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BMO-space for non-absolute integrable functions

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Abstract. The functions with bounded mean oscillation (BMO) have been shown to be immense interest in several areas of analysis and probability. We introduce BMO-type space $BMO_{HK}(\mathbb{R}^n)$ for non-absolute integrable functions. Various properties and completion of $BMO_{HK}(\mathbb{R}^n)$ are included. Relations between the classical BMO space and $BMO_{HK}(\mathbb{R}^n)$ are investigated.

1. Introduction

Around 1962, F. John and L. Nirenberg introduced the class of functions with bounded mean oscillation, in view of its apparent interest in the Real Analysis as well as in Partial Differential Equations (see [14]). Chaeles Fefferman gave important link between BMO and Harmonic Analysis in several real variables. One can find *BMO* spaces in various areas of analysis like function theory, harmonic analysis, PDEs. BMO-spaces are consider as applicable replacement for $\mathcal{L}^\infty$. *BMO* spaces are applicable to conserve a broad class of dominant operators such as the Hardy-Littlewood maximal function, Hilbert transform. Various properties of *BMO* as well as for detailed of *BMO* spaces (see [19–22]). BMO spaces are not naturally Banach space. Umberto Neri [18] studied the completeness of the BMO spaces. In recent times Kwok-Pun-Ho, Lucas Chaffee, Peng Chun, Yanchang Han, Rodolfo H. Torres, Lesley A. Ward, Vagif S. Guliyer, Fatih Deringoz, Sabir G. Hasanov studied the characterizations of BMO and Lipschitz spaces by rearrangement-invariant Banach function spaces also provided the sharp function characterization of the rearrangement-invariant Banach function spaces. The commutators of bilinear Calderón-Zygmund operators and pointwise multiplication with a symbol in CMO are bilinear compact operators on products of Lebesgue spaces (see [4, 13]). The necessary and sufficient conditions of boundedness of the commutators of Riesz potential operator on Orlicz spaces when measurable functions are belongs to the BMO and Lipschitz spaces, respectively (see [11]). T.X. Duong and L. Yan instigated a new spaces $BMO_{\mathcal{L}}$, analogous with $\mathcal{L}$ being a generator of a semigroup fulfilling the Gaussian upper bounds (see [7, 8]). D. Deng et.al. [6] investigated the applications of the new $BMO_{\mathcal{L}}$ space in the theory of singular integration. They obtained $BMO_{\mathcal{L}}$ estimates as well as interpolation results for fractional powers also purely imaginary powers and spectral multipliers of self adjoint operators, also demonstrated that the space $BMO_{\mathcal{L}}$ might coincide with or might be essentially different from the classical BMO space.

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In 1957, Jaroslav Kurzweil discussed about a new integral in one of his publication, while unaware of the work of Kurzweil, Ralph Henstock published an article on integration theory in which he discussed the same integration as J. Kurzweil. This new integral can integrate a substantial class of functions than the Riemann or Lebesgue integral. In the honors of these mathematicians, nowadays this integral is called Henstock-Kurzweil integral in short HK-integral. Measure theory is not essential in the definition of HK-integral. In quantum theory and nonlinear analysis, HK-integrals are aid for highly oscillatory functions to integrate. Moreover, HK integrability encloses improper integrals (see [1–3, 10, 12, 15, 17]). Tepper L.Gill and W.W. Zachary in [9] introduced a class of Banach space of Henstock-Kurzweil integral. They called this space as Kuelbs-Steadman space in short $\mathbb{K}S^p(\mathbb{R}^n)$. In Feynman operator Calculus, path integral $\mathbb{K}S^p(\mathbb{R}^n)$ is directly related. We motivate the work of Donggao Deng, Xuan Thinh Duong, Adam Sikora and Lixin Yan of [6] to introduce a new BMO type space and investigate the classical BMO space should be a subset as a continuous embedding.

In Section 2, we survey $\mathbb{K}S^p(\mathbb{R}^n)$ of [9] with a new norm that can be generated from weighted l^p spaces. We found this new norm is equivalent with the norm given by T. L. Gill and W.W. Zachary for their study. We introduce a new space $\mathbb{K}S^\infty_c(\mathbb{R})$ and study $\mathbb{K}S^\infty_c(\mathbb{R})$ is contained in locally Henstock-Kurzweil integrable function spaces like $\mathcal{L}^\infty_c(\mathbb{R}^n)$ defined in the Section 3.2 of [16] for their purpose. We define the locally Henstock-Kurzweil integrable function in the Definition 2.2 of the preliminaries section.

2. Preliminaries

Throughout the article, we assume $\overline{\Omega} \neq \emptyset$ is an abstract space. The class of all subsets of $\overline{\Omega}$ will be denoted by $P(\overline{\Omega})$, Σ is $\sigma-$algebra.

Definition 2.1. *[2] For a function $\xi : [a,b] \subset \overline{\Omega} \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable (briefly, Henstock integrable) on a set $A_o \in \Sigma$ if there is an element $\mathbb{I}_{A_o} \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a gauge δ on A_o with $|S(\xi, \mathfrak{D}) - \mathbb{I}_{A_o}| \leq \varepsilon$ whenever $\mathfrak{D}$ is a δ-fine partition of A_o such that $S(\xi, \mathfrak{D})$ exists in $\mathbb{R}$.*

Let $HK(\overline{\Omega})$ be the space of Henstock-Kurzweil integrable functions over $\overline{\Omega}$. Recalling the Henstock-Kurzweil integral properly contains the union of $\mathcal{L}^1$ and the Cauchy-Lebesgue integrable functions. It is very unfortunate that $HK(\overline{\Omega})$ is not a Banach space with Alexiewicz norm (see [1–3, 9]). We introduce the locally Henstock-Kurzweil integrable functions as follows:

Definition 2.2. *A measurable function $\xi : \mathbb{G} \subset \mathbb{R}^n \to \mathbb{R}$ is called locally Henstock-Kurzweil integrable if ξ_{χ_K} is Henstock-Kurzweil integrable over a compact set $K \subseteq \mathbb{G}$. We denote the set of locally Henstock-Kurzweil integrable functions as HK_{loc}.*

With easy analogous, $\mathcal{L}^1_{loc}(\mathbb{R}^n) \subset HK_{loc}(\mathbb{R}^n)$. On the assumption of the Euclidean space $\mathbb{R}^n$, a function f is claimed to be in $BMO(\mathbb{R}^n)$ if

$$\|\xi\|_{BMO(\mathbb{R}^n)} = \sup_{\mathfrak{Q}} \frac{1}{|\mathfrak{Q}|} \int_{\mathfrak{Q}} |\xi(x) - \xi_{\mathfrak{Q}}| dx < \infty,$$

where $\xi_{\mathfrak{Q}}$ is average value of ξ on the cube $\mathfrak{Q}$ and the supremum is taken over all cubes $\mathfrak{Q}$ in $\mathbb{R}^n$.

Definition 2.3. *[9, Definition 3.50] Suppose $\xi \in \mathcal{L}_{loc}(\mathbb{R}^n)$. Let $\mathcal{B}(x,r)$ is a cube in $\mathbb{R}^n$.*

(a) We define the average of ξ over $\mathcal{B}(x,r)$ by

$$\xi_{\mathcal{B}(x,r)} = \frac{1}{\mu(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} \xi(y) d\mu(y).$$

(b) We define the sharp function $M^{\bowtie}(\xi)(x)$ by

$$M^{\bowtie}(\xi)(x) = \sup_{\mathcal{B}(x,r)} \frac{1}{\mu(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} |\xi(y) - \xi_{\mathcal{B}(x,r)}| d\mu(y).$$

(c) For $M^{\bowtie}(\xi)(x) \in \mathcal{L}^{\infty}(\mathbb{R}^n)$, f is said to be bounded mean oscillation. Accurately,

$$BMO(\mathbb{R}^n) = \{\xi \in \mathcal{L}_{loc}(\mathbb{R}^n) : \ M^{\bowtie}(\xi) \in \mathcal{L}^{\infty}(\mathbb{R}^n)\}. \tag{1}$$

is the space of functions of bounded mean oscillation.

Proposition 2.4. *[16, Remark 3.10] If X is a Banach space then a mapping $J : \mathcal{L}^{\infty}_c(\mathbb{G}) \to X$ is continuous if and only if its precomposition with the inclusion mapping $\mathcal{L}^{\infty}(K) \to \mathcal{L}^{\infty}_c(\mathbb{G})$ is continuous for all compact subset K.*

3. Spaces $\mathbb{K}S^p(\mathbb{R}^n)$ and $\mathbb{K}S^{\infty}_c(\mathbb{R}^n)$

Recalling that all closed cubes $\{\mathcal{B}_j(x_i) | (j,i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$ are in the set $\{\mathcal{B}_k : k \in \mathbb{N}\}$. Suppose $\mathcal{E}_k(x)$ is the characteristic function of $\mathcal{B}_k$, with the aim that $\mathcal{E}_k(x) \in \mathcal{L}^p(\mathbb{R}^n) \cap \mathcal{L}^{\infty}(\mathbb{R}^n)$ for $1 \leq p < \infty$, with the assumption $|\mathcal{E}_k(x)| = \mathcal{E}_k(x) \leq 1$. Define $\mathfrak{F}_k(.)$ on $\mathcal{L}^1(\mathbb{R}^n)$ by

$$F_k(\xi) = \int_{\mathbb{R}^n} \mathcal{E}_k(x)\xi(x)dx.$$

Then $\mathfrak{F}_k(.)$ is a bounded linear functional on $\mathcal{L}^p(\mathbb{R}^n)$ for each k, $\|\mathfrak{F}_k\|_{\infty} \leq 1$ and if $\mathfrak{F}_k(\xi) = 0$, $\forall\, k$, $\xi = 0$. That is, it is fundamental on $\mathcal{L}^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. For $t_k > 0$ such that $\sum_{k=1}^{\infty} t_k = 1$, then $\mathbb{K}S^2(\mathbb{R}^n)$ is formed with the inner product as follows:

$$(\xi, g) = \sum_{k=1}^{\infty} \left[\int_{\mathbb{R}^n} \mathcal{E}_k(x)\xi(x)dx\right]\left[\int_{\mathbb{R}^n} \mathcal{E}_k(x)g(y)dy\right]^c.$$

Clearly $\mathbb{K}S^2(\mathbb{R}^n)$ is completion of $\mathcal{L}^1(\mathbb{R}^n) \subseteq HK(\mathbb{R}^n)$ with a small variant $\mathcal{E}_k(x) = 1$.

Let $\{\mathcal{B}_k\}_{k=1}^{\infty}$ be the countable collection of balls in $\mathbb{R}^n$ such that radius of $\mathcal{B}_r = \mathbb{G}(\mathcal{B}_l)$ is of the form 2^{-l}, $l \in \mathbb{N}$, and the center of $\mathcal{B}_k$ is contained in $\mathbb{Q}^n$. Let $\tau = \{t_k\}$ be a non negative real sequence such that $\sum_{k=1}^{\infty} t_k = 1$.

$$\|\xi\|_{\mathbb{K}S^p} = \left(\sum_{k=1}^{\infty} t_k \left|\int_{\mathcal{B}_k} \xi(x)dx\right|^p\right)^{\frac{1}{p}}. \tag{2}$$

$\mathbb{K}S^p(\mathbb{R}^n)$ is the closure of $\mathcal{L}^p(\mathbb{R}^n)$ with the norm (2) (see [9]). We write this by defining the weighted l^p space $l^p(\tau)$:

$$\|\{\sigma_k\}\|_{l^p(\tau)} = \left(\sum_{k=1}^{\infty} t_k|\sigma_k|^p\right)^{\frac{1}{p}}.$$

Then

$$\|\xi\|_{KS^p} = \|\left\{\int_{\mathcal{B}_k} \xi(x)dx\right\}\|_{l^p(\tau)} = \|\{\xi(B_k)\}\|_{l^p(\tau)}. \tag{3}$$

Clearly (3) gives us that norm of Kuelbs-Steadman space can be formed from the weighted l^p spaces. Throughout the work we use (2) as the norm of the space $\mathbb{K}S^p(\mathbb{R}^n)$. The norm (3) can be used to construct Kuelbs-Steadman spaces for variable exponent spaces.

Theorem 3.1. *With the norm (2) for $\mathbb{K}S^p$, we have the following:*

(i) If $1 \leq p \leq \infty$, $\mathcal{L}^p \subset \mathbb{K}S^p$ as continuous dense embeddings.

(ii) If $1 \leq p \leq \infty$ and $\xi, g \in \mathbb{K}S^p$, then

$$\left\|\xi + g\right\|_{\mathbb{K}S^p} \leqslant \|\xi\|_{\mathbb{K}S^p} + \left\|g\right\|_{\mathbb{K}S^p} \text{ (Minkowski inequality).}$$

(iii) If $1 \leq p \leq \infty$, K_o is a weakly sequentially compact subset of $\mathcal{L}^p$, it is a compact subset of $\mathbb{K}S^p$.

(iv) If $1 < p < \infty$, then $\mathbb{K}S^p$ is uniformly convex.

(v) If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of $\mathbb{K}S^p$ is $\mathbb{K}S^q$.

Recall the fact (see [16, Lemma 3.8]) that if $\mathcal{M}(\mathbb{R}^n)$ is a translation invariant Banach function space then the following are equivalents:

1. $C_c(\mathbb{R}^n) \subseteq \mathcal{M}(\mathbb{R}^n)$.
2. $C_c(\mathbb{R}^n) \cap \mathcal{M}(\mathbb{R}^n) \neq \emptyset$.

Then for any compact $K \subseteq \mathbb{R}^n$ we have $\chi_K \in \mathcal{M}$ so, $\mathbb{K}S^\infty(K) \subseteq \mathcal{M}$, also $\mathbb{K}S^\infty(K)$ in $\mathcal{M}$ is continuous. This motivates us to introduce $\mathbb{K}S^\infty_c(\mathbb{R}^n)$. From [16, Definition 3.9], we introduce $\mathbb{K}S^\infty_c(\mathbb{R}^n)$ of compactly supported essentially bounded functions on $\mathbb{R}^n$ as

$$\mathbb{K}S^\infty_c(\mathbb{R}^n) = \bigoplus_{K\subseteq\mathbb{R}^n} \mathbb{K}S^\infty(K), \text{ where } K \text{ is compact} \tag{4}$$

as the direct limit of $\mathbb{K}S^\infty(K)$, where $K \subseteq \mathbb{R}^n$ ranges over the compact sets. Since $\mathbb{K}S^\infty_c(\mathbb{R}^n)$ is a direct limit of topological vector spaces, it is naturally equipped with the inductive topology. So, $\mathbb{K}S^\infty_c(K) \subset \mathbb{K}S^\infty_c(M)$ for $K \subseteq M$. If K is a weakly sequentially compact subset of $\mathcal{L}^p$, it is a compact subset of $\mathbb{K}S^p$.

We leave few open questions in this section are

(1) Whether if K is weakly sequential compact subset of $\mathcal{L}^\infty_c(K)$ is strongly compact in $\mathbb{K}S^\infty_c(K)$?

(2) Whether following is possible:

$$\mathbb{K}S^\infty_c(\mathbb{R}^n) = \bigcup \bigoplus_{K\subseteq\mathbb{R}^n} \mathcal{L}^\infty(K), \text{ where } K \text{ is compact.}$$

Theorem 3.2. *$\mathbb{K}S^\infty_c(\mathbb{R}^n)$ is a linear space.*

Theorem 3.3. *Let E be translation invariant Banach function space such that $E \cap C_c(\mathbb{R}^n) \neq \{0\}$ then $\mathbb{K}S^\infty_c(\mathbb{R}^n) \subset E(\mathbb{R}^n)$ with continuous inclusion.*

Proof. For any $K \subseteq E$ we have $\chi_K \in E$. Since E is an ideal in $HK^0(\mathbb{R}^n)$, $\mathbb{K}S^\infty(\mathbb{R}^n) \subseteq E(\mathbb{R}^n)$. Let $\xi \in \mathbb{K}S^\infty(K)$ be arbitrary and suppose $j : \mathbb{K}S^\infty(K) \to E$ with the inclusion map

$$\begin{aligned} \|j(\xi)\|_E &= \|j(|\xi|)\|_E \\ &\leq \|j(\|\xi\|_{KS^\infty}\chi_K)\|_E \\ &= \|\xi\|_{KS^\infty}\|\chi_K\|_E. \end{aligned}$$

Therefore, the inclusion of $\mathbb{K}S^\infty(K)$ in E is continuous. Also,

$$\mathbb{K}S^\infty_c(\mathbb{R}^n) = \bigoplus_{K\subseteq\mathbb{R}^n} \mathbb{K}S^\infty(K), \text{ where } K \text{ is compact.} \tag{5}$$

So, $\mathbb{K}S^\infty_c(\mathbb{R}^n) \subseteq E(\mathbb{R}^n)$ with continuous inclusion. □

Theorem 3.4. $\mathbb{K}S^\infty_c(\mathbb{R}) \subset HK_{loc}(\mathbb{R}^n)$.

4. Introduction of $BMO_{HK}(\mathbb{R}^n)$ spaces

We motivate the importance of locally Henstock-Kurzweil integral for essentially bounded approach for BMO type spaces.

We introduce the definition of the space of $BMO_{HK}(\mathbb{R}^n)$ in terms of locally Henstock-Kurzweil integral. We discuss few preliminaries results related to this BMO type space.

Definition 4.1. *Let $\xi \in HK_{loc}(\mathbb{R}^n)$ and $\mathcal{B}(x,r)$ be a cube in $\mathbb{R}^n$.*

(a) We define the average of ξ over $\mathcal{B}(x,r)$ by

$$\xi_{\mathcal{B}(x,r)} = \frac{1}{\mu(\mathcal{B}(x,r))}(HK)\int_{\mathcal{B}(x,r)} \xi(y)d\mu(y).$$

(b) We define the sharp function $M^{\bowtie}(\xi)(x)$ by

$$M^{\bowtie}(\xi)(x) = \sup_{\mathcal{B}(x,r)} \frac{1}{\mu(\mathcal{B}(x,r))}(HK)\int_{\mathcal{B}(x,r)} |\xi(y) - \xi_{\mathcal{B}(x,r)}|d\mu(y).$$

(c) If $M^{\bowtie}(\xi)(x) \in \mathbb{K}S^{\infty}(\mathbb{R}^n)$, we say that ξ is bounded mean oscillation. More precisely, the space of functions of bounded mean oscillation is defined by

$$BMO_{HK}(\mathbb{R}^n) = \{\xi \in HK_{loc}(\mathbb{R}^n) : M^{\bowtie}(\xi) \in \mathbb{K}S^{\infty}(\mathbb{R}^n)\}, \tag{6}$$

Suppose $\xi \in HK_{loc}(\mathbb{R}^n)$. Let

$$\|\xi\|_* = \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{|\mathcal{B}(x,r)|}(HK)\int_{\mathcal{B}(x,r)} |\xi(y) - \xi_{\mathcal{B}(x,r)}|d\mu(y), \tag{7}$$

where $\xi_{\mathcal{B}(x,r)} = \frac{1}{|\mathcal{B}(x,r)|}(HK)\int_{\mathcal{B}(x,r)} \xi(y)d\mu(y)$ and $\|\xi\|_{BMO_{HK}} = \|M^{\bowtie}(\xi)\|_{\mathbb{K}S^{\infty}}$.

If $\sum_{k=1}^{n} t_k = 2^{-k}$ then clearly $\|\xi\|_*$ is equivalent to $\|M^{\bowtie}(\xi)\|_{\mathbb{K}S^{\infty}}$.

We can define a BMO type space as

$$BMO_{HK}(\mathbb{R}^n) = \Big\{\xi \in HK_{loc}(\mathbb{R}^n) : \|\xi\|_* < \infty\Big\}.$$

The function ξ is called of bounded mean oscillation if $\|\xi\|_* < \infty$ and $BMO_{HK}(\mathbb{R}^n)$ is the set of all locally Henstock-Kurzweil integrable functions ξ on $\mathbb{R}^n$ with $\|\xi\|_* < \infty$.

Theorem 4.2. *$BMO_{HK}(\mathbb{R}^n)$ is a linear space.*

Proof. Let $\xi, g \in BMO_{HK}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|\xi + g\|_* &= \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{|\mathcal{B}(x,r)|} \sum_{k=1}^{\infty} t_k |(HK)\int_{\mathcal{B}(x,r)} ((\xi+g)(y) - (\xi+g)_{\mathcal{B}(x,r)})|d\mu(y) \\
&= \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{|\mathcal{B}(x,r)} \sum_{k=1}^{\infty} t_k |(HK)\int_{\mathcal{B}(x,r)} (\xi(y) + g(y) - \xi_{\mathcal{B}(x,r)} - g_{\mathcal{B}(x,r)})|d\mu(y) \\
&\le \|\xi\|_* + \|g\|_* \\
&< \infty.
\end{aligned}$$

Therefore $\xi + g \in BMO_{HK}(\mathbb{R}^n)$. Again, for $\alpha \in \mathbb{C}$,

$$\|\alpha\xi\|_* = |\alpha|\|\xi\|_* < \infty.$$

So, $\alpha\xi \in BMO_{HK}(\mathbb{R}^n)$. Hence the proof. □

If $\|\xi\|_* = 0$ of (7), this never gives $\xi = 0$, so $\|\xi\|_*$ is not a norm.

Lemma 4.3. *If $\|\xi\|_* = 0$ then ξ is a constant a.e..*

Proof. Let $\|\xi\|_* = 0$. This gives

$$\frac{1}{|\mathcal{B}(x,r)|}\sum_{k=1}^{\infty} t_k |(HK)\int_{\mathcal{B}(x,r)} (\xi(y) - \xi_{|\mathcal{B}(x,r)})| = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} t_k |(HK)\int_{\mathcal{B}(x,r)} (\xi(y) - \xi_{\mathcal{B}(x,r)})| d\mu(y) = 0$$

$$\Rightarrow \xi(y) - \xi_{\mathcal{B}(x,r)} = 0 \text{ a.e.}$$

So, $\xi = \xi_{\mathcal{B}(x,r)}$ a.e. Hence ξ is a.e. equal to a constant. □

Using the Lemma 4.3, we can conclude the following:

Theorem 4.4. *The space $BMO_{HK}(\mathbb{R}^n)$ is a Banach space with reference to the norm (7).*

Proof. If $\{\xi_n\}$ is a Cauchy sequence in $BMO_{HK}(\mathbb{R}^n)$. Then $\{\xi_n\}$ is a Cauchy sequence in $\mathcal{L}^1(K)$ for any compact subset K of $\mathbb{R}^n$. Clearly $\mathcal{L}^1(K) \subset HK_{loc}(K)$. So, $\{\xi_n\} \in HK_{loc}(K)$ is a Cauchy sequence. Now using diagonal process, we can reduce $\{\xi_n\}$ converges in $BMO_{HK}(\mathbb{R}^n)$. □

Theorem 4.5. *$BMO(\mathbb{R}^n) \subseteq BMO_{HK}(\mathbb{R}^n)$ as continuous embeddings.*

Proof. Let $\xi \in BMO_{HK}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|\xi\|_* &= \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{|\mathcal{B}(x,r)|}\sum_{k=1}^{\infty} t_k |(HK)\int_{\mathcal{B}(x,r)} \Big(\xi(y) - \xi_{\mathcal{B}(x,r)}\Big)| d\mu(y)\\ &\le \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{|\mathcal{B}(x,r)|}\sum_{k=1}^{\infty} t_k \int_{\mathcal{B}(x,r)} |\xi(y) - \xi_{\mathcal{B}(x,r)}| d\mu(y)\\ &\le \|\xi\|_{BMO(\mathbb{R}^n)}.\end{aligned}$$

So, $BMO(\mathbb{R}^n) \subseteq BMO_{HK}(\mathbb{R}^n)$. □

Our next aim to show $BMO_{HK}(\mathbb{R}^n)$ is translational invariance in nature. Functions ξ and g are such that $\xi = g$ a.e., then ξ is Henstock-Kurzweil integrable if and only if g is Henstock-Kurzweil integrable and the integrals of $\xi = g$ if they exist. We consider $H^0(\mu)$ is an equivalence classes of measurable Henstock-Kurzweil integrals.

Definition 4.6. *(a) For $x \in \mathbb{R}^n$, we denote $\lambda_y : H^0(\mu) \to H^0(\mu)$ as left translation operator if $\lambda_y(g)(x) = g(y^{-1}x)$, $\forall\, g \in H^0(\mu)$, $y \in BMO_{HK}(\mathbb{R}^n)$.*

(b) For $x \in \mathbb{R}^n$, we denote $\rho_y : H^0(\mu) \to H^0(\mu)$ as right translation operator if $\rho_y(g)(x) = g(xy^{-1})$, $\forall\, g \in H^0(\mu)$, $y \in BMO_{HK}(\mathbb{R}^n)$.

The space $BMO_{HK}(\mathbb{R}^n)$ is called translation invariant if it is both left and right translation invariant. Clearly an element $f \in BMO_{HK}(\mathbb{R}^n)$ if $\lim_{y\to x} \|\lambda_y\xi - \lambda_x\xi\|_* = 0$, $\forall x \in \mathbb{R}^n$.

Definition 4.7. *(a) An element $\xi \in BMO_{HK}(\mathbb{R}^n)$ is called left strongly continuous if $\lim_{y\to x} \|\lambda_y\xi - \lambda_x\xi\|_* = 0$ $\forall x \in \mathbb{R}^n$.*

(b) An element $\xi \in BMO_{HK}(\mathbb{R}^n)$ is called right strongly continuous if $\lim_{y\to x} \|\rho_y\xi - \rho_x\xi\|_* = 0$, $\forall x \in \mathbb{R}^n$.

We say that an element $\xi \in BMO_{HK}(\mathbb{R}^n)$ is strongly continuous if

$$\lim_{y\to e} \|\lambda_y\xi - \xi\|_* = \lim_{y\to e} \|\rho_y\xi - \xi\|_* \text{ for } e \in BMO_{HK}(\mathbb{R}^n).$$

Theorem 4.8. *The space $BMO_{HK}(\mathbb{R}^n)$ is translation invariant.*

Proof. Let $\xi \in BMO_{HK}(\mathbb{R}^n)$ vanish at infinity, and let $\varepsilon > 0$ and $y \in \mathbb{R}^n$ be arbitrary. Let $K \subset \mathbb{R}^n$ be compact such that

$$\|\xi\chi_{\mathbb{R}^n\setminus K}\|_* < \frac{\varepsilon}{\|\lambda_y\|_{B\left(BMO_{HK}(\mathbb{R}^n), BMO_{HK}(\mathbb{R}^n)\right)}},$$

where $B(BMO_{HK}(\mathbb{R}^n), BMO_{HK}(\mathbb{R}^n))$ is the set of bounded linear on $BMO_{HK}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|(\lambda_y\xi)\chi_{\mathbb{R}^n\setminus yK}\|_* &= \|\lambda(\xi\chi_{\mathbb{R}^n\setminus K}\|_* \\ &\leq \|\lambda_y\|_{B\left(BMO_{HK}(\mathbb{R}^n), BMO_{HK}(\mathbb{R}^n)\right)}\|\xi\chi_{\mathbb{R}^n\setminus K}\|_* \\ &< \varepsilon.\end{aligned}$$

As yK is compact and $\varepsilon > 0$ arbitrary, so, $\lambda_y\xi$ vanishes at infinity. Similarly we can find $\rho_y\xi$ vanish at infinity. Now, let $\xi \in BMO_{HK}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$. Since ξ is left and strongly continuous and commutative,

$$\lim_{x\to e}\|\lambda_x(\lambda_y\xi) - \lambda_y\xi\|_* = 0 = \lim_{x\to e}\|\rho_x(\lambda_y\xi) - \lambda_y\xi\|_*.$$

Now

$$\begin{aligned}\|\rho_x(\lambda_y\xi) - \lambda_y\xi\|_* &= \|\lambda_y(\rho_x\xi) - \xi\|_* \\ &\leq \|\lambda_y\|_{B\left(BMO_{HK}(\mathbb{R}^n), BMO_{HK}(\mathbb{R}^n)\right)}\|\rho_x\xi - \xi\|_*.\end{aligned}$$

So, $\lambda_y\xi \in BMO_{HK}(\mathbb{R}^n)$. An analogous proof shows that $\rho_y\xi \in BMO_{HK}(\mathbb{R}^n)$. Hence $BMO_{HK}(\mathbb{R}^n)$ is translation invariant. □

4.1. Properties of $BMO_{HK}(\mathbb{R}^n)$ as translation invariant Banach function spaces:

Let $(\overline{\Omega}, \Sigma, \mu)$ be a complete $\sigma-$finite measure space and let $H^0(\mu) = H^0(\overline{\Omega}, \Sigma, \mu)$ the space of all equivalence classes of $\mu-$ measurable real valued functions endowed with the topology of convergence in measure relative to each set of finite measure. The Banach space $BMO_{HK}(\mathbb{R}^n) \subset H^o(\mu)$ is called a Banach function space on $(\overline{\Omega}, \Sigma, \mu)$ if there exists a $u \in BMO_{HK}(\mathbb{R}^n)$ such that $u > 0$ a.e. constant and $BMO_{HK}(\mathbb{R}^n)$ satisfies the ideal property:

(a) $x \in H^0(\mu)$, $y \in BMO_{HK}(\mathbb{R}^n)$, $|x| \leq |y|$ $\mu-$a.e. constant then
(b) $x \in BMO_{HK}(\mathbb{R}^n)$ and $\|x\|_* \leq \|y\|_*$.

Proposition 4.9. *$BMO_{HK}(\mathbb{R}^n)$ is a Banach function space.*

Proof. Functions that are differing for a constant $c > 0$ can contribute the same as $BMO_{HK}(\mathbb{R}^n)$ norm value if their disparity is not zero a.e. as constant functions have zero mean oscillation. □

Remark 4.10. *Note that $\mathbb{K}S^\infty(\mathbb{R}^n)$ is contained in $BMO_{HK}(\mathbb{R}^n)$ and we have $\|\xi\|_* \leq 2\|\xi\|_{\mathbb{K}S^\infty}$. Moreover $BMO_{HK}(\mathbb{R}^n)$ contains unbounded functions, in fact the function $log|x|$ on $\mathbb{R}$, is in BMO_{HK} but it is not bounded, so $\mathbb{K}S^\infty(\mathbb{R}^n) \subset BMO_{HK}(\mathbb{R}^n)$.*

Function spaces are important and natural examples of abstract Banach lattice.

Proposition 4.11. *$BMO_{HK}(\mathbb{R}^n)$ is an order continuous Banach lattice with a weak unit.*

There exists a probability space $(\overline{\Omega}, \Sigma, \mu)$ and a Banach function space $BMO_{HK}(\mathbb{R}^n)$ such that $BMO_{HK}(\mathbb{R}^n)$ is isometrically lattice isomorphic to $BMO_{HK}(\mathbb{R}^n)$ and $\mathbb{K}S^\infty(\mathbb{R}^n) \subset BMO_{HK}(\mathbb{R}^n) \subset HK(\mu)$ with a continuous inclusions. We denote $BMO^s_{HK}(\mathbb{R}^n)$ of strongly continuous part of Banach function space $BMO_{HK}(\mathbb{R}^n)$.

Definition 4.12. *For a Banach function space $BMO_{HK}(\mathbb{R}^n)$, the set of strongly continuous elements of $BMO_{HK}(\mathbb{R}^n)$ that vanish at infinity as:*

$$BMO^{s,o}_{HK}(\mathbb{R}^n) = \left\{\xi \in BMO^{s}_{HK}(\mathbb{R}^n) : \forall\ \varepsilon > 0\ \exists K \subseteq \mathbb{R}^n \text{ compact s.t. } \|\xi \chi_{\mathbb{R}^n \setminus K}\|_* < \varepsilon\right\}. \tag{8}$$

For a continuous function $\xi : \mathbb{G} \subset \mathbb{R}^n \to \mathbb{R}$ we write

$$\text{supp}(\xi) = \overline{\{x \in \mathbb{G} : \ \xi(x) \neq 0\}}$$

for its support, we denote $C_c(\mathbb{G})$, the set of all continuous function from $\mathbb{G}$ to $\mathbb{R}$ with compact support. For any measurable subset $\mathcal{H} \subseteq \mathbb{G}$ we denote $HK(\mathcal{H})$ and $KS^{\infty}(\mathcal{H})$ the $\mu-$Henstock-Kurzweil integrable and essentially bounded element of $H^0(\mu)$, respectively that vanish almost everywhere outside $\mathcal{H}$.

Lemma 4.13. *$BMO^{s}_{HK}(\mathbb{R}^n)$ is closed in $BMO_{HK}(\mathbb{R}^n)$.*

Proof. Let $(j_n)_{n\in\mathbb{N}} \subseteq BMO^{s}_{HK}(\mathbb{R}^n)$ be a sequence converging to $j \in BMO_{HK}(\mathbb{R}^n)$ and let $\varepsilon > 0$ be arbitrary. We claim that if κ is a neighbourhood of e in $\mathbb{G} \subseteq BMO_{HK}(\mathbb{R}^n)$ such that $\|\lambda_y j - j\|_* < \varepsilon,\ \forall y \in \kappa$. Suppose κ_0 is a neighbourhood of e in $\mathbb{G}$ with compact closure and a constant $\omega \in \mathbb{R} > 0$ such that $\|\lambda_y\|_{B\left(BMO_{HK}(\mathbb{R}^n), BMO_{HK}(\mathbb{R}^n)\right)} < \omega,\ \forall\ y \in \kappa_0$. For all $y \in \kappa_0$, for $n \in \mathbb{N}$ such that $\|j - j_n\|_* < \frac{\varepsilon}{2(\omega+1)}$ and a neighbourhood κ_1 of e in $\mathbb{G}$ such that $\|\lambda_y j_n - j_n\|_* < \frac{\varepsilon}{2}\ \forall\ y \in \kappa_1$. Let $\kappa = \kappa_0 \cap \kappa_1$, with triangle inequality, we have

$$\begin{aligned}
\|\lambda_y j - j\|_* &\leq \|\lambda_y j - \lambda_y j_n\|_* + \|\lambda_y j_n - j_n\|_* + \|j_n - j\|_* \\
&< (\omega + 1)\frac{\varepsilon}{2(\omega + 1)} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, j is left strongly continuous. Similarly j is right strongly continuous, hence j is strongly continuous and consequently $j \in BMO^{s}_{HK}(\mathbb{R}^n)$. Hence the proof. □

Corollary 4.14. *$BMO^{s,o}_{HK}(\mathbb{R}^n)$ is closed in $BMO_{HK}(\mathbb{R}^n)$.*

Proposition 4.15. *$C_c(\mathbb{G})$ is a subset of $BMO_{HK}(\mathbb{R}^n)$.*

Theorem 4.16. *$\mathbb{K}S^{\infty}_{c}(\mathbb{G}) \subseteq BMO_{HK}(\mathbb{R}^n)$ with continuous inclusions.*

Proof. $BMO_{HK}(\mathbb{G})$ is translational invariant Banach function space with $BMO_{HK}(\mathbb{G}) \cap C_c(\mathbb{G}) \neq \{0\}$. Then using Theorem 3.3, we get $\mathbb{K}S^{\infty}_{c}(\mathbb{G}) \subseteq BMO_{HK}(\mathbb{G})$. □

4.2. Completion of $BMO_{HK}(\mathbb{R}^n)$:

The completion of smooth functions in $BMO_{HK}(\mathbb{R}^n)$ norm, namely the space $VMO_{HK}(\mathbb{R}^n)$(= vanishing mean oscillation) is the completion of smooth maps in the $BMO_{HK}(\mathbb{R}^n)$ norm.

If $\xi \in BMO_{HK}(\mathbb{R}^n)$ then there is a sequence (ξ_j) in $VMO_{HK}(\mathbb{R}^n)$ of smooth functions such that $\|\xi_j - \xi\|_* \to 0$. An elementary argument establish that $VMO_{HK}(\mathbb{R}^n)$ is a closed subspace of $BMO_{HK}(\mathbb{R}^n)$. It is obvious that $VMO_{HK}(\mathbb{R}^n)$ contains all uniformly continuous functions in $BMO_{HK}(\mathbb{R}^n)$. The space $VMO_{HK}(\mathbb{R}^n)$ is a Banach space with the norm of $BMO_{HK}(\mathbb{R}^n)$.

Proposition 4.17. *$VMO(\mathbb{R}^n) \subset VMO_{HK}(\mathbb{R}^n)$.*

Proof. If $\xi \in VMO(\mathbb{R}^n)$, then there is a sequence (ξ_j) in $BMO(\mathbb{R}^n)$, such that $\|\xi_j - \xi\|_{BMO} \to 0$. The fact (ξ_j) in $BMO_{HK}(\mathbb{R}^n)$, gives $\|\xi_j - \xi\|_* \to 0$. Hence $\xi \in VMO_{HK}(\mathbb{R}^n)$. Therefore the proof. □

Theorem 4.18. *The space $VMO_{HK}(\mathbb{R}^n)$ is translation invariant.*

Proof. Proof is alike as Theorem 4.8. □

Remark 4.19. *The functions in $VMO_{HK}(\mathbb{R}^n)$ are those with the additional property that their mean oscillations over small cubes are small. Here in this paper we introduce $VMO_{HK}(\mathbb{R}^n)$ for the purpose to to investigate this space is translation invariant like $BMO_{HK}(\mathbb{R}^n)$. Under the assumption ξ is a Borel measurable function of $\mathbb{C}$ to itself.*

We can introduce the more general form of Fominykh-Chevalier Theorem as follows:

Theorem 4.20. *The following are equivalent:*

(a) $\sup_{x,y\in\mathbb{C}}(1+|x-y|)^{-1}|\xi(x)-\xi(y)|<\infty.$

(b) $T_\xi[VMO_{HK}(\mathbb{R}^n)] \subseteq VMO_{HK}(\mathbb{R}^n)$*, where* T_ξ *maps bounded subset of* $VMO_{HK}(\mathbb{R}^n)$ *to bounded subset of* $VMO_{HK}(\mathbb{R}^n)$.

Interested researcher can extend the result of Brezis and Nirenberg with the space $VMO_{HK}(\mathbb{R}^n)$. In our next work we will investigate Degree Theory and VMO_{HK} with compact manifolds without boundaries.

Declaration

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