}_2$) there exists $\alpha > 0$ such that

$$A(t,x)\xi\cdot\xi \geq \alpha|\xi|^2, \text{ for a.e } (t,x)\in Q_T \text{ and for all } \xi\in\mathbb{R}^N. \tag{2}$$

($\mathcal{A}_3$) f is a measurable function periodic and belonging to $L^2(Q_T)$.

($\mathcal{A}_4$) $g : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function periodic and satisfying

$$s \mapsto g(t,x,s) \text{ is an increasing function for a.e } (t,x) \in Q_T, \tag{3}$$

$$g(t,x,s)s \geq 0 \text{ for a.e } (t,x) \in Q_T \text{ and for all } s \in \mathbb{R}. \tag{4}$$

($\mathcal{A}_5$) There exists a nonnegative function H belonging to $L^2(Q_T)$ and a nonnegative constant β such that

$$|g(t,x,s)| \leq H(t,x) + \beta|s| \text{ for a.e } (t,x) \in Q_T \text{ and for all } s \in \mathbb{R}. \tag{5}$$

($\mathcal{A}_6$) $s \mapsto g(t,x,s)$ is differentiable such that $\partial_s g(t,x,s)$ belongs to $L^\infty(Q_T)$.

2.2. Functional framework and definition

For the reader's convenience, we will use the following notation

$$\mathbb{V} = H_0^1(\Omega), \quad \mathbb{H} = L^2(\Omega), \quad \mathbb{V}^* = H^{-1}(\Omega).$$

To introduce the functional framework involving our work, we set

$$\mathcal{X}_T := L^2(0,T;\mathbb{V}),$$

we equipped with the following norm

$$\|u\|_{\mathcal{X}_T} := \left(\int_{Q_T} |\nabla u|^2 \, dx \, dt\right)^{\frac{1}{2}}.$$

Furthermore, we set

$$\mathcal{X}_T^* := L^2(0,T;\mathbb{V}^*),$$

the dual space of $\mathcal{X}_T$. The above spaces lead to define the following functional space

$$\mathcal{W}_T := \{u \in \mathcal{X}_T, \quad \partial_t u \in \mathcal{X}_T^*\},$$

we equipped with the following norm

$$\|u\|_{\mathcal{W}_T} := \|u\|_{\mathcal{X}_T} + \|\partial_t u\|_{\mathcal{X}_T^*}.$$

Throughout this paper, we will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathbb{V}^*$ and $\mathbb{V}$ and we introduce the notion of weak periodic solution to clarify in which sense we want to solve problem (1).

Definition 2.1. *We call weak periodic solution to* (1) *all measurable function* $u : Q_T \to \mathbb{R}$ *that satisfies*

$$\begin{gathered} u \in \mathcal{W}_T, \quad u(0,x) = u(T,x) \text{ in } \mathbb{H}, \\ \int_0^T \langle \partial_t u, \varphi \rangle \, dt + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi \, dx \, dt + \int_{Q_T} g(t,x,u)\varphi \, dx \, dt = \int_{Q_T} f\varphi \, dx \, dt, \end{gathered} \tag{6}$$

for every test function $\varphi \in \mathcal{X}_T$.

Remark 2.2. *From assumptions ($\mathcal{A}_1$)-($\mathcal{A}_5$), we verify easily that all the terms of* (6) *are well defined. Moreover, by using the continuous embedding* $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}^*$, *we know that*

$$\mathcal{W}_T \hookrightarrow C([0,T];\mathbb{H}),$$

which means that the periodic condition makes a sense in Definition 2.1.

Let us mention that the existence and uniqueness of a weak periodic solution to problem (1) can be achieved via monotone operators theory see Theorem 1.1 p.316 of [21]. In this work, we consider a new approach based on the minimization of a cost function which helps us to simulate numerically the periodic solution of (1).

3. Optimization problem

In this section, we formulate the problem of the existence of a weak periodic solution to (1) into a minimization problem. First of all, we introduce the following least-squares cost function

$$\mathcal{J}(v) = \frac{1}{2} \int_{\Omega} (u(T,x) - v(x))^2 \, dx, \tag{7}$$

where u presents the weak solution to the following initial boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x)\nabla u) + g(t,x,u) = f(t,x) & \text{in } Q_T \\ u(0,x) = v(x) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \Sigma_T. \end{cases} \tag{8}$$

Furthermore, from the result of Theorem 3.1 p. 282 in [25] we assure that for any $v \in \mathbb{H}$, problem (8) has a unique weak solution u which satisfying the following variational formulation

$$\begin{gathered} u \in \mathcal{W}_T, \quad u(0,x) = v(x) \text{ in } \mathbb{H}, \\ \int_0^T \langle \partial_t u, \varphi \rangle \, dt + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi \, dx\, dt + \int_{Q_T} g(t,x,u)\varphi \, dx\, dt = \int_{Q_T} f\varphi \, dx\, dt, \end{gathered} \tag{9}$$

for all $\varphi \in \mathcal{X}_T$. As a result, we conclude that the cost function $\mathcal{J}$ is well-defined for all $v \in \mathbb{H}$. Thus, we are in the setting to introduce the following minimization problem

$$\begin{cases} \text{Find } v^* \in \mathcal{N}_{ad} \\ \mathcal{J}(v^*) = \min\limits_{v \in \mathcal{N}_{ad}} \mathcal{J}(v), \end{cases} \tag{10}$$

where $\mathcal{N}_{ad}$ designates the set of admissible functions which will be constructed in the next section. A further interesting result is that the corresponding minimum of the cost function $\mathcal{J}$ coincides with u the weak periodic solution of (1). On the other hand, when the cost function $\mathcal{J}$ goes to zero the solution u of the initial problem (8) satisfies the periodic condition $u(0,\cdot) = u(T,\cdot)$ in Ω. Which leads to deduce that we have reached the equivalence between the periodic parabolic problem (1) and the optimization problem (10).

3.1. Existence of an optimal solution

In this section, we aim to establish that the minimization problem (10) has at least one optimal solution in $\mathcal{N}_{ad}$. We start by introducing the set

$$\mathcal{N}_{ad} := \{v \in H^1(\Omega), \ \|v\|_{H^1(\Omega)} \leq C\}, \tag{11}$$

where C is a nonnegative constant. We consider on $\mathcal{N}_{ad}$ the topology defined by the strong convergence in $\mathbb{H}$. We would like to mention that the choice of $\mathbb{H}$ can seem to be consistent for $\mathcal{N}_{ad}$. But for our case, this choice does not guarantee the existence of an optimal solution to the minimization problem (10). Among the obtained advantages through the consideration of (11), we find a very interesting compactness result. More specifically, the Rellich-Kondrachov injection [5] allows us to ensure that

$$\mathcal{N}_{ad} \overset{\text{compact}}{\hookrightarrow} \mathbb{H}.$$

Consequently the existence of an optimal solution to (10) requires only the lower semi-continuity of the cost function $\mathcal{J}$ on $\mathbb{H}$.

Theorem 3.1. *Assume that* ($\mathcal{A}_1$)-($\mathcal{A}_5$) *hold true. Then, there exists at least one solution in* $\mathcal{N}_{ad}$ *to the optimization problem* (10).

Proof. From the previous discussion, we found that the existence of an optimal solution to (10) requires the lower semi-continuity of $\mathcal{J}$ on $\mathcal{N}_{ad}$. To do this, we consider (v_n) a sequence in $\mathbb{H}$ such that (v_n) converges to v strongly in $\mathbb{H}$. We recall that

$$\mathcal{J}(v_n) = \frac{1}{2}\int_\Omega (u_n(T,x) - v_n(x))^2\,dx, \tag{12}$$

where u_n is the unique weak solution to the following initial boundary value problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t,x)\nabla u_n) + g(t,x,u_n) = f & \text{in } Q_T \\ u_n(0,\cdot) = v_n(\cdot) & \text{in } \Omega \\ u_n = 0 & \text{on } \Sigma_T. \end{cases} \tag{13}$$

We multiply the first equation of (13) by u_n and we integrate over Q_T, one gets

$$\frac{1}{2}\int_\Omega |u_n(T)|^2\,dx + \int_{Q_T} A(t,x)\nabla u_n.\nabla u_n\,dx\,dt + \int_{Q_T} g(t,x,u_n)u_n\,dx\,dt = \int_{Q_T} f u_n\,dx\,dt + \frac{1}{2}\int_\Omega |v_n|^2\,dx. \tag{14}$$

Thanks to (2),(4) and by applying Hölder's inequality, the relation (14) becomes

$$\alpha\int_{Q_T} |\nabla u_n|^2\,dx\,dt + \int_{Q_T} g(t,x,u_n)u_n\,dx\,dt \le \|f\|_{L^2(Q_T)}\|u_n\|_{L^2(Q_T)} + \|v_n\|^2_{\mathbb{H}}. \tag{15}$$

By using the strong convergence of (v_n) in $\mathbb{H}$, we derive that (v_n) is bounded in $\mathbb{H}$. Furthermore, by using Young's inequality in the right-hand side of (15) and with the helps of (4), we conclude that (u_n) is bounded in $\mathcal{X}_T$. From the growth assumption (5), one may get the existence of a nonnegative constant C independent of n such that

$$\int_{Q_T} \left|g(t,x,u_n)\right|^2\,dx\,dt \le C\left(\|H\|^2_{L^2(Q_T)} + \beta^2\|u_n\|^2_{L^2(Q_T)}\right). \tag{16}$$

Thanks to the continuous embedding $\mathcal{X}_T \hookrightarrow L^2(Q_T)$ and by employing the boundness of (u_n) in $L^2(Q_T)$, we can derive from (16) that $(g(t,x,u_n))$ is bounded in $L^2(Q_T)$. On the other hand, by using the equation satisfied by (u_n), we deduce that $(\partial_t u_n)$ is bounded $\mathcal{X}_T^*$. From Rellich-Kondrachov injection [5], we know that

$$\mathbb{V} \overset{\text{compact}}{\hookrightarrow} \mathbb{H} \hookrightarrow \mathbb{V}^*.$$

Then, by applying Aubin's compactness result (see e.g [21]), we deduce the existence of a measurable function $u \in \mathcal{X}_T$ and a subsequence of (u_n) still denoted by (u_n) for simplicity such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

As a result, we obtain that $g(t,x,u_n)$ converge to $g(t,x,u)$ almost everywhere in Q_T. Then, the above results yield the following convergences

$$\begin{aligned} v_n &\to v \text{ strongly in } \mathbb{H} \text{ and a.e. in } Q_T, \\ u_n &\to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \nabla u_n &\rightharpoonup \nabla u \text{ weakly in } L^2(Q_T)^N, \\ \partial_t u_n &\rightharpoonup \partial_t u \text{ weakly in } \mathcal{X}_T^*, \\ g(t,x,u_n) &\to g(t,x,u) \text{ strongly in } L^2(Q_T). \end{aligned}$$

The latter convergence result is obtained via the application of Lebesgue dominated theorem. By using these convergences, we pass to the limit as $n \to +\infty$ in the weak formulation of (13), we get

$$u \in \mathcal{W}_T, \quad u(0,x) = v(x) \text{ in } \mathbb{H},$$
$$\int_0^T \langle \partial_t u, \varphi \rangle \, dt + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi \, dx\, dt + \int_{Q_T} g(t,x,u)\varphi \, dx\, dt = \int_{Q_T} f\varphi \, dx\, dt, \tag{17}$$

for all $\varphi \in \mathcal{X}_T$. Which is equivalent to say that the limit u is a weak solution to the problem (17). On the other hand, by using the fact that (17) has a unique solution, one may deduce that

$$\lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}(v).$$

Which implies that $\mathcal{J}$ is continuous on $\mathbb{H}$ and therefore lower semi-continuous on $\mathbb{H}$. Furthermore, a direct application of the theory of variations calculus [17] permits us to achieve the existence of an optimal solution to (10). □

3.2. *Derivative of the cost function*

Since we are looking to solve numerically the optimization problem (10), we propose to use a numerical method based on the gradient of the cost function $\mathcal{J}$. Hence, this section tackles the computation of the derivative of $\mathcal{J}$ with respect to the state variable. We will follow Lagrangian's approach which guarantees a rapid derivative of $\mathcal{J}$. The following theorem sum up the main result of this section.

Theorem 3.2. *We assume that ($\mathcal{A}_1$)-($\mathcal{A}_6$) holds. Then, the cost function $\mathcal{J}$ is differentiable for all $v \in \mathbb{H}$. Furthermore, we have*

$$\mathcal{J}'(v).\eta = \int_\Omega (v - u(T) - p(0))\, \eta \, dx, \tag{18}$$

for all $\eta \in \mathbb{H}$. Where u is the solution to the state equation (8) *and p is the solution to the following adjoint equation*

$$\begin{cases} \partial_t p + \operatorname{div}(A^*(t,x)\nabla p) = \partial_s g(t,x,u) & \text{in } Q_T \\ p(T) = v - u(T) & \text{in } \Omega \\ p = 0 & \text{in } \Sigma_T, \end{cases} \tag{19}$$

with A^* is the transpose matrix of A.

Proof. First of all, we define the Lagrangian $\mathcal{L}$ as follows

$$\begin{aligned} \mathcal{L}(u,p,v,\sigma) :=& \frac{1}{2}\int_\Omega (u(T)-v)^2\, dx + \int_0^T \langle \partial_t u, p\rangle \, dt + \int_{Q_T} A(t,x)\nabla u.\nabla p \, dx\, dt + \int_{Q_T} g(t,x,u)p\, dx\, dt \\ & - \int_{Q_T} fp\, dx\, dt + \int_\Omega \sigma(u(0)-v)\, dx. \end{aligned}$$

for all $(u,p,v,\sigma) \in \mathcal{W}_T \times \mathcal{W}_T \times \mathbb{H} \times \mathbb{H}$. At this stage, the variables u, p and v are independents. We mention that the main objective of adding the auxiliary function σ is to recover the initial boundary condition of the adjoint equation and its value will be discussed later. We start the computation by deriving the Lagrangian $\mathcal{L}$ with respect to u. For all $\varphi \in \mathcal{W}_T$, we obtain

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle =& \int_\Omega \varphi(T)(u(T)-v)\, dx + \int_0^T \langle \partial_t\varphi, p\rangle \, dt + \int_{Q_T} A(t,x)\nabla\varphi.\nabla p\, dx\, dt \\ & + \int_{Q_T} \partial_s g(t,x,u)\varphi p\, dx\, dt + \int_\Omega \sigma\varphi(0)\, dx. \end{aligned} \tag{20}$$

Through integration by part, one gets

$$\int_0^T \langle \partial_t \varphi, p\rangle \, dt = -\int_0^T \langle \partial_t p, \varphi\rangle \, dt + \int_\Omega (p(T)\varphi(T) - p(0)\varphi(0)) \, dx, \tag{21}$$

$$\int_{Q_T} A(t,x)\nabla\varphi.\nabla p \, dx \, dt = -\int_0^T \langle div(A^*(t,x)\nabla p, \varphi\rangle \, dt, \tag{22}$$

where A^* is the transpose matrix of A. From (21) and (22) equation (20) becomes

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle = &\int_\Omega \varphi(T)(u(T)-v)\, dx - \int_0^T \langle \partial_t p, \varphi\rangle \, dt + \int_\Omega (p(T)\varphi(T) - p(0)\varphi(0)) \, dx - \int_0^T \langle div(A^*(t,x)\nabla p, \varphi\rangle \, dt \\ &+ \int_{Q_T} \partial_s g(t,x,u)\varphi p \, dx \, dt + \int_\Omega \sigma\varphi(0)\, dx,\end{aligned} \tag{23}$$

To deduce the expression of the adjoint equation, we look for the equation satisfied by $p \in \mathcal{W}_T$ for which the equality (23) vanishes for all $\varphi \in \mathcal{W}_T$. From (23), it comes that p satisfies in Q_T the following equation

$$\partial_t p + \text{div}\,(A^*(t,x)\nabla p) = \partial_s g(t,x,u) \text{ in } Q_T. \tag{24}$$

To obtain the initial condition p in Ω, we fix $\sigma = p(0)$ in (23), we conclude that

$$p(T) = v - u(T) \text{ in } \Omega. \tag{25}$$

According to(24) and (25), we deduce that p satisfies the following adjoint equation

$$\begin{cases} \partial_t p + \text{div}(A^*(t,x)\nabla p) = \partial_s g(t,x,u) & \text{in } Q_T \\ p(T) = v - u(T) & \text{in } \Omega \\ p = 0 & \text{in } \Sigma_T. \end{cases}$$

Let $\eta \in \mathbb{H}$, we derive the Lagrangian $\mathcal{L}$ with respect to v, we obtain

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}, \eta \right\rangle = -\int_\Omega (u(T)-v)\eta \, dx - \int_\Omega p(0)\eta \, dx = \int_\Omega (v - u(T) - p(0))\, \eta \, dx.$$

As we can see, to deduce the expression of the derivative of $\mathcal{J}$, we will take u as the solution to the state equation (9). Thus, we get

$$\mathcal{L}(u,p,v,\sigma) = \mathcal{J}(v).$$

We therefore have

$$\mathcal{J}'(v).\eta = \int_\Omega (v - u(T) - p(0))\, \eta \, dx,$$

where $p(0)$ is the solution to the adjoint equation (19) computed at the initial period $t = 0$ and $u(T)$ is the solution to the state equation (8) evaluated at the final period T. □

4. The finite element approximation

All over this section we consider that Ω is a bounded convex N-polyhedron, that is a bounded interval if $N = 1$, a convex polygon if $N = 2$ and a convex polyhedron if $N = 3$. For $h > 0$, we consider $\mathcal{T}_h$ a regular triangulation of Ω, $\mathcal{T}_h$ covers Ω exactly. We have the P_1 finite element space as shown below:

$$\mathcal{V}_h = \left\{ v_h \in C^0(\overline{\Omega}), \;\; v_h \text{ is affine on every N-simplex of } \mathcal{T}_h \right\}.$$

$\mathcal{V}_h$ is a finite dimensional subspace of $\mathcal{V} = H^1(\Omega)$. The finite element approximation of problem (10) reads:

$$\begin{cases} \text{Find } v_h^* \in \mathcal{N}_{ad}^h \\ \mathcal{J}_h(v_h^*) = \min\limits_{v_h \in \mathcal{N}_{ad}^h} \mathcal{J}_h(v_h), \end{cases} \tag{26}$$

where $\mathcal{N}_{ad}^h := \{v_h \in \mathcal{V}_h,\ \|v_h\|_{H^1(\Omega)} \leq C\}$ is the set of admissible functions,
and

$$\mathcal{J}_h(v_h) = \frac{1}{2}\int_\Omega (u_h(T,x) - v_h(x))^2\, dx, \tag{27}$$

with u_h is the solution to the following initial problem

$$\begin{cases} u_h(0,x) = v_h(x) \quad a.e.\, x \in \Omega \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} u_h(t,x)\phi_h(x)\, dx + \int_{\mathcal{T}_h} A(t,x)\nabla u_h(t,x).\nabla\phi_h(x)\, dx + \int_{\mathcal{T}_h} g(t,x,u)\phi_h(x) dx = \int_{\mathcal{T}_h} f(t,x)\phi_h(x)\, dx \\ \forall t \in]0,T[, \forall \phi_h \in \mathcal{V}_h. \end{cases} \tag{28}$$

According to the previous paragraph, the expression of the differential of $\mathcal{J}_h$ is given by:

$$\mathcal{J}_h'(v_h)(x) = v_h(x) - p_h(0,x) - u_h(T,x), \tag{29}$$

where p_h is a solution of the adjoint model:

$$\begin{cases} p_h(T,x) = v_h(x) - u_h(T,x) \quad a.e.\, x \in \Omega \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} p_h(t,x)\phi_h(x)\, dx - \int_{\mathcal{T}_h} A^*(t,x)\nabla p_h(t,x).\nabla\phi_h(x)\, dx = \int_{\mathcal{T}_h} \partial_s g(t,x,u)\phi_h(x)\, dx \\ \forall t \in]0,T[, \forall \phi_h \in \mathcal{V}_h, \end{cases} \tag{30}$$

where u_h is the solution of (28).

5. Numerical simulations

We have also performed numerical simulations with the software FreeFem ++ ([18]) in two spatial dimensions. For a bounded domain Ω of $\mathbb{R}^2$ with smooth boundary and fix $\mu > 0$ a step of descent. Our algorithm reads as follows, (see Algorithm 1). We use an implicit method in time to solve the equation (28). In the same way, we used an implicit method in time for the resolution of the linear retrograde adjoint equation (30).

Algorithm 1

Input: a mesh $\mathcal{T}_h$ which gives a triangulation of Ω_h (a polygonal approximation of Ω) and an initial estimate $u_0^0 \in \mathcal{V}_h$ (for example a constant C_0). Compute $\mathcal{J}_h^0 = \mathcal{J}_h(u_0^0)$

For $k = 0, ..., k_{max} - 1$;

Solve the state equation

$$\begin{cases} u_h^k(0,x) = u_0^k(x) \quad a.e.\, x \in \Omega, \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} u_h^k(t,x)\phi_h(x)\,dx + \int_{\mathcal{T}_h} A(t,x)\nabla u_h^k(t,x).\nabla\phi_h(x)\,dx + \int_{\mathcal{T}_h} g(t,x,u)\phi_h(x)\,dx = \int_{\mathcal{T}_h} f(t,x)\phi_h(x)\,dx \\ \forall t \in]0,T[, \forall \phi_h \in \mathcal{V}_h. \end{cases} \tag{31}$$

Compute the value of $u_h^k(T,x)$;

Solve the adjoint equation

$$\begin{cases} p_h^k(T,x) = u_0^k(x) - u_h^k(T,x) \quad a.e.\, x \in \Omega, \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} p_h^k(t,x)\phi_h(x)\,dx - \int_{\mathcal{T}_h} A^*(t,x)\nabla p_h^k(t,x).\nabla\phi_h(x)\,dx = \int_{\mathcal{T}_h} \partial_s g(t,x,u)(t,x)\phi_h(x)\,dx \\ \forall t \in]0,T[, \forall \phi_h \in \mathcal{V}_h. \end{cases} \tag{32}$$

Update the new initial function u_0^{k+1} and a new value of $\mathcal{J}_h$ by computing

$$u_0^{k+1}(x) = (1-\mu)u_0^k(x) + \mu(p_h^k(0,x) + p_h^k(T,x)).$$

$$\mathcal{J}_h^{k+1} = \mathcal{J}_h(u_0^{k+1}).$$

Output: $u_h^{k_{max}}, \mathcal{J}_h^{k_{max}}$.

5.1. A numerical simulation

In view to reinforce our method, we computed the numerical solution reached in the coming two examples:

5.1.1. Example: A radial test case with regular coefficients

We consider the problem (1) on the unit disc Ω of $\mathbb{R}^2$ with:

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\},$$

and

$$A(x,y) = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $r = \sqrt{x^2+y^2}$ and

$$u(x,y) = 1 - r^2, \quad f(t,x,y) = 2\frac{2+r^2}{(1+r^2)^{\frac{3}{2}}}, \quad g(t,x,u) = 0.$$

Then u is the exact solution of (1) with $T = 1$.

Table 1: L^2 error and mesh characteristics for Example 5.1.1

Nb vertices	3633	14003	31564
$h_{\min}$	0.025	0.012	0.008
$h_{\max}$	0.053	0.032	0.019
L^2 error	0.09032	0.09035	0.09037
$\mathcal{J}_h$	8.719.e − 05	8.719.e − 05	8.718.e − 05

The Table 1, gives the number of vertices in the mesh $\mathcal{T}_h$ as well as the minimum and maximum length of the edges of the used triangulation. we present also
L^2 error $= \left\| u - u_h^{k_{max}} \right\|_{L^2(\Omega_h)}$ and $\mathcal{J}_h = \mathcal{J}_h^{k_{max}}$ obtained for $k_{max} = 100$ and different value of the mesh size h . The initial estimate is taken $u_h^0 = 1$. The solutions corresponding to the initial $u_h^{kmax}(0, \cdot)$ and the final time $u_h^{kmax}(T, \cdot)$ are shown respectively in Figure (1) and Figure (2).
Figure 3 shows objective function $\mathcal{J}_h$ value decreases along with the increase of the iteration number.

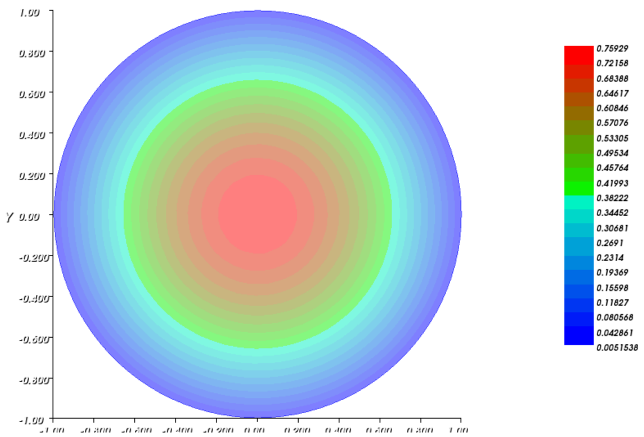

Figure 1: Output initial $u_h^{kmax}(0, \cdot)$.

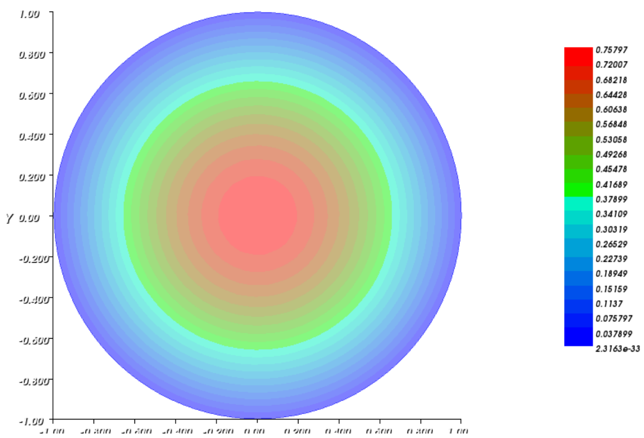

Figure 2: Output initial $u_h^{kmax}(T, \cdot)$.

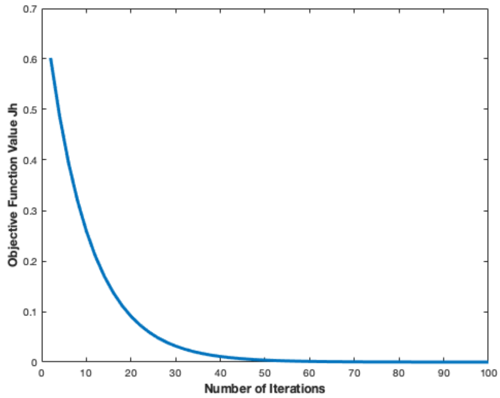

Figure 3: The decrease of the objective function $\mathcal{J}_h$ according to the number of iterations.

5.1.2. Example: Numerical simulation with a discontinuous matrix

To illustrate our method in the case of a discontinuous elliptic matrix, we computed the numerical solution obtained on the unit disc Ω in $\mathbb{R}^2$ for the values

$$A(x,y) = \begin{pmatrix} \sqrt{(1+x^2+y^2)} & 0 \\ 0 & \sqrt{(1+x^2+y^2)} \end{pmatrix}$$

by taking

$$u(t,x) = \cos(\pi t)\left(1 - x^2 - y^2\right), \quad g(t,x,u) = \mathrm{Arctan}(u), \quad T = 2$$

$$f(t,x,y) = \left(1 + x^2 + y^2\right)\sin(\pi t)$$

Table 2: L^2-error and mesh characteristics for Example 5.1.2

Nb vertices	162	601	934
$h_{\min}$	0.295212	0.131497	0.110537
$h_{\max}$	0.48218	0.250919	0.219081
L^2 error	9.22117×10^{-3}	9.26387×10^{-3}	9.29288×10^{-3}
$\mathcal{J}_h$	4.70111×10^{-3}	4.72577×10^{-3}	4.74206×10^{-3}

The Table 2, gives the number of vertices in the mesh $\mathcal{T}_h$ as well as the minimum and maximum length of the edges of the used triangulation. We present also L^2 error = $\left\|u - u_h^{k_{max}}\right\|_{L^2(\Omega_h)}$ and $\mathcal{J}_h = \mathcal{J}_h^{k_{max}}$ obtained for $k_{max} = 90$ and different value of the mesh size h . The initial estimate is taken $u_h^0 = 1$. The solutions corresponding to the initial $u_h^{kmax}(0,\cdot)$ and the final time $u_h^{kmax}(T,\cdot)$ are shown respectively in Figure 4 and 5. Figure 6 shows objective function $\mathcal{J}_h$ value decreases along with the increase of the iteration number.

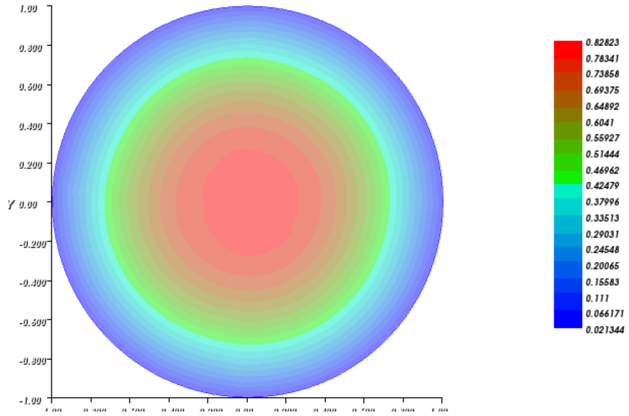

Figure 4: Output initial $u_h^{kmax}(0, \cdot)$.

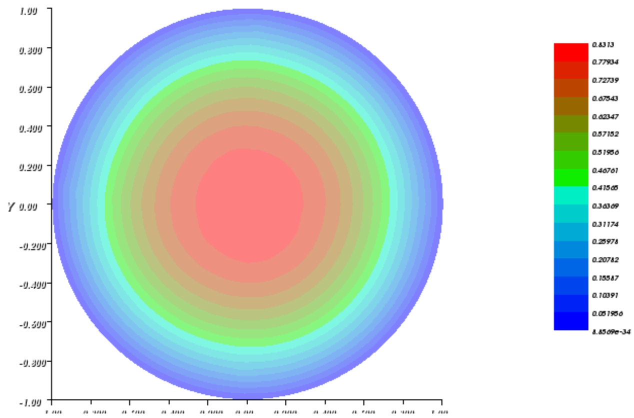

Figure 5: Output initial $u_h^{kmax}(T, \cdot)$.

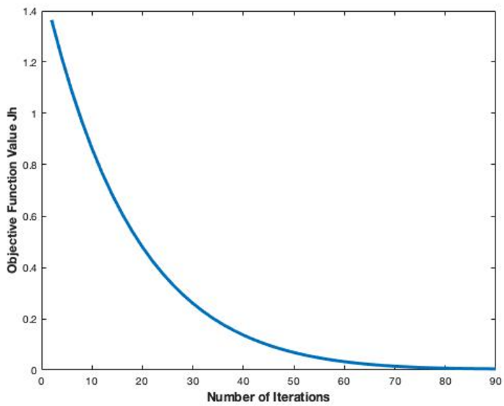

Figure 6: The decrease of the objective function $\mathcal{J}_h$ according to the number of iterations.

Conclusions

We have considered a semilinear periodic parabolic equation with a discontinuous coefficient. We have proposed a new method to theoretically analyze and numerically simulate the weak periodic solution to the studied problem. Our method was based on the introduction of a least-squares criterion. The consideration of this cost function points out in the formulation of the periodic problem into an equivalent optimization problem. We have proved that the minimization problem has at least one optimal solution in a suitable set of admissible functions. By following the Lagrangian approach, we have computed the derivative of the cost function with respect to the state variable and then we have presented an iterative algorithm to solve the minimization problem. After that, we have used the finite element method to discretize the equations of our problem. Finally, we have made some numerical examples to show the efficiency and the robustness of our proposed method. In view of the obtained numerical results, we have concluded that the presented method gives more feasibility concerning the numerical simulations of periodic solutions to semilinear parabolic equations with discontinuous coefficients. In addition, the obtained numerical results confirmed the theoretical analysis presented in this work. As a conclusion, the proposed method proves great promise as a numerical tool for the simulation of the periodic solution to a partial differential equation with discontinuous coefficient.

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