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Approximation of operators related to squared Szász-Mirakjan basis functions

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Abstract. The main objective of this paper is to define a sequence of positive linear operators by means of the squared Szász-Mirakjan basis functions. We estimate the rate of convergence in terms of the modulus of continuity and the class of Lipschitz functions. Furthermore, we have shown the comparison and convergence of these operators with the help of some illustrative graphics.

1. Introduction and Preliminaries

The basis of the theory of approximation is the theorem discovered by Weierstrass in 1885. The first constructive proof of this theorem was given by Bernstein [5] in 1912. He introduced a sequence of polynomials $B_m : C[0,1] \to C[0,1]$ defined by

$$B_m(h;y) = \sum_{k=0}^{m} p_{m,k}(y) h\left(\frac{k}{m}\right), \ y \in [0,1], \tag{1}$$

where Bernstein basis function is given by

$$p_{m,k}(y) = \binom{m}{k} y^k (1-y)^{m-k}, \quad m \in \mathbb{N}.$$

Later it was discovered that Bernstein polynomials have numerous noteworthy properties, so new applications and generalizations are being found of it. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as computer-aided geometric design, numerical analysis and solutions of differential equations. Szász [16] and Mirakjan [12] generalized the Bernstein polynomials to an infinite interval as

$$S_m(h;y) = \sum_{k=0}^{\infty} s_{m,k}(y) h\left(\frac{k}{m}\right), h \in C[0,\infty), \tag{2}$$

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where

$$s_{m,k}(y) = e^{-my}\frac{(my)^k}{k!}, \;\; y \geq 0.$$

In the recent past, there is a growing interest in studying the monotonicity and convexity properties of the sums of squared basis function in approximation theory (see[1, 8, 14, 15, 17]).
Gavrea and Ivan [9] in 2017 considered the rational functions

$$b_{m,k}(y) = \frac{p^2_{m,k}(y)}{\sum\limits_{i=0}^{m} p^2_{m,i}(y)} \tag{3}$$

and studied the approximation properties of the positive linear Bernstein-type rational operators $\mathcal{L}_m : C[0,1] \to C[0,1]$ defined by

$$L_m(h;y) = \sum_{k=0}^{m} b_{m,k}(y)h\Big(\frac{k}{m}\Big), \; m \in \mathbb{N}. \tag{4}$$

In the present paper, we will define a generalization of the operators (4) to the infinite interval and study about their convergence and approximation properties. Gavrea and Ivan [8] proved that the sum of the squared basis function

$$\sum_{k=0}^{\infty}\Big(\frac{e^{-my}(my)^k}{k!}\Big)^2, \;\; y \geq 0,$$

is convex of any even order and concave of any odd order.

Holhoş [10] defined the positive linear Szász-Mirakjan type rational operators $\mathcal{T}_m : C[0,\infty) \to C[0,\infty)$ by

$$\mathcal{T}_m(h;y) = \sum_{k=0}^{\infty} t_{m,k}(y)h\Big(\frac{k}{m}\Big), \;\; m \in \mathbb{N} \tag{5}$$

where

$$t_{m,k}(y) = \frac{s^2_{m,k}(y)}{\sum\limits_{i=0}^{\infty} s^2_{m,i}(y)}.$$

We note that the operators $\mathcal{T}_m$ are linear, positive and preserve constants. Abel [2] mentioned without proof a complete asymptotic expansion of these operators and some estimations of the rate of convergence in terms of the usual modulus of continuity. Independently, Holhoş [10] presented with proofs an estimate of the rate of convergence in weighted spaces and a quantitative Voronovskaya type theorem. Explicit upper bounds were also obtained by Holhoş [10]. Note that these operators extend the class of Szász-Mirakjan type operators which preserve some polynomial functions [7], [11], [18]. For most recent variants of such operators, we refere to [3, 4, 6, 13].

This paper is organized as follows. In Section 2, we give some auxiliary results. In Section 3, we prove the main results wherein we obtain the second central moments and rate of convergence of these operators. In Section 4, we give some examples with illustrative graphics.

2. Auxiliary Results

In order to prove the main results, the following function will be the indispensable tool. For $m \in \mathbb{N}$ and $y \in [0, \infty)$ define

$$g_m(y) = \frac{\int_0^1 \frac{e^{-4myt} yt}{\sqrt{t(1-t)}} dt}{\int_0^1 \frac{e^{-4myt}}{\sqrt{t(1-t)}} dt}. \tag{6}$$

Integration by parts shows that the integral in (6) are differentiable with respect to parameter y.
Let

$$M_m = \sup_{y \in [0,\infty)} g_m(y), \quad m \in \mathbb{N}. \tag{7}$$

In order to study the convergence results, we prove the following Lemmas:

Lemma 2.1. *The sum of squared Szász-Mirakjan basis function satisfies the following equality*

$$\sum_{k=0}^{\infty} \left(\frac{e^{-my}(my)^k}{k!} \right)^2 = \frac{1}{\pi} \int_0^1 \frac{e^{-4myt} yt}{\sqrt{t(1-t)}} dt,$$

for $y \in [0, \infty)$ and $m \in \mathbb{N}$.

Proof. Since

$$e^{-my} \sum_{k=0}^{\infty} \frac{(my)^k}{k!} e^{iku} = e^{my(e^{iu}-1)},$$

by Parseval's identity, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(\frac{e^{-my}(my)^k}{k!} \right)^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{my(e^{iu}-1)}|^2 du \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2my(\cos u - 1)} du \\
&= \frac{1}{\pi} \int_0^{\pi} e^{2my(\cos u - 1)} du \\
&= \frac{1}{\pi} \int_0^{\pi} e^{-4my \sin^2(\frac{u}{2})} du.
\end{aligned}$$

Putting $\frac{u}{2} = \theta$, we get

$$\sum_{k=0}^{\infty} \left(\frac{e^{-my}(my)^k}{k!} \right)^2 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-4my \sin^2 \theta} d\theta$$

and with $\sin^2 \theta = t$, we have

$$\sum_{k=0}^{\infty} \left(\frac{e^{-my}(my)^k}{k!} \right)^2 = \frac{1}{\pi} \int_0^1 \frac{e^{-4myt} yt}{\sqrt{t(1-t)}} dt.$$

□

Lemma 2.2. *For all $y \in [0, \infty)$ and $m \geq 1$, the following inequality holds:*

$$mg_m(y) \geq \frac{1}{8} - \frac{1}{8(1+2my)}, \tag{8}$$

where $g_m(y)$ is defined by (6).

Proof. Since $\frac{1}{\sqrt{t(1-t)}} = (2\arcsin\sqrt{t})'$, using integration by parts, we obtain

$$\begin{aligned} \int_0^1 \frac{e^{-4myt}}{\sqrt{t(1-t)}}dt &= \int_0^1 e^{-4myt}(2\arcsin\sqrt{t})'dt \\ &= \pi e^{-4my} + 8my\int_0^1 e^{-4myt}\arcsin\sqrt{t}dt \\ &= (\epsilon_m(y)+1)8my\int_0^1 e^{-4myt}\arcsin\sqrt{t}dt, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \epsilon_m(y) &= \frac{\pi e^{-4my}}{8my\int_0^1 e^{-4myt}\arcsin\sqrt{t}dt} \\ &< \frac{\pi e^{-4my}}{8my\int_0^1 e^{-4my}\arcsin\sqrt{t}dt} \\ &= \frac{1}{2my}, \end{aligned} \tag{10}$$

for $y \in [0, \infty)$ and m≥ 1.
From (9), using the inequality

$$\arcsin\sqrt{t} \leq \frac{t}{\sqrt{t(1-t)}}, \quad t \in [0,1),$$

we obtain

$$\begin{aligned} \int_0^1 \frac{e^{-4myt}}{\sqrt{t(1-t)}}dt &\leq (\epsilon_m(y)+1)8my\int_0^1 \frac{e^{-4myt}t}{\sqrt{t(1-t)}}dt \\ \frac{1}{8(\epsilon_m(y)+1)} &\leq m\frac{\int_0^1 \frac{e^{-4myt}yt}{\sqrt{t(1-t)}}dt}{\int_0^1 \frac{e^{-4myt}}{\sqrt{t(1-t)}}dt} \\ \frac{1}{8(\epsilon_m(y)+1)} &\leq mg_m(y). \end{aligned} \tag{11}$$

By (10) and (11), we get

$$mg_m(y) \geq \frac{1}{8} - \frac{1}{8(1+2my)}.$$

□

3. Main Results

Theorem 3.1. *The second central moments of the operators $\mathcal{T}_m$ defined by* (5) *is given by*

$$\mathcal{T}_m((t-y)^2;y) = 4yg_m(y),$$

where $g_m(y)$ is defined by equation (6).

Proof. Starting from the equality

$$e^{-my}\sum_{k=0}^{\infty}\frac{(my)^k}{k!}e^{(y-\frac{k}{m})u}e^{ik\theta} = e^{my(e^{(i\theta-\frac{u}{m})}-1)+uy}$$

and differentiating both side with respect to u, we obtain

$$e^{-my}\sum_{k=0}^{\infty}\frac{(my)^k}{k!}\Big(y-\frac{k}{m}\Big)e^{(y-\frac{k}{m})u}e^{ik\theta} = ye^{-my}e^{mye^{(i\theta-\frac{u}{m})}+uy}(1-e^{i\theta-\frac{u}{m}}).$$

Taking $u=0$, we get

$$e^{-my}\sum_{i=0}^{\infty}\frac{(my)^k}{k!}\Big(y-\frac{k}{m}\Big)e^{ik\theta} = ye^{my(e^{i\theta}-1)}(1-e^{i\theta}).$$

Using Parseval's equality, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(e^{-my}\frac{(my)^k}{k!}\Big)^2\Big(y-\frac{k}{m}\Big)^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|ye^{my(e^{i\theta}-1)}(1-e^{i\theta})|^2 d\theta\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}y^2e^{2my(\cos\theta-1)}(1-\cos\theta)d\theta\\ &= \frac{4}{\pi}\int_{0}^{\pi}y^2e^{-4my\sin^2(\frac{\theta}{2})}\sin^2(\frac{\theta}{2})d\theta,\end{aligned}$$

putting $\frac{\theta}{2}=\phi$, we get

$$\sum_{k=0}^{\infty}\Big(e^{-my}\frac{(my)^k}{k!}\Big)^2\Big(y-\frac{k}{m}\Big)^2 = \frac{8}{\pi}\int_{0}^{\frac{\pi}{2}}y^2e^{-4my\sin^2\phi}\sin^2\phi d\phi$$

and with $\sin^2\phi = t$, we obtain

$$\sum_{k=0}^{\infty}\Big(e^{-my}\frac{(my)^k}{k!}\Big)^2\Big(y-\frac{k}{m}\Big)^2 = \frac{4}{\pi}\int_{0}^{1}\frac{y^2e^{-4myt}t}{\sqrt{t(1-t)}}dt. \tag{12}$$

Now from (5), (12) and Lemma 2.1, we have

$$\begin{aligned}\mathcal{T}_m((t-y)^2;y) &= \frac{4y\int_0^1\frac{e^{-4myt}yt}{\sqrt{t(1-t)}}dt}{\int_0^1\frac{e^{-4myt}}{\sqrt{t(1-t)}}dt}\\ &= 4yg_m(y),\end{aligned}$$

which completes the proof. □

Theorem 3.2. *The second central moments of the operators $\mathcal{T}_m$ defined by* (5) *possess the following properties:*

$$\frac{1}{2}\cdot\frac{y}{m}\leq\mathcal{T}_m((t-y)^2;y)\leq 4pM_p\cdot\frac{y}{m}, \tag{13}$$

for all $x\geq 0$ and $m\geq p\geq 1$.

Proof. Since

$$m\mathcal{T}_m((t-y)^2;y)=4ymg_m(y),$$

using Lemma 2.2, we have

$$\begin{aligned} m\mathcal{T}_m((t-y)^2;y) &\geq \frac{y}{2}-\frac{y}{2(1+2my)}\\ \Rightarrow \mathcal{T}_m((t-y)^2;y) &\geq \frac{1}{2}\cdot\frac{y}{m}. \end{aligned} \tag{14}$$

Since the sequence $\{mM_m\}_{m\geq 1}$ is non decreasing, we have

$$\begin{aligned} \mathcal{T}_m((t-y)^2;y) &= 4yg_m(y)\\ &\leq 4yM_m\leq 4pM_p\cdot\frac{y}{m}, \end{aligned} \tag{15}$$

for all $m\geq p\geq 1$. From equations (14) and (15), the proof is completed. □

In the approximation of a function by positive linear operators not only the convergence of operators is required but also the speed of convergence is important. The rate of convergence depends on the smoothness properties of the function and appropriate tool for estimating the smoothness of function are represented by the modulus of continuity. We compute the rate of convergence of the constructed operators in terms of modulus of continuity and class of Lipschitz function:
For any $\delta>0$ and $h\in C[a,b]$, the modulus of continuity $\omega(h,\delta)$ is defined by

$$\omega(h,\delta)=\sup_{|t-y|\leq\delta,\ \ t,y\in[a,b]}|h(t)-h(y)|. \tag{16}$$

Also,

$$|h(t)-h(y)|\leq\omega(h,\delta)\left(1+\frac{|t-y|}{\delta}\right). \tag{17}$$

If h is uniformly continuous then it is necessary and sufficient that

$$\lim_{\delta\to 0}\omega(h,\delta)=0.$$

Theorem 3.3. *For any $h\in C_B[0,\infty)$(space of all bounded and uniformly continuous functions on $[0,\infty)$), we have the estimate*

$$|\mathcal{T}_m(h;y)-h(y)|\leq\left(1+2\sqrt{pM_py}\right)\cdot\omega\left(h,\frac{1}{\sqrt{m}}\right),$$

where $m\geq p\geq 1$, M_m is given by (7) *and $\omega(h,.)$ is the modulus of continuity.*

Proof. Since

$$\begin{aligned} |\mathcal{T}_m(h;y)-h(y)| &= \left|\sum_{k=0}^{\infty}t_{m,k}(y)h\left(\frac{k}{m}\right)-h(y)\right|\\ &\leq \sum_{k=0}^{\infty}t_{m,k}(y)\left|h\left(\frac{k}{m}\right)-h(y)\right|. \end{aligned}$$

By using Cauchy-Schwartz inequality and equation (17), we get

$$\begin{aligned}
|\mathcal{T}_m(h;y)-h(y)| &\leq \sum_{k=0}^{\infty} t_{m,k}(y)\Big\{1+\frac{1}{\delta}\Big|\frac{k}{m}-y\Big|\Big\}\omega(h,\delta)\\
&\leq \Big\{1+\frac{1}{\delta}\Big(\sum_{k=0}^{\infty} t_{m,k}(y)\Big(\frac{k}{m}-y\Big)^2\Big)^{\frac{1}{2}}\Big(\sum_{k=0}^{\infty} t_{m,k}(y)\Big)^{\frac{1}{2}}\Big\}\omega(h,\delta)\\
&\leq \Big\{1+\frac{1}{\delta}\Big(\mathcal{T}_m((.-y)^2;y)\Big)^{\frac{1}{2}}\Big(\mathcal{T}_m(1;y)\Big)^{\frac{1}{2}}\Big\}\omega(h,\delta).
\end{aligned}$$

Using the fact that $\mathcal{T}_m(1;y)=1$ and Theorem 3.2, we have

$$|\mathcal{T}_m(h;y)-h(y)| \leq \Big\{1+\frac{1}{\delta}\Big(4pM_p\cdot\frac{y}{m}\Big)^{\frac{1}{2}}\Big\}\omega(h,\delta),$$

choosing $\delta=\frac{1}{\sqrt{m}}$, we obtain the result. □

From Theorem 3.3, we deduce the following corollary:

Corollary 3.4. *For any $h\in C_B[0,\infty)$, the sequence $\{\mathcal{T}_m h\}_{m\in\mathbb{N}}$ converges uniformly to h on $[0,\infty)$.*

The rate of convergence of operators $\mathcal{T}_m(h;y)$ defined by equation (5) in terms of the element of the usual Lipschitz class $Lip_M(\mu)$ is as follows:
Let $h\in C_B[0,\infty)$, $M>0$ and $0<\mu\leq 1$. The class of $Lip_M(\mu)$ is defined as

$$Lip_M(\mu)=\{h:|h(\zeta_1)-h(\zeta_2)|\leq M|\zeta_1-\zeta_2|^{\mu},\zeta_1,\zeta_2\in[0,\infty)\}. \tag{18}$$

Theorem 3.5. *For each $h\in Lip_M(\mu)$ $(M>0,\ \ 0<\mu\leq 1)$, we have*

$$|\mathcal{T}_m(h;y)-h(y)|\leq M(\delta_m(y))^{\frac{\mu}{2}},$$

where $\delta_m(y)=\mathcal{T}_m((t-y)^2;y)$.

Proof. We prove this theorem by using the definition of Lipschitz function (18) and Hölder's inequality.

$$\begin{aligned}
|\mathcal{T}_m(h;y)-h(y)| &\leq \mathcal{T}_m(|h(t)-h(y)|;y)\\
&\leq M\mathcal{T}_m(|t-y|^{\mu};y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathcal{T}_m(h;y)-h(y)| &\leq M\sum_{k=0}^{\infty} t_{m,k}(y)\Big|\frac{k}{m}-y\Big|^{\mu}\\
&\leq M\sum_{k=0}^{\infty}(t_{m,k}(y))^{\frac{2-\mu}{2}}(t_{m,k}(y))^{\frac{\mu}{2}}\Big|\frac{k}{m}-y\Big|^{\mu}\\
&\leq M\Big\{\Big(\sum_{k=0}^{\infty} t_{m,k}(y)\Big)^{\frac{2-\mu}{2}}\Big(\sum_{k=0}^{\infty} t_{m,k}(y)\Big|\frac{k}{m}-y\Big|^{2}\Big)^{\frac{\mu}{2}}\Big\}\\
&= M\Big(\mathcal{T}_m((.-y)^2;y)\Big)^{\frac{\mu}{2}}.
\end{aligned}$$

Choosing $\delta_m(y)=\mathcal{T}_m((.-y)^2;y)$, proof is completed. □

4. Graphical Analysis and Error Estimation

In this section, we will give some numerical examples with illustrative graphics and also compare error estimation with the help of MATLAB.

Example 4.1. *Let* $h(y) = ye^{-3y}$ *and* $m \in \{10, 20, 30\}$. *The convergence of the operator* $\mathcal{T}_m(h; y)$ *towards the function* $h(y)$ *and the absolute error* $|\mathcal{T}_m(h; y) - h(y)|$ *is shown in Figure 1 and 2, respectively. Also, the absolute error of the operators at certain points is computed in the Table 1.*

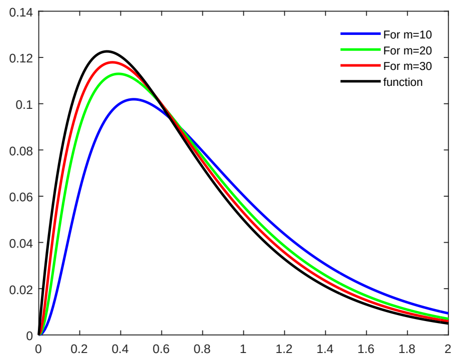

Figure 1:

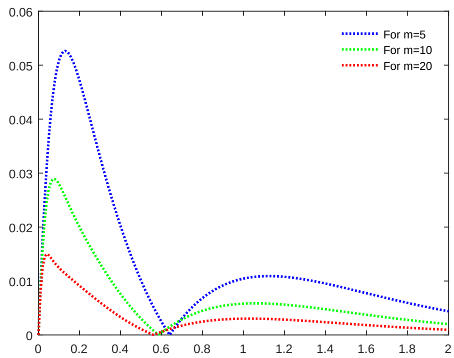

Figure 2:

Table 1: Absolute Error of the operators $\mathcal{T}_m(h; y)$ with function $h(y) = ye^{-3y}$

y	For $m = 5$	For $m = 10$	For $m = 20$
1.0	0.0104	0.0058	0.0030
1.2	0.0180	0.0056	0.0028
1.4	0.0095	0.0048	0.0024
1.6	0.0077	0.0037	0.0018
1.8	0.0059	0.0028	0.0013
2.0	0.0044	0.0020	0.0009

Example 4.2. *Let* $h(y) = y^2 \sin(4\pi y)$ *and* $m \in \{20, 30, 50\}$. *The convergence of the operators* $\mathcal{T}_m(h; y)$ *towards the function* $h(y)$ *and the absolute error of the operators with the function is shown in Figure 3 and 4, respectively. In Table 2, we compute the absolute error of the operators at a certain point in the interval.*

Table 2: Absolute Error of the operators $\mathcal{T}_m(h; y)$ with function $h(y) = y^2 \sin(4\pi y)$

y	For $m = 20$	For $m = 30$	For $m = 50$
0.2	0.0180	0.0134	0.0087
0.4	0.0963	0.0735	0.0491
0.6	0.2313	0.1787	0.1205
0.8	0.2688	0.2050	0.1342
1.0	0.0369	0.0594	0.0703
1.2	0.7923	0.7234	0.5885

From these examples, we observe that approximation of $h(y)$ by $\mathcal{T}_m(h; y)$ becomes better when we take larger values of m.

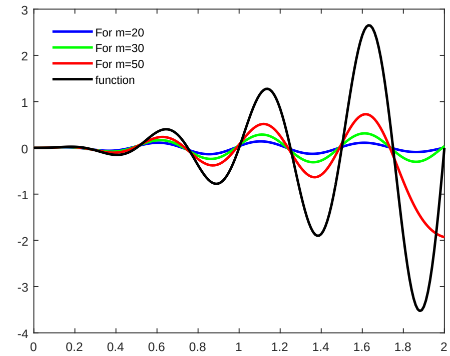

Figure 3:

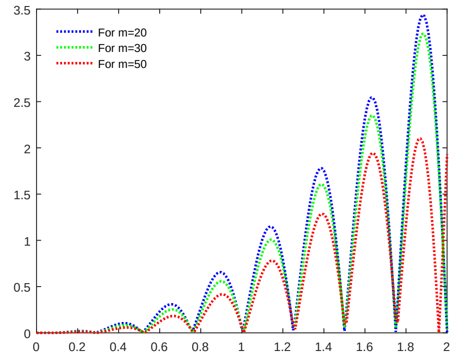

Figure 4:

Example 4.3. *Let $h(y) = y^3 e^{-y^2}$. The comparison of convergence of Szász Mirakjan rational type operators $\mathcal{T}_{10}(h;y)$ (pink), $\mathcal{T}_{20}(h;y)$ (red) and Szász Mirakjan operators $S_{10}(h;y)$ (blue), $S_{20}(h;y)$ (green) towards the function $h(y)$ (black–) is illustrated in Figure 5. The Absolute Error for these operators is shown in Figure 6. From these figures, it is clear our constructed operator $\mathcal{T}_m(h;y)$ gives a better approximation to $h(y)$ than classical Szász Mirakjan operators $S_m(h;y)$.*

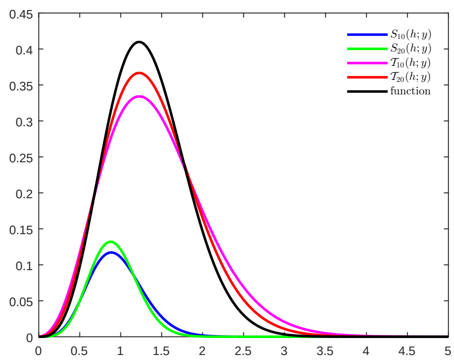

Figure 5:

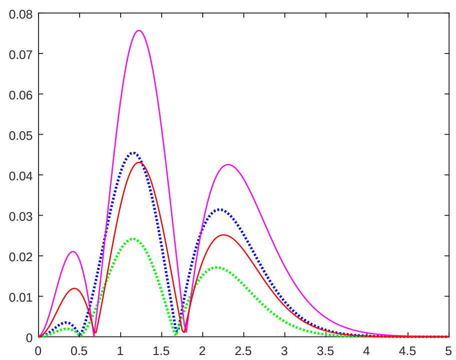

Figure 6:

Table 3: Absolute Error of the defined operators $\mathcal{T}_m(h;y)$ and Szász operators $S_m(h;y)$ with the function $h(y) = y^3 e^{-y^2}$ for $m = 10$ and $m = 20$

y	$\mathcal{T}_{10}(h;y)$	$S_{10}(h;y)$	$\mathcal{T}_{20}(h;y)$	$S_{20}(h;y)$
1.0	0.0407	0.0585	0.0216	0.0326
1.5	0.0223	0.0513	0.0112	0.0281
2.0	0.0268	0.0279	0.0152	0.0187
2.5	0.0252	0.0389	0.0128	0.0213
3.0	0.0086	0.0173	0.0037	0.0076
3.5	0.0017	0.0047	0.0006	0.0015

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