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Reynolds operators on Hom-Leibniz algebras

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Abstract. In this paper, we first introduce the notion of Reynolds operators on Hom-Leibniz algebras and give some constructions. Furthermore, we define the cohomology of Reynolds operators, and use this cohomology to study deformations of Reynolds operators. As applications, we introduce and study NS-Hom-Leibniz algebras as the underlying structure of Reynolds operators.

1. Introduction

The study of Hom-algebras can be traced back to Hartwig, Larsson and Silvestrov's work in [14], where the notion of Hom-Lie algebra in the context of q-deformation theory of Witt and Virasoro algebras [15] was introduced, which plays an important role in physics, mainly in conformal field theory. The notion of a Leibniz algebra was introduced by Loday [16, 17] with the motivation in the study of the periodicity in algebraic K-theory. Leibniz algebras were studied from different aspects due to applications in both mathematics and physics. The notion of a Hom-Leibniz algebra was introduced by Makhlouf and Silvestrov [18], generalizing both Hom-Lie algebras and Leibniz algebras. Hom-Leibniz algebras were widely studied in the following aspects: representation and cohomology theory [5], deformation theory [21], Hom-Leibniz cohomology [24]. For more interesting Hom-algebra structures, see [2–4, 12, 13, 19] and references cited therein.

Our main objective is to study Reynolds operators on Hom-Leibniz algebras. The notion of Rota-Baxter operators on associative algebras was introduced in 1960 by Baxter [1] in his study of fluctuation theory in probability. Recently, it has been found many applications, including in Connes-Kreimer's algebraic approach to the renormalization in perturbative quantum field theory [7]. For further details on Rota-Baxter operators, see [11]. The study of Reynolds operators has its origin in the well-known work of Reynolds [22] on fluid dynamics in 1895 and has since found broad applications. It also has close relationship with important linear operators such as algebra endomorphisms, derivations and Rota-Baxter operators. Also it was closely related to the probability theory. For further details on Reynolds operators, see [23]. Recently, Das [8] introduced twisted Rota-Baxter operators on Lie algebras and considers NS-Lie algebras as the underlying structure Motivated by Uchino [25]. Later, Das and Guo [9] introduced twisted Rota-Baxter

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operators on Leibniz algebras and considers NS-Leibniz algebras as the underlying structure. In fact, twisted Rota-Baxter operators are all generalizations of Reynolds operators.

Recently, Mishra and Naolekar [20] studied $\mathcal{O}$-operators, also known as relative or generalized Rota-Baxter operators on Hom-Lie algebras. Das and Sen [10] studied Nijenhuis operators on Hom-Lie algebras. Later, Chtioui, Mabrouk and Makhlouf [6] introduced the cohomology theory and deformations of $\mathcal{O}$-operators on Hom-associative algebras. Zhang, Gao and Guo [26] gave the construction of free objects of Reynolds algebras by bracketed words and rooted trees. However, so far, there is little research on Reynolds operators on Leibniz algebras, not to mention studying about the Reynolds operators on Hom-Leibniz algebras. In this paper, we study the deformation and cohomology theory of Reynolds operators on Hom-Leibniz algebras. The specific structure is as follows. In Section 2, we introduce the notion of Reynolds operators on Hom-Leibniz algebras and give some constructions. In Section 3, we define the cohomology of Reynolds operators. In Section 4, we use this cohomology to study deformations of Reynolds operators. In Section 5, as applications, we introduce and study NS-Hom-Leibniz algebras as the underlying structure of Reynolds operators.

2. Preliminaries

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$ and finite-dimensional. We now recall some useful definitions in [5].

Definition 2.1. *A Hom-Leibniz algebra is a triple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a bilinear operation $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ and a linear map $\phi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ satisfying*

$$[\phi_{\mathfrak{g}}(x), [y,z]_{\mathfrak{g}}]_{\mathfrak{g}} = [[x,y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), [x,z]_{\mathfrak{g}}]_{\mathfrak{g}}, \quad \forall x,y,z \in \mathfrak{g}.$$

A Hom-Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ is said to be regular (involutive), if $\phi_{\mathfrak{g}}$ is nondegenerate (satisfies $\phi_{\mathfrak{g}}^2 = Id$).

Definition 2.2. *A representation of a Hom-Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ is a quadruple $(V, \phi_V, \rho^L, \rho^R)$, where V is a vector space, $\phi_V \in gl(V)$, $\rho^L, \rho^R : \mathfrak{g} \to gl(V)$ are three linear maps such that the following equalities hold for all $x,y \in \mathfrak{g}$:*

$$\begin{aligned}
&(1) \qquad \rho^L(\phi_{\mathfrak{g}}(x))\circ\phi_V = \phi_V\circ\rho^L(x), \quad \rho^R(\phi_{\mathfrak{g}}(x))\circ\phi_V = \phi_V\circ\rho^R(x);\\
&(2) \qquad \rho^L([x,y]_{\mathfrak{g}})\circ\phi_V = \rho^L(\phi_{\mathfrak{g}}(x))\circ\rho^L(y) - \rho^L(\phi_{\mathfrak{g}}(y))\circ\rho^L(x);\\
&(3) \qquad \rho^R([x,y]_{\mathfrak{g}})\circ\phi_V = \rho^L(\phi_{\mathfrak{g}}(x))\circ\rho^R(y) - \rho^R(\phi_{\mathfrak{g}}(y))\circ\rho^L(x);\\
&(4) \qquad \rho^R([x,y]_{\mathfrak{g}})\circ\phi_V = \rho^L(\phi_{\mathfrak{g}}(x))\circ\rho^R(y) + \rho^R(\phi_{\mathfrak{g}}(y))\circ\rho^R(x).
\end{aligned}$$

Define the left multiplication $L : \mathfrak{g} \to gl(\mathfrak{g})$ and the right multiplication $R : \mathfrak{g} \to gl(\mathfrak{g})$ by $L_x y = [x,y]_{\mathfrak{g}}$ and $R_x y = [y,x]_{\mathfrak{g}}$ respectively for all $x,y \in \mathfrak{g}$. Then $(\mathfrak{g}, \phi_{\mathfrak{g}}, L, R)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$, which is called a regular representation.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Leibniz algebra and $(V, \phi_V, \rho^L, \rho^R)$ be a representation of it. The cohomology of the Hom-Leibniz algebra $\mathfrak{g}$ with coefficients in V is the cohomology of the cochain complex $\{C^*(\mathfrak{g}, V), \partial\}$, where $C^n(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}^{\otimes n}, V)$ for $n \geq 0$, and the coboundary operator $\partial^n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ given by

$$\begin{aligned}
&(\partial^n f)(x_1,\ldots,x_{n+1})\\
&= \sum_{i=1}^{n}(-1)^{i+1}\rho^L(\phi_{\mathfrak{g}}^{n-1}(x_i))f(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}) + (-1)^{n+1}\rho^R(\phi_{\mathfrak{g}}^{n-1}(x_{n+1}))f(x_1,\ldots,x_n)\\
&+ \sum_{1\leq i<j\leq n+1}(-1)^i f(\phi_{\mathfrak{g}}(x_1),\ldots,\widehat{\phi_{\mathfrak{g}}(x)}_i,\ldots,\phi_{\mathfrak{g}}(x_{j-1}),[x_i,x_j]_{\mathfrak{g}},\phi_{\mathfrak{g}}(x_{j+1}),\ldots,\phi_{\mathfrak{g}}(x_{n+1})),
\end{aligned}$$

for $x_1,\ldots,x_{n+1} \in \mathfrak{g}$. The corresponding cohomology groups are denoted by $H^*(\mathfrak{g}, V)$.

3. Reynolds operators on Hom-Leibniz algebras

In this section, we introduce the notion of Reynolds operators on Hom-Leibniz algebras and give some constructions.

Definition 3.1. *Let (V,ϕ_V,ρ^L,ρ^R) be a representation of a Hom-Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$. A Reynolds operator on $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ with respect to the representation (V,ϕ_V,ρ^L,ρ^R) is a linear map $K: V\to\mathfrak{g}$ such that*

$$\begin{aligned}&K\circ\phi_V=\phi_{\mathfrak{g}}\circ K,\\ &[Ku,Kv]_{\mathfrak{g}}=K(\rho^L(Ku)v+\rho^R(Kv)u-[Ku,Kv]_{\mathfrak{g}}),\forall u,v\in V.\end{aligned}$$

Example 3.2. *Suppose d is a derivation of a Hom-Leibniz algebra $(g,[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ such that $(id+d):\mathfrak{g}\to\mathfrak{g}$ is invertible, then $(id+d)^{-1}$ is a Reynolds operator respect to the regular representation.*

Example 3.3. *Consider the three-dimensional Hom-Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ given with respect to a basis $\{e_1,e_2,e_3\}$ together with the following nonvanishing operations:*

$$[e_1,e_1]=e_3,\quad \phi_{\mathfrak{g}}(e_1)=-e_1,\quad \phi_{\mathfrak{g}}(e_2)=e_2,\quad \phi_{\mathfrak{g}}(e_3)=e_3.$$

Then $K=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ is a Reynolds operator on $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ with respect to the regular representation if and only if

$$[Ke_i,Ke_j]_{\mathfrak{g}}=K([Ke_i,e_j]_{\mathfrak{g}}+[e_i,Ke_j]_{\mathfrak{g}}-[Ke_i,Ke_j]_{\mathfrak{g}}),\qquad \forall i,j=1,2,3.$$

Since $[Ke_1,Ke_1]_{\mathfrak{g}}=[a_{11}e_1+a_{21}e_2+a_{31}e_3,a_{11}e_1+a_{21}e_2+a_{31}e_3]_{\mathfrak{g}}=a_{11}^2e_3$ and

$$\begin{aligned}&K([Ke_1,e_1]_{\mathfrak{g}}+[e_1,Ke_1]_g-[Ke_1,Ke_1]_{\mathfrak{g}})\\=\;&K([a_{11}e_1+a_{21}e_2+a_{31}e_3,e_1]_{\mathfrak{g}}+[e_1,a_{11}e_1+a_{21}e_2+a_{31}e_3]_{\mathfrak{g}}-a_{11}^2e_3)\\=\;&(2a_{11}-a_{11}^2)Ke_3\\=\;&(2a_{11}-a_{11}^2)a_{13}e_1+(2a_{11}-a_{11}^2)a_{23}e_2+(2a_{11}-a_{11}^2)a_{33}e_3.\end{aligned}$$

Thus, by $[Ke_1,Ke_1]_{\mathfrak{g}}=K([Ke_1,e_1]_{\mathfrak{g}}+[e_1,Ke_1]_{\mathfrak{g}}-[Ke_1,Ke_1]_{\mathfrak{g}})$, we have

$$(2a_{11}-a_{11}^2)a_{13}=0,(2a_{11}-a_{11}^2)a_{23}=0,a_{11}^2=(2a_{11}-a_{11}^2)a_{33}.$$

Similarly, we obtain

$$\begin{array}{lll}a_{11}a_{12}=(a_{12}-a_{11}a_{12})a_{33}, & (a_{12}-a_{11}a_{12})a_{13}=0, & (a_{12}-a_{11}a_{12})a_{23}=0;\\ a_{11}a_{13}=(a_{13}-a_{11}a_{13})a_{33}, & (a_{13}-a_{11}a_{13})a_{13}=0, & (a_{13}-a_{11}a_{13})a_{23}=0;\\ a_{12}a_{11}=(a_{12}-a_{11}a_{12})a_{33}, & (a_{12}-a_{11}a_{12})a_{13}=0, & (a_{12}-a_{11}a_{12})a_{23}=0;\\ a_{13}a_{11}=(a_{13}-a_{11}a_{13})a_{33}, & (a_{13}-a_{11}a_{13})a_{13}=0, & (a_{13}-a_{11}a_{13})a_{23}=0,\\ a_{12}^2+a_{12}^2a_{33}=0, & a_{12}^2a_{13}=0, & a_{12}^2a_{13}=0.\end{array}$$

Summarize the above discussion, we have the following two cases:

(1)If $a_{11}=a_{12}=a_{13}=0$, then any $K=\begin{pmatrix}0&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ is a Reynolds operator on $(g,[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ with respect to the regular representation.

(2) If $a_{12}=a_{13}=a_{23}=0$ and $a_{11}\neq 0,2$, then any $K=\begin{pmatrix}a_{11}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&\frac{a_{11}}{2-a_{11}}\end{pmatrix}$ is a Reynolds operator on $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ with respect to the regular representation.

Let $K : V \to \mathfrak{g}$ be a Reynolds operator. Suppose $(V', \phi_{V'}, \rho'^L, \rho'^R)$ is a representation of another Hom-Leibniz algebra $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, \phi_{\mathfrak{g}'})$. Let $K' : V' \to \mathfrak{g}'$ be a Reynolds operator.

Definition 3.4. *A morphism of Reynolds operators from K to K' consists of a pair (φ, ψ) of a Hom-Leibniz algebra morphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ and a linear map $\psi : V \to V'$ satisfying*

$$\begin{aligned}
&\varphi \circ K = K' \circ \psi,\\
&\psi(\rho^L(x)u) = \rho'^L(\varphi(x))\psi(u), \quad \psi(\rho^R(x)u) = \rho'^R(\varphi(x))\psi(u),\\
&\phi_{\mathfrak{g}'} \circ \varphi = \varphi \circ \phi_{\mathfrak{g}}, \ \ \phi_{V'} \circ \psi = \psi \circ \phi_V, \quad \textit{for } x \in \mathfrak{g}, u \in V.
\end{aligned}$$

One can construct the semidirect product algebra. More precisely, the direct sum $\mathfrak{g} \oplus V$ carries a Hom-Leibniz algebra structure with the bracket given by

$$\begin{aligned}
&[(x,u),(y,v)] := ([x,y]_{\mathfrak{g}}, \rho^L(x)v + \rho^R(y)u - [x,y]_{\mathfrak{g}}),\\
&(\phi_{\mathfrak{g}} + \phi_V)(x,u) := (\phi_{\mathfrak{g}}(x), \phi_V(u)), \quad \text{for } x, y \in \mathfrak{g}, u, v \in V.
\end{aligned}$$

We denote this semidirect product Hom-Leibniz algebra by $(\mathfrak{g} \ltimes V, \phi_{\mathfrak{g}} + \phi_V)$. Using this semidirect product, one can characterize Reynolds operators by their graph.

Proposition 3.5. *A linear map $K : V \to \mathfrak{g}$ is a Reynolds operator if and only if its graph $Gr(K) = \{(Ku, u) | \ u \in V\}$ is a subalgebra of the semidirect product $(\mathfrak{g} \ltimes V, \phi_{\mathfrak{g}} + \phi_V)$.*

The proof of the above proposition is straightforward, hence we omit the details. Since $Gr(K)$ is isomorphic to V as a vector space, as a consequence, we get the following result.

Proposition 3.6. *Let $K : V \to \mathfrak{g}$ be a Reynolds operator. Then the vector space V carries a Hom-Leibniz algebra structure with the bracket*

$$[u,v]_K := \rho^L(Ku)v + \rho^R(Kv)u - [Ku, Kv]_{\mathfrak{g}}, \ \ \textit{for } u, v \in V.$$

4. Cohomology of Reynolds operators

In this section, we define the cohomology of a Reynolds operator K as the cohomology of the Hom-Leibniz algebra $(V, [\cdot,\cdot]_K, \phi_V)$ that is constructed in Proposition 3.6 with coefficients in a suitable representation on $\mathfrak{g}$.

Proposition 4.1. *Let $K : V \to \mathfrak{g}$ be a Reynolds operator. Define maps $\overline{\rho}^L, \overline{\rho}^R : V \to gl(\mathfrak{g})$ by*

$$\overline{\rho}^L(u)x = [Ku, x]_{\mathfrak{g}} - K(\rho^R(x)u) + K[Ku, x]_{\mathfrak{g}} \quad \textit{and} \quad \overline{\rho}^R(u)x = [x, Ku]_{\mathfrak{g}} - K(\rho^L(x)u) + K[x, Ku]_{\mathfrak{g}},$$

for $u \in V$ and $x \in \mathfrak{g}$. Then $(\mathfrak{g}, \phi_{\mathfrak{g}}, \overline{\rho}^L, \overline{\rho}^R)$ is a representation of the Hom-Leibniz algebra $(V, [\cdot,\cdot]_K, \phi_V)$.

Proof. For $u, v \in V$ and $x \in \mathfrak{g}$, we have

$$\begin{aligned}
&\overline{\rho}^L(\phi_V(u))\overline{\rho}^L(v)x - \overline{\rho}^L(\phi_V(v))\overline{\rho}^L(u)x\\
&= \overline{\rho}^L(\phi_V(u))([Kv, x]_{\mathfrak{g}} - K(\rho^R(x)v) + K[Kv, x]_{\mathfrak{g}}) - \overline{\rho}^L(\phi_V(v))([Ku, x]_{\mathfrak{g}} - K(\rho^R(x)u) + K[Ku, x]_{\mathfrak{g}})\\
&= [K\phi_V(u), [Kv, x]_{\mathfrak{g}}]_{\mathfrak{g}} - [K\phi_V(u), K(\rho^R(x)v)]_{\mathfrak{g}} + [K\phi_V(u), K[Kv, x]_{\mathfrak{g}}]_{\mathfrak{g}} - K(\rho^R([Kv, x])\phi_V(u))\\
&\quad + K(\rho^R(K\rho^R(x)v)\phi_V(u)) - K(\rho^R(K[Kv, x]_{\mathfrak{g}})\phi_V(u)) + K[K\phi_V(u), [Kv, x]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&\quad - K[K\phi_V(u), K(\rho^R(x)v)]_{\mathfrak{g}} + K[K\phi_V(u), K[Kv, x]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&\quad - [K\phi_V(v), [Ku, x]_{\mathfrak{g}}]_{\mathfrak{g}} + [K\phi_V(v), K(\rho^R(x)u)]_{\mathfrak{g}} + [K\phi_V(v), K[Ku, x]_{\mathfrak{g}}]_{\mathfrak{g}} + K(\rho^R([Ku, x])\phi_V(v))\\
&\quad - K(\rho^R(K\rho^R(x)u)\phi_V(v)) + K(\rho^R(K[Ku, x]_{\mathfrak{g}})\phi_V(v)) - K[K\phi_V(v), [Ku, x]_{\mathfrak{g}}]_{\mathfrak{g}}
\end{aligned}$$

$$
\begin{aligned}
&+ K[K\phi_V(v), K(\rho^R(x)u)]_{\mathfrak{g}} - K[K\phi_V(v), K[Ku, x]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&= [[Ku, Kv]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}} - K(\rho^R(\phi_{\mathfrak{g}}(x))\rho^L(Ku)v) - K(\rho^R(\phi_{\mathfrak{g}}(x))\rho^R(Kv)u)\\
&\quad + K(\rho^R(\phi_{\mathfrak{g}}(x))[Ku, Kv]_{\mathfrak{g}}) + K[K[u, v]_K, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}\\
&= [K[u, v]_K, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}} - K(\rho^R(\phi_{\mathfrak{g}}(x))[u, v]_K) + K[K[u, v]_K, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}\\
&= \overline{\rho}^L([u, v]_K)\phi_{\mathfrak{g}}(x).
\end{aligned}
$$

Thus, we deduce that

$$\overline{\rho}^L([u, v]_K) \circ \phi_{\mathfrak{g}} = \overline{\rho}^L(\phi_V(u))\overline{\rho}^L(v) - \overline{\rho}^L(\phi_V(v))\overline{\rho}^L(u).$$

Also

$$
\begin{aligned}
&\overline{\rho}^L(\phi_V(u))\overline{\rho}^R(v)x - \overline{\rho}^R(\phi_V(v))\overline{\rho}^L(u)x\\
&= \overline{\rho}^L(\phi_V(u))([x, Kv]_{\mathfrak{g}} - K(\rho^L(x)v) + K[x, Kv]_{\mathfrak{g}}) - \overline{\rho}^R(v)([Ku, x]_{\mathfrak{g}} - K(\rho^R(x)u) + K[Ku, x]_{\mathfrak{g}})\\
&= [K\phi_V(u), [x, Kv]_{\mathfrak{g}}]_{\mathfrak{g}} - [K\phi_V(u), K(\rho^L(x)v)]_{\mathfrak{g}} + [K\phi_V(u), K[x, Kv]_{\mathfrak{g}}]_{\mathfrak{g}} - K(\rho^R([x, Kv]_{\mathfrak{g}})\phi_V(u))\\
&\quad + K(\rho^R(K\rho^L(x)v)\phi_V(u)) - K(\rho^R(K[x, Kv]_{\mathfrak{g}})\phi_V(u)) + K[K\phi_V(u), [x, Kv]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&\quad - K[K\phi_V(u), K(\rho^L(x)v)]_{\mathfrak{g}} + K[K\phi_V(u), K[x, Kv]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&\quad - [[Ku, x]_{\mathfrak{g}}, K\phi_V(v)]_{\mathfrak{g}} + [K(\rho^R(x)u), K\phi_V(v)]_{\mathfrak{g}} - [K[Ku, x]_{\mathfrak{g}}, K\phi_V(v)]_{\mathfrak{g}}\\
&\quad + K(\rho^L([Ku, x])\phi_V(v)) - K(\rho^L(K\rho^R(x)u)\phi_V(v)) + K(\rho^L(K[Ku, x]_{\mathfrak{g}})\phi_V(v)) - K[[Ku, x]_{\mathfrak{g}}, K\phi_V(v)]_{\mathfrak{g}}\\
&\quad + K[K(\rho^R(x)u), K\phi_V(v)]_{\mathfrak{g}} - K[K[Ku, x]_{\mathfrak{g}}, K\phi_V(v)]_{\mathfrak{g}}\\
&= [\phi_{\mathfrak{g}}(x), [Ku, Kv]_{\mathfrak{g}}]_{\mathfrak{g}} - K(\rho^L(\phi_{\mathfrak{g}}(x))\rho^L(Ku)v) - K(\rho^L(\phi_{\mathfrak{g}}(x))\rho^R(Kv)u)\\
&\quad + K\rho^L(\phi_{\mathfrak{g}}(x))[Ku, Kv]_{\mathfrak{g}}) + K[\phi_{\mathfrak{g}}(x), K[u, v]_K]_{\mathfrak{g}}\\
&= [\phi_{\mathfrak{g}}(x), K[u, v]_K]_{\mathfrak{g}} - K(\rho^L(\phi_{\mathfrak{g}}(x))[u, v]_K) + K[\phi_{\mathfrak{g}}(x), K[u, v]_K]_{\mathfrak{g}}\\
&= \overline{\rho}^R([u, v]_K)\phi_{\mathfrak{g}}(x),
\end{aligned}
$$

which shows that

$$\overline{\rho}^R([u, v]_K) \circ \phi_{\mathfrak{g}} = \overline{\rho}^L(\phi_V(u))\overline{\rho}^R(v) - \overline{\rho}^R(\phi_V(v))\overline{\rho}^L(u).$$

Similarly, we can show that

$$\overline{\rho}^R([u, v]_K) \circ \phi_{\mathfrak{g}} = \overline{\rho}^R(\phi_V(v)) \circ \overline{\rho}^R(u) + \overline{\rho}^L(\phi_V(u)) \circ \overline{\rho}^R(v).$$

Therefore, $(\mathfrak{g}, \phi_{\mathfrak{g}}, \overline{\rho}^L, \overline{\rho}^R)$ is a representation of the Hom-Leibniz algebra $(V, [\cdot, \cdot]_K, \phi_V)$. □

It follows from the above proposition that we may consider the cohomology of the Hom-Leibniz algebra $(V, [\cdot, \cdot]_K, \phi_V)$ with coefficients in the representation $(\mathfrak{g}, \phi_{\mathfrak{g}}, \overline{\rho}^L, \overline{\rho}^R)$. More precisely, we define

$$C^n_K(V, \mathfrak{g}) := \mathrm{Hom}(V^{\otimes n}, \mathfrak{g}), \quad \text{for } n \geq 0$$

and the differential $\partial_K : C^n_K(V, \mathfrak{g}) \to C^{n+1}_K(V, \mathfrak{g})$ by

$$
\begin{aligned}
&(\partial_K f)(u_1, \ldots, u_{n+1})\\
&= \sum_{i=1}^{n}(-1)^{i+1}[K\phi_V^{n-1}(u_i), f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1})]_{\mathfrak{g}} - \sum_{i=1}^{n}(-1)^{i+1}K(\rho^R(f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}))\phi_V^{n-1}(u_i))\\
&\quad + \sum_{i=1}^{n}(-1)^{i+1}K[K\phi_V^{n-1}(u_i), f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1})]_{\mathfrak{g}} + (-1)^{n+1}[f(u_1, \ldots, u_n), K\phi_V^{n-1}(u_{n+1})]_{\mathfrak{g}}\\
&\quad + (-1)^n K(\rho^L(f(u_1, \ldots, u_n))\phi_V^{n-1}(u_{n+1})) - (-1)^n K[f(u_1, \ldots, u_n), K\phi_V^{n-1}(u_{n+1})]_{\mathfrak{g}}\\
&\quad + \sum_{1\leq i<j\leq n+1}(-1)^i f(\phi_V(u_1), \ldots, \widehat{\phi_V(u_i)}, \ldots, \phi_V(u_{j-1}), \rho^L(Ku_i)u_j + \rho^R(Ku_j)u_i\\
&\quad - [Ku_i, Ku_j]_{\mathfrak{g}}, \phi_V(u_{j+1}), \ldots, \phi_V(u_{n+1})),
\end{aligned}
$$

for $f \in C^n_K(V,\mathfrak{g})$ and $u_1,\dots,u_{n+1} \in V$. Then $\{C^*_K(V,\mathfrak{g}),\partial_K\}$ is a cochain complex. We denote by

$$Z^n_K(V,\mathfrak{g}) = \{f \in C^n_K(V,\mathfrak{g}) \mid \partial_K f = 0\} \quad \text{and} \quad B^n_K(V,\mathfrak{g}) = \{\partial_K g \mid g \in C^{n-1}_K(V,\mathfrak{g})\},$$

the spaces of n-cocycles and n-coboundaries, respectively. The corresponding quotients

$$H^n_K(V,\mathfrak{g}) := \frac{Z^n_K(V,\mathfrak{g})}{B^n_K(V,\mathfrak{g})}, \text{ for } n \geq 0$$

are called the cohomology of the Reynolds operator K.

5. Deformations of Reynolds operators

In this section, we will apply the classical deformation theory of Reynolds operators on Hom-Leibniz algebras.

5.1. Linear deformations

Let $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\phi_\mathfrak{g})$ be a Hom-Leibniz algebra, (V,ϕ_V,ρ^L,ρ^R) be a representation of it. Suppose $K : V \to \mathfrak{g}$ is a Reynolds operator.

Definition 5.1. *A linear map $K_1 : V \to \mathfrak{g}$ is said to generate a linear deformation of the Reynolds operator K if for all $t \in \mathbb{C}$, the sum $K_t = K + tK_1$ is still a Reynolds operator. In this case, $K_t = K + tK_1$ is said to be a linear deformation of K.*

Suppose K_1 generates a linear deformation of K. Then we have

$$K_t \circ \phi_V = \phi_\mathfrak{g} \circ K_t,$$
$$[K_tu, K_tv]_\mathfrak{g} = K_t\Big(\rho^L(K_tu)v + \rho^R(K_tv)u - [K_tu, K_tv]_\mathfrak{g}\Big), \text{ for } u, v \in V.$$

This is equivalent to the following conditions

$$K_1 \circ \phi_V = \phi_\mathfrak{g} \circ K_1, \tag{5. 1}$$
$$\begin{aligned}[Ku, K_1v]_\mathfrak{g} + [K_1u, Kv]_\mathfrak{g} = {} & K_1(\rho^L(Ku)v + \rho^R(Kv)u - [Ku, Kv]_\mathfrak{g}) \\ & + K(\rho^L(K_1u)v + \rho^R(K_1v)u - [K_1u, Kv]_\mathfrak{g} - [Ku, K_1v]_\mathfrak{g}),\end{aligned} \tag{5. 2}$$
$$[K_1u, K_1v]_\mathfrak{g} = K_1(\rho^L(K_1u)v + \rho^R(K_1v)u - [Ku, K_1v]_\mathfrak{g} - [K_1u, Kv]_\mathfrak{g}) - K[K_1u, K_1v]_\mathfrak{g}, \tag{5. 3}$$
$$K_1([K_1(u), K_1(v)]_\mathfrak{g}) = 0. \tag{5. 4}$$

Note that Eq. (4.2) means that K_1 is a 1-cocycle in the cohomology of K. Hence K_1 induces an element in $H^1_K(V,\mathfrak{g})$.

Definition 5.2. *Two linear deformations $K_t = K + tK_1$ and $K'_t = K + tK'_1$ of K are said to be equivalent if there exists an element $x \in \mathfrak{g}$ such that $\phi_\mathfrak{g}(x) = x$ and*

$$(\phi_t = \mathrm{Id}_\mathfrak{g} + tL_x,\ \psi_t = \mathrm{Id}_V + t(\rho^L(x) - [x, K-]_\mathfrak{g})$$

is a morphism of Reynolds operators from K_t to K'_t.

The condition that $\phi_t = \mathrm{Id}_\mathfrak{g} + tL_x$ is a Hom-Leibniz algebra morphism of $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\phi_\mathfrak{g})$ is equivalent to

$$[[x,y]_\mathfrak{g},[x,z]_\mathfrak{g}]_\mathfrak{g} = 0, \text{ for } y, z \in \mathfrak{g}. \tag{5. 5}$$

Further, the conditions $\psi_t(\rho^L(y)u) = \rho^L(\phi_t(y))\psi_t(u)$ and $\psi_t(\rho^R(y)u) = \rho^R(\phi_t(y))\psi_t(u)$, for $y \in \mathfrak{g}, u \in V$ are respectively equivalent to

$$\begin{cases}[x, K(\rho^L(y)u)]_\mathfrak{g} = \rho^L(y)[x, Ku]_\mathfrak{g}, \\ \rho^L([x,y])(\rho^L(x)u - [x, Ku]_\mathfrak{g}) = 0,\end{cases} \tag{5. 6}$$

$$\begin{cases}[x,K(\rho^R(y)u)]_{\mathfrak{g}}=\rho^R(y)[x,Ku]_{\mathfrak{g}},\\ \rho^R([x,y])(\rho^L(x)u-[x,Ku]_{\mathfrak{g}})=0.\end{cases} \tag{5. 7}$$

Similarly, the conditions $\psi_t([y,z]_{\mathfrak{g}})=[\phi_t(y),\phi_t(z)]_{\mathfrak{g}}$ and $\phi_t\circ K_t=K'_t\circ\psi_t$ are respectively equivalent to

$$\begin{cases}-\rho^L(x)[y,z]_{\mathfrak{g}}+[x,K[y,z]_{\mathfrak{g}}]_{\mathfrak{g}}=-[x,[y,z]_{\mathfrak{g}}]_{\mathfrak{g}}-[y,[x,z]_{\mathfrak{g}}]_{\mathfrak{g}},\\ [[x,y]_{\mathfrak{g}},[x,z]_{\mathfrak{g}}]_{\mathfrak{g}}=0,\end{cases} \tag{5. 8}$$

$$\begin{cases}K_1(u)+[x,Ku]_{\mathfrak{g}}=K(\rho^L(x)u-[x,Ku]_{\mathfrak{g}})+K'_1(u),\\ [x,K_1u]_{\mathfrak{g}}=K'_1(\rho^L(x)u-[x,Ku]_{\mathfrak{g}}).\end{cases} \tag{5. 9}$$

It follows from the first identity in (5. 9) that $K_1(u)-K'_1(u)=\partial_K(x)(u)$. Hence we obtain the following result.

Theorem 5.3. *If two linear deformations $K_t=K+tK_1$ and $K'_t=K+tK'_1$ of a Reynolds operator K are equivalent, then K_1 and K'_1 are in the same cohomology class of $H^1_K(V,\mathfrak{g})$.*

Definition 5.4. *A linear deformation $K_t=K+tK_1$ of a Reynolds operator K is said to be trivial if K_t is equivalent to the undeformed deformation $K'_t=K$.*

We will now define Nijenhuis elements associated with a Reynolds operator K in a way that a trivial deformation of K induces a Nijenhuis element.

Definition 5.5. *Let K be a Reynolds operator. An element $x\in\mathfrak{g}$ such that $\phi_{\mathfrak{g}}(x)=x$ is called a Nijenhuis element associated with K if x satisfies*

$$[x,\overline{\rho}^R(u)(x)]_{\mathfrak{g}}=0,\quad \textit{for } u\in V$$

and Equations (5. 5), (5. 6), (5. 7), (5. 8) hold.

The set of all Nijenhuis elements associated with K is denoted by $\mathrm{Nij}(K)$. As mentioned earlier that a trivial deformation induces a Nijenhuis element. In the next subsection, we give a sufficient condition for the rigidity of a Reynolds operator in terms of Nijenhuis elements.

5.2. Formal deformations

Let $\mathbb{C}[[t]]$ be the ring of power series in one variable t. For any $\mathbb{C}$-linear space V, let $V[[t]]$ denote the vector space of formal power series in t with coefficients in V. Moreover, if $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ is a Hom-Leibniz algebra over $\mathbb{C}$, then one can extend the Hom-Leibniz bracket on $\mathfrak{g}[[t]]$ by $\mathbb{C}[[t]]$-bilinearity. Furthermore, if (V,ϕ_V,ρ^L,ρ^R) is a representation of the Hom-Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$, then there is a representation $(V[[t]],\phi_V,\rho^L,\rho^R)$ of the Hom-Leibniz algebra $\mathfrak{g}[[t]]$. Here, ρ^L and ρ^R are also extended by $\mathbb{C}[[t]]$-bilinearity.

Let $K:V\to\mathfrak{g}$ be a Reynolds operator on the Hom-Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ with respect to the representation (V,ϕ_V,ρ^L,ρ^R). We consider a power series of the form

$$K_t=\sum_{i=0}^{+\infty}K_it^i,\quad \text{for } K_i\in\mathrm{Hom}(V,\mathfrak{g})\ \text{ with } K_0=K.$$

Extend K_t to a linear map from $V[[t]]$ to $\mathfrak{g}[[t]]$ by $\mathbb{C}[[t]]$-linearity, which we still denote by K_t.

Definition 5.6. *A formal deformation of K is given by a formal power series $K_t=\sum_{i=0}^{+\infty}K_it^i$ with $K_0=K$ satisfying*

$$K_t\circ\phi_V=\phi_{\mathfrak{g}}\circ K_t, \tag{5. 10}$$

$$[K_tu,K_tv]_{\mathfrak{g}}=K_t\Big(\rho^L(K_tu)v+\rho^R(K_tv)u-[K_t(u),K_t(v)]_{\mathfrak{g}}\Big),\quad \textit{for } u,v\in V. \tag{5. 11}$$

It follows that K_t is a Reynolds operator on the Hom-Leibniz algebra $\mathfrak{g}[[t]]$ with respect to the representation $V[[t]]$.

Remark 5.7. *If $K_t = \sum_{i=0}^{+\infty} K_i t^i$ is a formal deformation of the Reynolds operator K on the Hom-Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ with respect to the representation $(V, \phi_V, \rho^L, \rho^R)$, then $[\cdot,\cdot]_{K_t}$ defined by*

$$[u,v]_{K_t} := \sum_{i=0}^{+\infty} \Big(\rho^L(K_i u)v + \rho^R(K_i v)u - \sum_{j+k=i} [K_j u, K_k v]_{\mathfrak{g}}\Big) t^i, \ \ \textit{for } u, v \in V,$$

is a formal deformation of the associated Hom-Leibniz algebra $(V, [\cdot,\cdot]_K, \phi_V)$.

By expanding the identity (5. 11) and comparing coefficients of various powers of t, we obtain for $n \geq 0$,

$$\sum_{i+j=n} [K_i u, K_j v]_{\mathfrak{g}} = \sum_{i+j=n} K_i(\rho^L(K_j u)v + \rho^R(K_j v)u) - \sum_{i+j+k=n} K_i[K_j(u), K_k(v)]_{\mathfrak{g}},$$

for $u, v \in V$. It holds for $n = 0$ as K is a Reynolds operator. For $n = 1$, we obtain

$$\begin{aligned}[Ku, K_1 v]_{\mathfrak{g}} + [K_1 u, Kv]_{\mathfrak{g}} = \ & K_1(\rho^L(Ku)v + \rho^R(Kv)u - [Ku, Kv]_{\mathfrak{g}}) \\ & + K(\rho^L(K_1 u)v + \rho^R(K_1 v)u - [K_1(u), Kv]_{\mathfrak{g}} - [K(u), K_1 v]_{\mathfrak{g}}).\end{aligned}$$

This condition is equivalent to $(\partial_K(K_1))(u,v) = 0$, for $u, v \in V$.

Next, we define an equivalence between two formal deformations of a Reynolds operator.

Definition 5.8. *Two formal deformations $K_t = \sum_{i=0}^{+\infty} K_i t^i$ and $K'_t = \sum_{i=0}^{+\infty} K'_i t^i$ of a Reynolds operator K are said to be equivalent if there exists an element $x \in \mathfrak{g}$ such that $\phi_{\mathfrak{g}}(x) = x$, linear maps $\phi_i \in gl(\mathfrak{g})$ and $\psi_i \in gl(V)$ for $i \geq 2$ such that the pair*

$$\Big(\phi_t = \mathrm{Id}_{\mathfrak{g}} + tL_x + \sum_{i=2}^{+\infty} \phi_i t^i, \ \psi_t = \mathrm{Id}_V + t(\rho^L(x) - [x, K-]_{\mathfrak{g}}) + \sum_{i=2}^{+\infty} \psi_i t^i\Big)$$

is a morphism of Reynolds operators from K_t to K'_t.

By equating coefficients of t from both sides of the identity $\phi_t \circ K_t = K'_t \circ \psi_t$, we obtain

$$K_1(u) - K'_1(u) = K(\rho^L(x)u - [x, Ku]_{\mathfrak{g}}) - [x, Ku]_{\mathfrak{g}} = \partial_K(x)(u), \ \text{for } u \in V.$$

As a summary, we get the following result.

Theorem 5.9. *The linear term of a formal deformation of a Reynolds operator K is a* 1*-cocycle in the cohomology of K, and the corresponding cohomology class depends only on the equivalence class of the deformation of K.*

Definition 5.10. *A Reynolds operator K is said to be rigid if any formal deformation of K is equivalent to the undeformed deformation $K'_t = K$.*

In the next theorem, we give a sufficient condition for the rigidity of a Reynolds operator in terms of Nijenhuis elements.

Theorem 5.11. *Let K be a Reynolds operator. If $Z^1_K(V, \mathfrak{g}) = \partial_K(\mathrm{Nij}(K))$, then K is rigid.*

Proof. Let $K_t = \sum_{i=0}^{+\infty} K_i t^i$ be any formal deformation of K. I t follows from Theorem 5.9 that the linear term K_1 is a 1-cocycle in the cohomology of K, i.e., $K_1 \in Z^1_K(V, \mathfrak{g})$. Thus, by the hypothesis, there is a Nijenhuis element $x \in \mathrm{Nij}(K)$ such that $K_1 = -\partial_K(x)$. We take

$$\phi_t = \mathrm{Id}_{\mathfrak{g}} + tL_x \quad \text{and} \quad \psi_t = \mathrm{Id}_V + t(\rho^L(x) - [x, K-]_{\mathfrak{g}}),$$

and define $K'_t = \phi_t \circ K_t \circ \psi_t^{-1}$. Then K'_t is a formal deformation equivalent to K_t. For $u \in V$, we observe that

$$\begin{aligned} K'_t(u) &= (\mathrm{Id}_{\mathfrak{g}} + tL_x)(K_t(u - t\rho^L(x)u + t[x, Ku]_{\mathfrak{g}} + \text{ power of } t^{\geq 2})) \\ &= K(u) + t(K_1 u - K\rho^L(x)u + K[x, Ku]_{\mathfrak{g}} + [x, Ku]_{\mathfrak{g}}) + \text{ power of } t^{\geq 2}. \\ &= K(u) + t^2 K'_2(u) + \cdots \qquad (\text{as } K_1 = -\partial_K(x)). \end{aligned}$$

Hence the coefficient of t in the expression of K'_t is trivial. Applying the same process repeatedly, we get that K_t is equivalent to K. Therefore, K is rigid. □

6. NS-Hom-Leibniz algebras

In this section, we introduce NS-Hom-Leibniz algebras as the underlying structure of Reynolds operators. We study some properties of NS-Hom-Leibniz algebras and give some examples.

Definition 6.1. *An NS-Hom-Leibniz algebra is a quintuple $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ consisting of a vector space A together with three bilinear operations $\triangleright, \triangleleft, \diamond : A \otimes A \to A$ and an algebra homomorphism $\phi_A : A \to A$ satisfying for all $x, y, z \in A$,*

$$\begin{aligned} &(A1) \qquad \phi_A(x) \triangleright (y * z) = (x \triangleright y) \triangleright \phi_A(z) + \phi_A(y) \triangleleft (x \triangleright z), \\ &(A2) \qquad \phi_A(x) \triangleleft (y \triangleright z) = (x \triangleleft y) \triangleright \phi_A(z) + \phi_A(y) \triangleright (x * z), \\ &(A3) \qquad \phi_A(x) \triangleleft (y \triangleleft z) = (x * y) \triangleleft \phi_A(z) + \phi_A(y) \triangleleft (x \triangleleft z), \\ &(A4) \qquad \phi_A(x) \triangleleft (y \diamond z) + \phi_A(x) \diamond (y * z) = (x \diamond y) \triangleright \phi_A(z) + (x * y) \diamond \phi_A(z) \\ &\qquad\qquad + \phi_A(y) \triangleleft (x \diamond z) + \phi_A(y) \diamond (x * z), \end{aligned}$$

where $x * y = x \triangleright y + x \triangleleft y + x \diamond y$.

NS-Hom-Leibniz algebras are more general than NS-Leibniz algebras introduced in [9]. More precisely, an NS-Hom-Leibniz algebra $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ in which $\phi_A = Id$ is an NS-Leibniz algebra.

In the following, we show that NS-Hom-Leibniz algebras split Hom-Leibniz algebras.

Proposition 6.2. *Let $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ be an NS-Hom-Leibniz algebra. Then the vector space A with the bilinear operation*

$$[\cdot, \cdot]_* : A \otimes A \to A, \ [x, y]_* := x * y$$

is a Hom-Leibniz algebra.

Proof. By summing up the left hand sides of the identities (A1)-(A4), we get $[\phi_A(x), [y, z]_*]_*$. On the other hand, by summing up the right hand sides of the identities (A1)-(A4), we have $[[x, y]_*, \phi_A(z)]_* + [\phi_A(y), [x, z]_*]_*$. Hence the result follows. □

Proposition 6.3. *Let (A, ϕ_A) be a Hom-associative algebra and $P : A \to A$ be a linear map satisfying $P(x)P(y) = P(P(x)y) = P(xP(y))$ and $\phi_A \circ P = P \circ \phi_A$, for any $x, y \in A$. Define bilinear operations $\triangleright, \triangleleft, \diamond : A \otimes A \to A$ by*

$$x \triangleright y = -yP(x), x \triangleleft y = P(x)y, \text{ and } x \diamond y = 0, \text{ for } x, y \in A.$$

Then $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ is an NS-Hom-Leibniz algebra.

Proof. For any $x, y, z \in A$, we have

$$\begin{aligned} (x \triangleright y) \triangleright \phi_A(z) + \phi_A(y) \triangleleft (x \triangleright z) &= \phi_A(z)P(yP(x)) - P(\phi_A(y))(zP(x)) \\ &= (zP(y))P(\phi_A(x)) - (P(y)z)P(\phi_A(x)) \\ &= \phi_A(x) \triangleright (y * z). \end{aligned}$$

Also, we have

$$\begin{aligned}\phi_A(x) \triangleleft (y \triangleright z) - (x \triangleleft y) \triangleright \phi_A(z) &= -P(\phi_A(x))(zP(y)) + \phi_A(z)P(P(x)y)\\ &= -P(x)(zP(y)) + (zP(x))P(\phi_A(y))\\ &= \phi_A(y) \triangleright (x * z).\end{aligned}$$

Thus (A1) and (A2) hold. Similarly, we can check that (A3) and (A4) hold obviously. This completes the proof. □

NS-Hom-Leibniz algebras also arise from weighted Rota-Baxter operators on Hom-Leibniz algebras. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Leibniz algebra. A linear map $T : \mathfrak{g} \to \mathfrak{g}$ is said to be a Rota-Baxter operator of weight λ on the Hom-Leibniz algebra if T satisfies

$$\begin{aligned}&T \circ \phi_{\mathfrak{g}} = \phi_{\mathfrak{g}} \circ T,\\ &[Tx, Ty]_{\mathfrak{g}} = T([Tx, y]_{\mathfrak{g}} + [x, Ty]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}), \text{ for } x, y \in \mathfrak{g}.\end{aligned}$$

Note that the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$ is a Rota-Baxter operator of weight -1. If T is a Rota-Baxter operator of weight λ, then $-\lambda\mathrm{Id} - T$ is so. In the following result, we show that Rota-Baxter operators of weight λ induce NS-Hom-Leibniz algebras.

Proposition 6.4. *Let $T : \mathfrak{g} \to \mathfrak{g}$ be a Rota-Baxter operator of weight λ on the Hom-Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$. Then there is an NS-Hom-Leibniz algebra structure on the vector space $\mathfrak{g}$ with bilinear operations*

$$x \triangleright y = [x, Ty]_{\mathfrak{g}},\ x \triangleleft y = [Tx, y]_{\mathfrak{g}} \text{ and } x \diamond y = \lambda[x, y]_{\mathfrak{g}}, \quad \text{for } x, y \in \mathfrak{g}.$$

Proof. For any $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}(x \triangleright y) \triangleright \phi_{\mathfrak{g}}(z) + \phi_{\mathfrak{g}}(y) \triangleleft (x \triangleright z) &= [[x, Ty]_{\mathfrak{g}}, T\phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [T\phi_{\mathfrak{g}}(y), [x, Tz]_{\mathfrak{g}}]_{\mathfrak{g}}\\ &= [\phi_{\mathfrak{g}}(x), [Ty, Tz]_{\mathfrak{g}}]_{\mathfrak{g}}\\ &= [\phi_{\mathfrak{g}}(x), T(y * z)]_{\mathfrak{g}}\\ &= \phi_{\mathfrak{g}}(x) \triangleright (y * z).\end{aligned}$$

Also,

$$\begin{aligned}\phi_{\mathfrak{g}}(x) \triangleleft (y \triangleright z) - (x \triangleleft y) \triangleright \phi_{\mathfrak{g}}(z) &= [T\phi_{\mathfrak{g}}(x), [y, Tz]_{\mathfrak{g}}]_{\mathfrak{g}} - [[x, Ty]_{\mathfrak{g}}, T\phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}\\ &= [\phi_{\mathfrak{g}}(y), [Tx, Tz]_{\mathfrak{g}}]_{\mathfrak{g}}\\ &= [\phi_{\mathfrak{g}}(y), T(x * z)]_{\mathfrak{g}}\\ &= \phi_{\mathfrak{g}}(y) \triangleright (x * z).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\phi_{\mathfrak{g}}(x) \triangleleft (y \triangleleft z) - \phi_{\mathfrak{g}}(y) \triangleleft (x \triangleleft z) &= [T\phi_{\mathfrak{g}}(x), [Ty, z]_{\mathfrak{g}}]_{\mathfrak{g}} - [T\phi_{\mathfrak{g}}(y), [Tx, z]_{\mathfrak{g}}]_{\mathfrak{g}}\\ &= [[Tx, Ty]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}\\ &= [T(x * y), \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}\\ &= (x * y) \triangleleft \phi_{\mathfrak{g}}(z).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&(x \diamond y) \triangleright \phi_{\mathfrak{g}}(z) + (x * y) \diamond \phi_{\mathfrak{g}}(z) + \phi_{\mathfrak{g}}(y) \triangleleft (x \diamond z) + \phi_{\mathfrak{g}}(y) \diamond (x * z)\\ =\ &\lambda[[x, y]_{\mathfrak{g}}, T\phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + \lambda[[Tx, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + \lambda[[x, Ty], \phi_{\mathfrak{g}}(z)] + \lambda^2[[x, y], \phi_{\mathfrak{g}}(z)]\\ &+\lambda[T\phi_{\mathfrak{g}}(y), [x, z]] + \lambda[\phi_{\mathfrak{g}}(y), [Tx, z]_{\mathfrak{g}}]_{\mathfrak{g}} + \lambda[\phi_{\mathfrak{g}}(y), [x, Tz]] + \lambda^2[\phi_{\mathfrak{g}}(y), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}\\ =\ &\lambda[T\phi_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} + \lambda[\phi_{\mathfrak{g}}(x), [y, Tz]]_{\mathfrak{g}}]_{\mathfrak{g}} + \lambda[\phi_{\mathfrak{g}}(x), [Ty, z]_{\mathfrak{g}}]_{\mathfrak{g}} + \lambda^2[\phi_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}}\\ =\ &\phi_{\mathfrak{g}}(x) \triangleleft (y \diamond z) + \phi_{\mathfrak{g}}(x) \diamond (y * z).\end{aligned}$$

This completes the proof. □

The Hom-Leibniz algebra $(A, [\cdot,\cdot]_*, \phi_A)$ of the above proposition is called the subadjacent Hom-Leibniz algebra of $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ and $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ is called a compatible NS-Hom-Leibniz algebra structure on $(A, [\cdot,\cdot]_*, \phi_A)$.

Proposition 6.5. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Leibniz algebra and $N : \mathfrak{g} \to \mathfrak{g}$ be a Nijenhuis operator on it. Then the bilinear operations*

$$x \triangleright y = [x, Ny]_{\mathfrak{g}}, \quad x \triangleleft y = [Nx, y]_{\mathfrak{g}} \quad \textit{and} \quad x \diamond y = -N[x, y]_{\mathfrak{g}}, \quad \textit{for } x, y \in \mathfrak{g}$$

defines an NS-Hom-Leibniz algebra structure on $\mathfrak{g}$.

Proof. For any $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}\phi_{\mathfrak{g}}(x) \triangleright (y * z) = [\phi_{\mathfrak{g}}(x), N(y * z)] &= [\phi_{\mathfrak{g}}(x), [Ny, Nz]] \\ &= [[x, Ny], N\phi_{\mathfrak{g}}(z)] + [N\phi_{\mathfrak{g}}(y), [x, Nz]] \\ &= (x \triangleright y) \triangleright \phi_{\mathfrak{g}}(z) + \phi_{\mathfrak{g}}(y) \triangleleft (x \triangleright z).\end{aligned}$$

Hence the identity (A1) of Definition 6.1 holds. Similarly, we get

$$\begin{aligned}\phi_{\mathfrak{g}}(x) \triangleleft (y \triangleright z) = [N\phi_{\mathfrak{g}}(x), [y, Nz]] &= [[Nx, y], N\phi_{\mathfrak{g}}(z)] + [\phi_{\mathfrak{g}}(y), [Nx, Nz]] \\ &= (x \triangleleft y) \triangleright \phi_{\mathfrak{g}}(z) + \phi_{\mathfrak{g}}(y) \triangleright (x * z),\end{aligned}$$

and

$$\begin{aligned}\phi_{\mathfrak{g}}(x) \triangleleft (y \triangleleft z) = [N\phi_{\mathfrak{g}}(x), [Ny, z]] &= [[Nx, Ny], \phi_{\mathfrak{g}}(z)] + [N\phi_{\mathfrak{g}}(y), [Nx, z]] \\ &= (x * y) \triangleleft \phi_{\mathfrak{g}}(z) + \phi_{\mathfrak{g}}(y) \triangleleft (x \triangleleft z).\end{aligned}$$

Therefore, the identities (A2) and (A3) also hold. To prove the identity (A4), we first recall from [5] that the given Hom-Leibniz bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and the deformed Hom-Leibniz bracket $[\cdot,\cdot]_N$ are compatible in the sense that their sum also defines a Hom-Leibniz bracket on $\mathfrak{g}$. This is equivalent to the fact that

$$\begin{aligned}&[\phi_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}]_N + [\phi_{\mathfrak{g}}(x), [y, z]_N]_{\mathfrak{g}} \\ &= [[x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_N + [[x, y]_N, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), [x, z]_{\mathfrak{g}}]_N + [\phi_{\mathfrak{g}}(y), [x, z]_N]_{\mathfrak{g}},\end{aligned} \tag{6. 1}$$

for $x, y, z \in \mathfrak{g}$. The identity (A4) of Definition 6.1 simply follows from (6. 1). Hence $(\mathfrak{g}, \phi_{\mathfrak{g}}, \triangleright, \triangleleft, \diamond)$ is an NS-Hom-Leibniz algebra. □

Let $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ be an NS-Hom-Leibniz algebra. Define two linear maps $L_\triangleleft : A \to gl(A)$, $R_\triangleright : A \to gl(A)$ by

$$L_\triangleleft(x)y = x \triangleleft y, \quad R_\triangleright(x)y = y \triangleright x \quad \text{and} \quad [x, y]_* = -x \diamond y, \quad \text{for } x, y \in A.$$

With these notations, we have the following result.

Proposition 6.6. *Let $(A, \phi_A, \triangleright, \triangleleft, \diamond)$ be an NS-Hom-Leibniz algebra. Then $(A, \phi_A, L_\triangleleft, R_\triangleright)$ is a representation of the subadjacent Hom-Leibniz algebra $(A, [\cdot,\cdot]_*, \phi_A)$. Moreover, the identity map $\mathrm{Id} : A \to A$ is a Reynolds operator on the Hom-Leibniz algebra $(A, [\cdot,\cdot]_*, \phi_A)$ with respect to the representation $(A, \phi_A, L_\triangleleft, R_\triangleright)$.*

Proof. For any $x, y, z \in A$, we have

$$\begin{aligned}L_\triangleleft([x, y]_*)\phi_A(z) = [x, y]_* \triangleleft \phi_A(z) &\overset{(A3)}{=} \phi_A(x) \triangleleft (y \triangleleft z) - \phi_A(y) \triangleleft (x \triangleleft z) \\ &= \Big(L_\triangleleft(\phi_A(x)) \circ L_\triangleleft(y) - L_\triangleleft(\phi_A(y)) \circ L_\triangleleft(x)\Big)z.\end{aligned}$$

Similarly, we get

$$\begin{aligned}R_\triangleright([x, y]_*)\phi_A(z) = \phi_A(z) \triangleright [x, y]_* &\overset{(A2)}{=} \phi_A(x) \triangleleft (z \triangleright y) - (x \triangleleft z) \triangleright \phi_A(y) \\ &= L_\triangleleft(\phi_A(x)) \circ R_\triangleright(y)z - R_\triangleright(\phi_A(y)) \circ L_\triangleleft(x)z,\end{aligned}$$

and

$$R_{\triangleright}([x,y]_*)\phi_A(z) = \phi_A(z) \triangleright [x,y]_* \overset{(A1)}{=} (z \triangleright x) \triangleright \phi_A(y) + \phi_A(x) \triangleleft (z \triangleright y) = \Big(R_{\triangleright}(\phi_A(y)) \circ R_{\triangleright}(x) + L_{\triangleleft}(\phi_A(x))R_{\triangleright}(y)\Big)z.$$

Therefore, $(A, \phi_A, L_{\triangleleft}, R_{\triangleright})$ is a representation of the subadjacent Hom-Leibniz algebra $(A, [\cdot,\cdot]_*, \phi_A)$. Finally, we have

$$\mathrm{Id}(L_{\triangleleft}(\mathrm{Id}\, x)y + R_{\triangleright}(\mathrm{Id}\, y)x - [\mathrm{Id}\, x, \mathrm{Id}\, y]_*) = x \triangleleft y + x \triangleright y + x \circ y = [\mathrm{Id}\, x, \mathrm{Id}\, y]_*,$$

which shows that $\mathrm{Id} : A \to A$ is a Reynolds operator on the Hom-Leibniz algebra $(A, [\cdot,\cdot]_*, \phi_A)$ with respect to the representation $(A, \phi_A, L_{\triangleleft}, R_{\triangleright})$. □

Proposition 6.7. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Leibniz algebra and $(V, \phi_V, \rho^L, \rho^R)$ be a representation. Let $K : V \to \mathfrak{g}$ be a Reynolds operator. Then there is an NS-Hom-Leibniz algebra structure on V with bilinear operations given by*

$$u \triangleright v := \rho^R(Kv)u, \quad u \triangleleft v := \rho^L(Ku)v \quad \text{and} \quad u \diamond v := -[Ku, Kv]_{\mathfrak{g}}, \quad \text{for } u, v \in V.$$

Proof. For any $u, v, w \in V$, we have

$$\begin{aligned}\phi_V(u) \triangleright (v * w) = \rho^R(K(v * w))\phi_V(u) &= \rho^R([Kv, Kw])\phi_V(u)\\ &= \rho^L(K\phi_V(v))\rho^R(Kw)u + \rho^R(K\phi_V(w))\rho^R(Kv)u\\ &= \phi_V(v) \triangleleft (u \triangleright w) + (u \triangleright v) \triangleright \phi_V(w).\end{aligned}$$

Similarly, we get

$$\begin{aligned}\phi_V(u) \triangleleft (v \triangleright w) = \rho^L(K\phi_V(u))\rho^R(Kw)v &= \rho^R([Ku, Kz])\phi_V(v) + \rho^R(K\phi_V(w))\rho^L(Ku)v\\ &= \phi_V(v) \triangleright (u * w) + (u \triangleleft v) \triangleright \phi_V(w),\end{aligned}$$

and

$$\begin{aligned}\phi_V(u) \triangleleft (v \triangleleft w) = \rho^L(K\phi_V(u))\rho^L(Kv)(w) &= \rho^L([Ku, Kv])\phi_V(w) + \rho^L(K\phi_V(v))\rho^L(Ku)w\\ &= (u * v) \triangleleft \phi_V(w) + \phi_V(v) \triangleleft (u \triangleleft w).\end{aligned}$$

Hence (A1), (A2) and (A3) of Definition 6.1 hold. Since $(\partial[\cdot,\cdot]_{\mathfrak{g}})(Ku, Kv, Kz) = 0$, i.e.,

$$\begin{aligned}&\rho^L(K\phi_V(u))[Kv, Kw]_{\mathfrak{g}} - \rho^L(K\phi_V(v))[Ku, Kw]_{\mathfrak{g}} - \rho^R(K\phi_V(w))[Ku, Kv]_{\mathfrak{g}}\\ &- [[Ku, Kv]_{\mathfrak{g}}, K\phi_V(w)]_{\mathfrak{g}} - [K\phi_V(v), [Ku, Kw]_{\mathfrak{g}}]_{\mathfrak{g}} + [K\phi_V(u), [Kv, Kw]_{\mathfrak{g}}]_{\mathfrak{g}} = 0.\end{aligned}$$

This is equivalent to the condition (A4) of Definition 6.1. Hence the proof is completed. □

Remark 6.8. *The subadjacent Hom-Leibniz algebra of the NS-Hom-Leibniz algebra constructed in Proposition 6.7 is given by*

$$[u, v]_* = \rho^L(Ku)v + \rho^R(Kv)u - [Ku, Kv]_{\mathfrak{g}}, \quad \text{for } u, v \in V.$$

This Hom-Leibniz algebra structure on V coincides with the one given in Proposition 3.6.

In the following, we give a necessary and sufficient condition for the existence of a compatible NS-Hom-Leibniz algebra structure on a Hom-Leibniz algebra.

Proposition 6.9. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Leibniz algebra. Then there is a compatible NS-Hom-Leibniz algebra structure on $\mathfrak{g}$ if and only if there exists an invertible Reynolds operator $K : V \to \mathfrak{g}$ on $\mathfrak{g}$ with respect to a representation $(V, \phi_V, \rho^L, \rho^R)$. Furthermore, the compatible NS-Hom-Leibniz algebra structure on $\mathfrak{g}$ is given by*

$$x \triangleright y := K(\rho^R(y)K^{-1}x), \quad x \triangleleft y := K(\rho^L(x)K^{-1}y) \quad \text{and} \quad x \diamond y = -K[x, y]_{\mathfrak{g}}, \quad \text{for } x, y \in \mathfrak{g}.$$

Proof. Let $K : V \to \mathfrak{g}$ be an invertible Reynolds operator on $\mathfrak{g}$ with respect to a representation $(V, \phi_V, \rho^L, \rho^R)$. By Proposition 6.7, there is an NS-Hom-Leibniz algebra structure on V given by

$$u \,\bar{\triangleright}\, v := \rho^R(Kv)u, \quad \bar{\triangleleft}\, v := \rho^L(Ku)v \quad \text{and} \quad u \,\bar{\diamond}\, v := -[Ku, Kv]_{\mathfrak{g}}, \quad \text{for } u, v \in V.$$

Since K is an invertible map, the bilinear operations

$$\begin{aligned} x \triangleright y &:= K(K^{-1}x \,\bar{\triangleright}\, K^{-1}y) = K(\rho^R(y)K^{-1}x), \\ x \triangleleft y &:= K(K^{-1}x \,\bar{\triangleleft}\, K^{-1}y) = K(\rho^L(x)K^{-1}y), \\ x \diamond y &:= K(K^{-1}x \,\bar{\diamond}\, K^{-1}y) = -K[x, y]_{\mathfrak{g}}, \quad \text{for } x, y \in \mathfrak{g}, \end{aligned}$$

defines an NS-Hom-Leibniz algebra on $\mathfrak{g}$. Moreover, we have

$$\begin{aligned} & x \triangleright y + x \triangleleft y + x \diamond y \\ &= K(\rho^R(y)K^{-1}x) + K(\rho^L(x)K^{-1}y) - K[x, y]_{\mathfrak{g}} \\ &= K(\rho^R(K \circ K^{-1}y)K^{-1}x) + K(\rho^L(K \circ K^{-1}x)K^{-1}y) - K[K \circ K^{-1}x, K \circ K^{-1}y]_{\mathfrak{g}} \\ &= [K \circ K^{-1}x, K \circ K^{-1}y]_* = [x, y]_*. \end{aligned}$$

Conversely, let $(\mathfrak{g}, \phi_{\mathfrak{g}}, \triangleright, \triangleleft, \diamond)$ be a compatible NS-Hom-Leibniz algebra structure on $\mathfrak{g}$. By Proposition 6.6, $(\mathfrak{g}, \phi_{\mathfrak{g}}, L_{\triangleleft}, R_{\triangleright})$ is a representation of the Hom-Leibniz algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$, and the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$ is a Reynolds operator on the Hom-Leibniz algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ with respect to the representation $(\mathfrak{g}, \phi_{\mathfrak{g}}, L_{\triangleleft}, R_{\triangleright})$. Hence the proof is finished. □

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