Filomat 37:7 (2023), 1997–2012
https://doi.org/10.2298/FIL2307997S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Homoclinic solutions and periodic solutions of complex-valued neural networks with time-varying delays

Lin Sun[a], **Fanchao Kong**[b]

[a]*School of Finance, Hunan University of Finance and Economics, Changsha, Hunan 410205, China*
[b]*School of Mathematics and Statistics, Anhui Normal University, Wuhu, Anhui 241000, China*

Abstract. In this paper, a class of complex-valued neural networks with time-varying delays is studied. By employing an extension of Mawhin's continuation theorem and an approximation technique, several sufficient conditions of the new results on the existence of homoclinic solutions and periodic solutions are established. Moreover, the asymptotic behavior of solutions via the Lyapunov function is also investigated. Finally, for the purpose of validity, an example is given to illustrate the effectiveness of main results.

1. Introduction

In this paper, we consider the following complex-valued neural networks with time-varying delays:

$$\frac{dz_p(t)}{dt} = -d_p(t)z_p(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_q(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_q(t-\tau_{pq}(t))) + H_p(t), \tag{1.1}$$

where $z_p(t) = x_p(t) + iy_p(t)$ denotes the complex-valued state vector associated with the p-th neuron, p, $q = 1, 2, \cdots, n$, n is the number of neurons. For convenience, $z_p(t)$, $x_p(t)$ and $y_p(t)$ are denoted as z_p, x_p and y_p, respectively. This model describes the continuous evolution process of the neural networks. $d_p(t) \in \mathbb{R}$ is the self-feedback connection weight, $a_{pq}(t)$, $b_{pq}(t)$ are complex-valued connection weight matrices without and with time delays respectively. $f_q(z_q)$, $g_q(z_q) : \mathbb{C} \to \mathbb{C}$ are the activation functions of the neurons. $H_p(t) \in \mathbb{C}$ is the external input vector. $\tau_{pq}(t) \geq 0$ correspond to the transmission delays.

The model (1.1) can be rewritten in vector form as follows,

$$\dot{z}(t) = -D(t)z(t) + A(t)f(z(t)) + B(t)f(z(t-\tau(t))) + H(t), \tag{1.2}$$

where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^\top \in \mathbb{C}^n$ is the state vector, $D(t) = \mathrm{diag}(d_1(t), d_2(t), ..., d_n(t)) \in \mathbb{R}^n$ with $d_p > 0 (p = 1, 2, ..., n)$ is the self-feedback connection weight matrix, $A(t) = (a_{pq})_{n\times n} \in \mathbb{C}^{n\times n}$ and $B(t) = (b_{pq})_{n\times n} \in \mathbb{C}^{n\times n}$ are, respectively, the connection weight matrix without and with time delay, $f(z) =$

2020 *Mathematics Subject Classification*. Primary 34C37; Secondary 34A60, 93D23
Keywords. Complex-valued neural networks, Homoclinic solution, Periodic solution, Globally asymptotic stability, An extension of Mawhin's continuation theorem.
Received: 21 February 2022; Revised: 22 April 2022; Accepted: 14 May 2022
Communicated by Dragan S. Djordjević
Research supported by the National Social Science Fund of China (20BJL061), National Natural Science Foundation of China (12001011), Natural Science Fund Project of the University in Anhui Province (No. 2022AH030023).
Email address: fanchaokong88@yahoo.com (Fanchao Kong)

$(f_1(z_1(t)), f_2(z_2(t)), ..., f_n(z_n(t)))^\top : \mathbb{C}^n \to \mathbb{C}^n$, and $f(z(t-\tau(t))) = (f_1(z_1(t-\tau_1(t))), f_2(z_2(t-\tau_2(t))), ..., f_n(z_n(t-\tau_n(t))))^\top : \mathbb{C}^n \to \mathbb{C}^n$ are the vector-valued activation functions without and with time delays whose elements consist of complex-valued nonlinear functions, $\tau(t) \geq 0$ correspond to the transmission delays, $H(t) = (H_1(t), H_2(t), ..., H_n(t))^\top \in \mathbb{C}^n$ is the external input vector-valued function.

A complex-valued neural network which can be regarded as the extension of real-valued neural networks in some sense is one that processes information in the complex plane; that is, its state, connection weight, and activation function are complex-valued. It has been discovered essentially useful in extending the scope of their applications in optoelectronics, filtering, imaging, speech synthesis, computer vision, remote sensing, quantum devices, spatiotemporal analysis of physiological neural devices and systems, and artificial neural information processing, see [2], [4], [6], [7], [8], [9], [13], [20], [28] and the references therein.

Compared with the real-valued neural networks, it is more difficult to study the dynamic behaviors of complex-valued neural networks since complex-valued neural networks are quite different and have more complicated properties than the real-valued neural networks. Moreover, a source of instability for neural networks is time delay which inevitably exists in the implementation of artificial neural networks due to the finite switching speed of amplifiers or network congestion. Therefore, stability analysis for delayed complex-valued neural networks has become an important research topic and various criteria have been developed in the literature in recent years, see [5], [8], [16], [17], [19], [21], [22], [23], [25], [26], [27], [29] and the references therein. For example, in [8], Hu and Wang studied global stability for the following complex-valued recurrent neural networks with time-delays of the form:

$$\dot{z}(t) = -Dz + Af(z(t)) + Bg(z(t-\tau)) + u,$$

where $z = (z_1, z_2, ..., z_n)^\top \in \mathbb{C}^n$ is the state vector, $D = diag(d_1, d_2, ..., d_n) \in \mathbb{R}^{n\times n}$ with $d_j > 0 (j = 1, 2, ..., n)$ is the self-feedback connection weight matrix, $A = (a_{jk})_{n\times n} \in \mathbb{C}^{n\times n}$ and $B = (b_{jk})_{n\times n} \in \mathbb{C}^{n\times n}$ are, respectively, the connection weight matrix without and with time delays, $f(z(t)) = (f_1(z_1(t)), f_2(z_2(t)), ..., f_n(z_n(t)))^\top : \mathbb{C}^n \to \mathbb{C}^n$ and $g(z(t-\tau)) = (g_1(z_1(t-\tau_1)), g_2(z_2(t-\tau_2)), ..., g_n(z_n(t-\tau_n)))^\top : \mathbb{C}^n \to \mathbb{C}^n$ are the vector-valued activation functions without and with time delays whose elements consist of complex-valued nonlinear functions, $\tau_j (j = 1, 2, ..., n)$ are constant time delays, $u = (u_1, u_2, ..., u_n)^\top \in \mathbb{C}^n$ is the external input vector. By considering two classes of activation functions and different approaches, the authors systematically studied the stability problem of the complex-valued recurrent neural networks.

In [17], Pan et al further discussed the exponential stability of a class of complex-valued neural networks with time-varying delays of the form:

$$\frac{dz_k(t)}{dt} = -d_k z_k(t) + \sum_{j=1}^{n} \Big(w_{kj} f_j(z_j(t)) + v_{kj} g_j(z_j(t-\tau_j(t)))\Big) + J_k,$$

where $z_k(t) = x_k(t) + iy_k(t)$, $k, j = 1, ..., n$. n is the number of units in the neural networks, $z_k(t)$ corresponds to the state variable, $d_k (d_k > 0)$ represents the neuron charging time constant, $f_j(z_j)$, $g_j(z_j) : \mathbb{C} \to \mathbb{C}$ are the activation functions of the neurons, w_{kj}, $v_{kj} \in \mathbb{C}$ stand for the weights of the neuron interconnections, $J_k \in \mathbb{C}$ is the external bias, and $\tau_j(t) (0 \leq \tau_j(t) \leq \tau)$ corresponds to the transmission delays. By using the conjugate system of the complex-valued neural networks and the Brouwer's fixed point theorem, sufficient conditions to guarantee the existence and uniqueness of an equilibrium are obtained.

As we all know, the problem of existence of homoclinic solutions and periodic solutions is one of the most important problems in qualitative theory of differential systems. However, to the best of our knowledge, the corresponding theory of homoclinic solutions for the complex-valued neural networks with time-varying delays is not investigated till now.

Motivated by the above discussion, in this paper, we aim to study the existence of homoclinic solutions and periodic solutions of (1.2) by applying an extension of Mawhin's continuation theorem and some analysis techniques. Moreover, we also study the asymptotic behavior results of the solutions for (1.2). It is worthy to point out that it is the first time to investigate the homoclinic solutions for the complex-valued neural networks with time-varying delays.

As is well known, a solution $u(t)$ of (1.1) is named homoclinic (to 0) if $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to +\infty$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

In order to study the existence of homoclinic solutions for the system (1.1), from the work in [12], [15], [18], [24], we can know that the existence of a homoclinic solution of (1.1) is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following equation:

$$\frac{dz_p(t)}{dt} = -d_p(t)z_p(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_q(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_q(t-\tau_{pq}(t))) + H_{p_k}(t), \tag{1.3}$$

where $k \in \mathbb{N}$, $H_{p_k} : \mathbb{R}^+ \to \mathbb{C}$ is a $2kT$-periodic function such that

$$H_{p_k}(t) = \begin{cases} H_p(t), & t \in [-kT, kT - \varepsilon_0), \\ H_p(kT-\varepsilon_0) + \frac{H_p(-kT)-H_p(kT-\varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0), & t \in [kT-\varepsilon_0, kT], \end{cases} \tag{1.4}$$

where $\varepsilon_0 \in (0, T)$ is a constant independent of k. The existence of $2kT$-periodic solutions to (1.3) is obtained by applying an extension of Mawhin's continuation theorem [14]. The contributions and novelties are stated as following:

- The existence of $2kT$-periodic solutions of time-varying delayed complex-valued neural networks has been studied, which is more generalized than the previous T-periodic solutions.
- Homoclinic solutions is firstly considered for the time-varying delayed complex-valued neural networks, which has never been studied before.
- Asymptotic behavior of the solutions is also investigated, some previous stability results on the complex-valued neural networks with constant delays can be extended.

The remainder of this paper is organized as follows. In Section 2, we will give some definitions and some useful lemmas. Section 3 and 4 is devoted to establishing some criteria for the existence and globally exponentially stability of homoclinic and periodic solution for (1.2). Finally, in section 5, an numerical example is given to illustrate the effectiveness of the obtained results.

2. Preliminary

Throughout this paper, we define

$$\begin{aligned} &\overline{a}_{pq} = \max_{1\le p,q\le n} \sup_{t\in\mathbb{R}} |a_{pq}(t)|,\ \overline{b}_{pq} = \max_{1\le p,q\le n} \sup_{t\in\mathbb{R}} |b_{pq}(t)|,\ \overline{d}_{pq} = \max_{1\le p,q\le n} \sup_{t\in\mathbb{R}} |d_{pq}(t)|, \\ &\underline{a}_{pq} = \min_{1\le p,q\le n} \inf_{t\in\mathbb{R}} |a_{pq}(t)|,\ \underline{b}_{pq} = \min_{1\le p,q\le n} \inf_{t\in\mathbb{R}} |b_{pq}(t)|,\ \underline{d}_{pq} = \min_{1\le p,q\le n} \inf_{t\in\mathbb{R}} |b_{pq}(t)|. \end{aligned}$$

Notations i denotes the imaginary unite, that is, $i = \sqrt{-1}$. $z(t)$ represents the complex-valued function, that is, $z(t) = x(t) + iy(t)$, where $x(t), y(t) \in \mathbb{R}^n$. $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional real spaces and complex vector spaces respectively.

For convenience, throughout this paper, $\|\cdot\|_0$ denotes the Euclidean norm on $\mathbb{R}$. For each $k \in \mathbb{N}$, let

$$C_{2kT} = \Big\{\varphi_p | \varphi_p \in C(\mathbb{R}, \mathbb{R}), \varphi_p(t + 2kT) \equiv \varphi_p(t), p = 1, 2, ..., n\Big\},$$

$$C^1_{2kT} = \Big\{\varphi_p | \varphi_p \in C^1(\mathbb{R}, \mathbb{R}), \varphi_p(t + 2kT) \equiv \varphi_p(t), p = 1, 2, ..., n\Big\}$$

and $\|\varphi_p\|_0 = \max\limits_{1\le p\le n} \sup\limits_{t\in[0,2kT]} |z_p(t)|$. If the norms of C_{2kT} and C^1_{2kT} are defined by $\|\cdot\|_{C_{2kT}} = \|\cdot\|_0$ and $\|\varphi\|_{C^1_{2kT}} = \max\{\|\varphi_p\|_0, \|\varphi'_p\|_0\}$, then C_{2kT} and C^1_{2kT} are all Banach spaces. Furthermore, for $\varphi_p \in C_{2kT}$, $\|\varphi_p\|_2 = \Big(\int_{-kT}^{kT} |\varphi_p(t)|^2 dt\Big)^{\frac{1}{2}}$.

Lemma 2.1. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}$, $a > 0$ is a constant, then for every $t \in \mathbb{R}$, the following inequality holds:*

$$|\varphi(t)| \leq (2a)^{-\frac{1}{2}}\Big(\int_{t-a}^{t+a} |\varphi(s)|^2 ds\Big)^{\frac{1}{2}} + a(2a)^{-\frac{1}{2}}\Big(\int_{t-a}^{t+a} |\varphi'(s)|^p ds\Big)^{\frac{1}{2}}.$$

This lemma is a special case of Lemma 2.2 in [24].

Lemma 2.2. *[24] Let $\varphi_k \in C^1_{2kT}$ be a $2kT$-periodic function for each $k \in \mathbb{N}$ with*

$$\|\varphi_k\|_0 \leq A_0, \quad \|\varphi'_k\|_0 \leq A_1, \quad \|\varphi''_k\|_0 \leq A_2,$$

where A_0, A_1 and A_2 are constants independent of $k \in \mathbb{N}$. Then there exists a function $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ such that for each interval $[c, d] \subset \mathbb{R}$, there is a subsequence $\{\varphi_{k_j}\}$ of $\{\varphi_k\}_{k\in\mathbb{N}}$ with $\varphi'_{k_j}(t) \to \varphi'_0(t)$ uniformly on $[c, d]$.

By separating the state, the connection weight, the activation function and the external input into its real and imaginary part, then system (1.3) can be rewritten as follows for $p = 1, 2, \cdots, n$,

$$\begin{aligned}\frac{dx_p(t)}{dt} = &-d_p(t)x_p(t) + \sum_{q=1}^{n} a^R_{pq}(t)f^R_q(z_q(t)) - \sum_{q=1}^{n} a^I_{pq}(t)f^I_q(z_q(t)) + \sum_{q=1}^{n} b^R_{pq}(t)f^R_q(z_q(t-\tau_{pq}(t)))\\ &- \sum_{q=1}^{n} b^I_{pq}(t)f^I_q(z_q(t-\tau_{pq}(t))) + H^R_{p_k}(t),\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\frac{dy_p(t)}{dt} = &-d_p(t)y_p(t) + \sum_{q=1}^{n} a^R_{pq}(t)f^I_q(z_q(t)) + \sum_{q=1}^{n} a^I_{pq}(t)f^R_q(z_q(t)) + \sum_{q=1}^{n} b^R_{pq}(t)f^I_q(z_q(t-\tau_{pq}(t)))\\ &+ \sum_{q=1}^{n} b^I_{pq}(t)f^R_q(z_q(t-\tau_{pq}(t))) + H^I_{p_k}(t),\end{aligned} \tag{2.2}$$

where $x_p(t) = \mathrm{Re}(z_p(t))$, $y_p(t) = \mathrm{Im}(z_p(t))$, $a^R_{pq}(t) = \mathrm{Re}(a_{pq}(t))$, $a^I_{pq}(t) = \mathrm{Im}(a_{pq}(t))$, $b^R_{pq}(t) = \mathrm{Re}(b_{pq}(t))$, $b^I_{pq}(t) = \mathrm{Im}(b_{pq}(t))$, $H^R_p(t) = \mathrm{Re}(H_p(t))$, $H^I_R(t) = \mathrm{Im}(H_p(t))$, $f^R_q(z_q(t)) = \mathrm{Re}(f_q(z_q(t)))$, $f^I_q(z_q(t)) = \mathrm{Im}(f_q(z_q(t)))$, $f^R_q(z_q(t-\tau_{pq}(t))) = \mathrm{Re}(f_q(z_q(t-\tau_{pq}(t))))$ $f^I_q(z_q(t-\tau_{pq}(t))) = \mathrm{Im}(f_q(z_q(t-\tau_{pq}(t))))$.

Define $U_k = V_k = \Big\{z_p(t) = x_p(t) + iy_p(t), x_p \in C_{2kT}, y_p \in C_{2kT}, p = 1, 2, ..., n\Big\}$ with the norm

$$\|z_p\|_{U_k} = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |z_p(t)| = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |x_p(t)| + \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |y_p(t)|.$$

Then U_k and V_k are Banach spaces when they are endowed with above norm.

For $p = 1, 2, \cdots, n$, we denote

$$\begin{aligned}\Phi_{p,x}(t) = &- d_p(t)x_p(t) + \sum_{q=1}^{n} a^R_{pq}(t)f^R_q(z_q(t)) - \sum_{q=1}^{n} a^I_{pq}(t)f^I_q(z_q(t)) + \sum_{q=1}^{n} b^R_{pq}(t)f^R_q(z_q(t-\tau_{pq}(t)))\\ &- \sum_{q=1}^{n} b^I_{pq}(t)f^I_q(z_q(t-\tau_{pq}(t))) + H^R_{p_k}(t),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\Psi_{p,y}(t) = &- d_p(t)y_p(t) + \sum_{q=1}^{n} a^R_{pq}(t)f^I_q(z_q(t)) + \sum_{q=1}^{n} a^I_{pq}(t)f^R_q(z_q(t)) + \sum_{q=1}^{n} b^R_{pq}(t)f^I_q(z_q(t-\tau_{pq}(t)))\\ &+ \sum_{q=1}^{n} b^I_{pq}(t)f^R_q(z_q(t-\tau_{pq}(t))) + H^I_{p_k}(t).\end{aligned} \tag{2.4}$$

Now, define the operator

$$L : D(L) \subset U_k \to V_k, \;\; Lz_p = z_p',$$

and define the nonlinear operator

$$N : \overline{\Omega} \subset U_k \to V_k, \;\; Nz_p = \begin{pmatrix} \Phi_{p,x}(t) \\ \Psi_{p,y}(t) \end{pmatrix}, \;\; z_p \in U_k.$$

where Ω is an open bounded subset of U_k. Clearly, the problem of the existence of a $2kT$-periodic solution to (1.3) is equivalent to the problem of the existence of a solution in $\overline{\Omega}$ for the equation $Lz_p = Nz_p$.

By simply calculating, we have

$$\ker L = \left\{ z_p \in U_k | z_p \equiv c \in \mathbb{R} \right\}, \; \mathrm{Im} L = \left\{ z_p \in V_k, \int_0^{2kT} z_p(s) ds = 0 \right\}. \diamond$$

Therefore, L is a Fredholm operator of index zero.

Define

$$P : U_k \to \ker L, \;\; Pz_p = \frac{1}{2kT} \int_0^{2kT} z_p(s) ds, \; Q : V_k \to V_k / \mathrm{Im} L, \;\; Qz_p = \frac{1}{2kT} \int_0^{2kT} z_p(s) ds.$$

It is easy to verify that P and Q are two continuous projections such that $\mathrm{Im} P = \ker L$, $\mathrm{Im} L = \ker Q = \mathrm{Im}(I - Q)$.

It follows that $L|_{D(L) \cap \ker P} : D(L) \cap \ker P \to \mathrm{Im} L$ is invertible, and the generalized inverse $K_P : \mathrm{Im} L \to D(L) \cap \ker P$ can be written by

$$(K_p z_p)(t) = \int_0^{2kT} G_k(t,s) z_p(s) ds, \; G_k(t,s) = \begin{cases} \frac{s - 2kT}{2kT}, & 0 \le t \le s; \\ \frac{s}{2kT}, & s \le t \le 2kT. \end{cases}$$

For all $\overline{\Omega}$ such that $\overline{\Omega} \subset U_k$, we can see that $K_p(I - Q)N(\overline{\Omega})$ is a relative compact set of U_k and $QN(\overline{\Omega})$ is a bounded set of V_k, so the operator N is L-compact in $\overline{\Omega}$.

Lemma 2.3. *[14] Assume that Ω is an open bounded set in U_k such that the following conditions are satisfied:*

(i) *For each $\lambda \in (0, 1)$, the equation*

$$\frac{dz_p(t)}{dt} + \lambda d_p(t) z_p(t) - \lambda \sum_{q=1}^{n} a_{pq}(t) f_q(z_q(t)) - \lambda \sum_{q=1}^{n} b_{pq}(t) f_q(z_q(t - \tau_{pq}(t))) - \lambda H_{p_k}(t) = 0$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$\Delta(c_p) := \frac{1}{2kT} \int_0^{2kT} \Big(- d_p(\xi) c_p + \sum_{q=1}^{n} a_{pq}(t) f_q(c_q) + \sum_{q=1}^{n} b_{pq}(t) f_q(c_q) + H_{p_k}(\xi) \Big) d\xi = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree*

$$d_B\{\Delta, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then (1.3) *has a $2kT$-periodic solution in $\overline{\Omega}$*

Definition 2.4. *[8] If $z^*(t) = (z_1^*(t), z_2^*(t), ..., z_n^*(t))^\top$ is a periodic solution of (1.1) and $z(t) = (z_1(t), z_2(t), ..., z_n(t))^\top$ is any solution of (1.1) satisfying*

$$\lim_{t\to+\infty}\sum_{i=1}^{n}|z_p(t)-z_p^*(t)| = 0,\ p = 1, 2, \cdots.$$

We call $z^*(t)$ is globally asymptotic stable.

For the sake of convenience, we list the following assumptions which will be used by us in studying the existence of homoclinic solutions and periodic solutions to the (1.2) in Section 3.

(H1) There exists constant $M_q \geq 0$ such that

$$|f_q(z_q)| \leq M_q,\ z_q \in \mathbb{C},\ q = 1, 2, ..., n.$$

(H2) For $p = 1, 2, \cdots, n$, $t \in \mathbb{R}$, $d_p(t)$ are positive continuous periodic real-functions and $d_p'(t) < 0$, $a_{pq}(t)$, $b_{pq}(t)$, $H_p(t)$ are all continuous periodic functions.

(H3) $H_p \in \mathbb{C}$ is a bounded function, $H_p(t) \neq 0$ for all $t \neq 0$ and

$$B := \left(\int_{\mathbb{R}}|H_p(t)|^2dt\right)^2 + \varepsilon_0^{1/2}\sup_{t\in\mathbb{R}}|H_p(t)| < +\infty,$$

where ε_0 is determined by (1.4).

(H4) $H_p \in \mathbb{C}$ is a bounded function, $H_p(t) \neq 0$ for all $t \neq 0$ and

$$\left(\int_{-T}^{T}|H_p(t)|^2dt\right)^2 + \varepsilon_0^{1/2}\sup_{t\in\mathbb{R}}|H_p(t)| < +\infty.$$

where ε_0 is determined by (1.4).

Remark 2.5. *It follows from (1.4) that $|H_{p_k}(t)| \leq \sup_{t\in\mathbb{R}}|H_p(t)|$. So if assumption (H3) holds, then for each $k \in \mathbb{N}$, $\left(\int_{-kT}^{kT}|H_{p_k}(t)|^2dt\right)^{1/2} < B$.*

3. Homoclinic and periodic solutions

In order to investigate the existence of $2kT$-periodic solutions to (1.3), we need to study some properties of all the possible $2kT$-periodic solutions to the following equations:

$$\begin{aligned}\frac{dx_p(t)}{dt} = \lambda\Big[&- d_p(t)x_p(t) + \sum_{q=1}^{n}a_{pq}^R(t)f_q^R(z_q(t)) - \sum_{q=1}^{n}a_{pq}^I(t)f_q^I(z_q(t)) + \sum_{q=1}^{n}b_{pq}^R(t)f_q^R(z_q(t-\tau_{pq}(t)))\\ &- \sum_{q=1}^{n}b_{pq}^I(t)f_q^I(z_q(t-\tau_{pq}(t))) + H_{p_k}^R(t)\Big],\ \lambda \in (0, 1],\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\frac{dy_p(t)}{dt} = \lambda\Big[&- d_p(t)y_p(t) + \sum_{q=1}^{n}a_{pq}^R(t)f_q^I(z_q(t)) + \sum_{q=1}^{n}a_{pq}^I(t)f_q^R(z_q(t)) + \sum_{q=1}^{n}b_{pq}^R(t)f_q^I(z_q(t-\tau_{pq}(t)))\\ &+ \sum_{q=1}^{n}b_{pq}^I(t)f_q^R(z_q(t-\tau_{pq}(t))) + H_{p_k}^I(t)\Big], \lambda \in (0, 1].\end{aligned}\tag{3.2}$$

For each $k \in \mathbb{N}$, let $\Gamma \subset U_k$ represents the set of all the $2kT$-periodic solutions to (1.3).

Theorem 3.1. *Suppose that the assumptions (H1)-(H3) hold, for each $k \in \mathbb{N}$, if $z_p \in \Gamma$, then there are positive constants A_1, A_2, A_3 and A_4, which are independent of k and λ, such that*

$$\|z_p\|_2 \le A_1,\ \|z_p'\|_2 \le A_2,\ \|z_p\|_{U_k} \le A_3,\ \|z_p'\|_{U_k} \le A_4,\ p = 1, 2, \ldots n.$$

Proof Multiplying both sides of (3.1) by $x_p'(t)$ and integrating on the interval $[-kT, kT]$, we have

$$\begin{aligned}\int_{-kT}^{kT}(x_p'(t))^2dt =& -\lambda\int_{-kT}^{kT}x_p'(t)d_p(t)x_p(t)dt + \lambda\sum_{q=1}^{n}\int_{-kT}^{kT}x_p'(t)a_{pq}^R(t)f_q^R(z_q(t))dt\\ &-\lambda\sum_{q=1}^{n}\int_{-kT}^{kT}x_p'(t)a_{pq}^I(t)f_q^I(z_q(t))dt + \lambda\sum_{q=1}^{n}\int_{-kT}^{kT}x_p'(t)b_{pq}^R(t)f_q^R(z_q(t-\tau_{pq}(t)))dt\\ &-\lambda\sum_{q=1}^{n}\int_{-kT}^{kT}x_p'(t)b_{pq}^I(t)f_q^I(z_q(t-\tau_{pq}(t)))dt + \lambda\int_{-kT}^{kT}x_p'(t)H_{p_k}^R(t)dt.\end{aligned} \tag{3.3}$$

Note that

$$\int_{-kT}^{kT}x_p'(t)d_p(t)x_p(t)dt = \frac{1}{2}\int_{-kT}^{kT}d_p(t)dx_p^2(t) = -\frac{1}{2}\int_{-kT}^{kT}d_p'(t)x_p^2(t)dt. \tag{3.4}$$

Substituting (3.4) into (3.3) and by (H1) and (H3), we have

$$\|x_p'\|_2^2 + \frac{\delta}{2}\|x_p\|_2^2 \le \sum_{q=1}^{n}\left(\|\overline{a}_{pq}^R M_q\|_2 + \|\overline{a}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + \sum_{q=1}^{n}\left(\|\overline{b}_{pq}^R M_q\|_2 + \|\overline{b}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + B\|x_p'\|_2. \tag{3.5}$$

where $\delta = \min_{t\in\mathbb{R}^+} -d_p'(t)$.

From the inequality above, we can see that

$$\|x_p'\|_2^2 \le \sum_{q=1}^{n}\left(\|\overline{a}_{pq}^R M_q\|_2 + \|\overline{a}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + \sum_{q=1}^{n}\left(\|\overline{b}_{pq}^R M_q\|_2 + \|\overline{b}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + B\|x_p'\|_2, \tag{3.6}$$

and

$$\frac{\delta}{2}\|x_p\|_2^2 \le \sum_{q=1}^{n}\left(\|\overline{a}_{pq}^R M_q\|_2 + \|\overline{a}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + \sum_{q=1}^{n}\left(\|\overline{b}_{pq}^R M_q\|_2 + \|\overline{b}_{pq}^I M_q\|_2\right)\cdot\|x_p'\|_2 + B\|x_p'\|_2. \tag{3.7}$$

It follows from (3.6) that

$$\|x_p'\|_2 \le \sum_{q=1}^{n}\left(\|\overline{a}_{pq}^R M_q\|_2 + \|\overline{a}_{pq}^I M_q\|_2 + \|\overline{b}_{pq}^R M_q\|_2 + \|\overline{b}_{pq}^I M_q\|_2 + B\right) := \widehat{w}_p. \tag{3.8}$$

Substituting (3.8) into (3.7), we can have

$$\|x_p\|_2 \le \sqrt{\frac{2}{\delta}}\left[\sum_{q=1}^{n}\left(\|\overline{a}_{pq}^R M_q\|_2 + \|\overline{a}_{pq}^I M_q\|_2 + \|\overline{b}_{pq}^R M_q\|_2 + \|\overline{b}_{pq}^I M_q\|_2 + B\right)\cdot\widehat{w}_p\right]^{1/2} := \widetilde{w}_p. \tag{3.9}$$

Furthermore, it follows from Lemma 2.1 that

$$\begin{aligned}|x_p(t)| \le&(2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}|x_p(s)|^2ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}|x_p'(s)|^2ds\right)^{\frac{1}{2}}\\ =&(2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}|x_p(s)|^2ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}|x_p'(s)|^2ds\right)^{\frac{1}{2}},\end{aligned}$$

which together with (3.8) and (3.9) yields

$$|x_p(t)| \leq (2T)^{-\frac{1}{2}} \left(\int_{-kT}^{kT} |x_p(s)|^2 ds \right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \left(\int_{-kT}^{kT} |x_p'(s)|^2 ds \right)^{\frac{1}{2}} \leq (2T)^{-\frac{1}{2}} \widetilde{w}_p + T(2T)^{-\frac{1}{2}} \widehat{w}_p.$$

thus, we can obtain

$$\|x_p\|_0 = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |x_p(t)| \leq \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} \left\{ (2T)^{-\frac{1}{2}} \widetilde{w}_p + T(2T)^{-\frac{1}{2}} \widehat{w}_p \right\} := \widehat{\rho}_p. \tag{3.10}$$

Clearly, $\widehat{\rho}_p$ is independent of k and λ.

Moreover, from (3.1), we can see that

$$\begin{aligned} |x_p'(t)| \leq & |d_p(t)||x_p(t)| + \sum_{q=1}^{n} |a_{pq}^R(t)||f_q^R(z_q(t))| + \sum_{q=1}^{n} |a_{pq}^I(t)||f_q^I(z_q(t))| + \sum_{q=1}^{n} |b_{pq}^R(t)||f_q^R(z_q(t-\tau_{pq}(t)))| \\ & + \sum_{q=1}^{n} |b_{pq}^I(t)||f_q^I(z_q(t-\tau_{pq}(t)))| + |H_{p_k}^R(t)|, \end{aligned}$$

which together with (H1), (H3) and (3.10) gives

$$\|x_p'\|_0 = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |x_p'(t)| \leq \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} \left\{ \overline{d}_p \widehat{\rho}_p + \sum_{q=1}^{n} \left(\overline{a}_{pq}^R + \overline{a}_{pq}^I + \overline{b}_{pq}^R + \overline{b}_{pq}^I \right) M_q + B \right\} := \widetilde{\rho}_p. \tag{3.11}$$

Clearly, $\widetilde{\rho}_p$ is independent of k and λ.

By using the same methods as above, we can see that there exists four positive constants $\widehat{\sigma}_p$, $\widetilde{\sigma}_p$, $\widehat{\delta}_p$ and $\widetilde{\delta}_p$, which are independent of k and λ such that

$$\|y_p'\|_2 \leq \widehat{\sigma}_p, \quad \|y_p\|_2 \leq \widetilde{\sigma}_p. \tag{3.12}$$

and

$$\|y_p\|_0 \leq \widehat{\delta}_p, \quad \|y_p'\|_0 \leq \widetilde{\delta}_p. \tag{3.13}$$

Note that $z_p(t) = x_p(t) + iy_p(t)$, then by (3.10), (3.11) and (3.13), we get

$$\|z_p\|_{U_k} = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |z_p(t)| = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |x_p(t)| + \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |y_p(t)| \leq \widehat{\rho}_p + \widehat{\delta}_p := A_3, \tag{3.14}$$

and

$$\|z_p'\|_{U_k} = \max_{1\leq p\leq n} \sup_{t\in[0,2kT]} |z_p'(t)| \leq \|x_p'\|_0 + \|y_p'\|_0 \leq \widetilde{\rho}_p + \widetilde{\delta}_p := A_4. \tag{3.15}$$

Moreover, by (3.8), (3.9) and (3.12), we have

$$\begin{aligned} \|z_p\|_2 = & \left(\int_{-kT}^{kT} |z_p(t)|^2 dt \right)^{1/2} = \left(\int_{-kT}^{kT} |x_p(t) + iy_p(t)|^2 dt \right)^{1/2} \\ \leq & \left(\int_{-kT}^{kT} |x_p(t)|^2 dt \right)^{1/2} + \left(\int_{-kT}^{kT} |y_p(t)|^2 dt \right)^{1/2} \leq \widetilde{w}_p + \widetilde{\sigma}_p := A_1, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \|z_p'\|_2 = & \left(\int_{-kT}^{kT} |z_p'(t)|^2 dt \right)^{1/2} = \left(\int_{-kT}^{kT} |x_p'(t) + iy_p'(t)|^2 dt \right)^{1/2} \\ \leq & \left(\int_{-kT}^{kT} |x_p'(t)|^2 dt \right)^{1/2} + \left(\int_{-kT}^{kT} |y_p'(t)|^2 dt \right)^{1/2} \leq \widehat{w}_p + \widehat{\sigma}_p := A_2, \end{aligned} \tag{3.17}$$

Therefore, from (3.14)-(3.17), we know that all the conclusions of Theorem 3.1 hold.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied, then for each $k \in \mathbb{N}$,* (1.3) *has at least one $2kT$-periodic solution $z_{p_k}(t)$ in $\Gamma \subset U_k$ such that*

$$\|z_{p_k}\|_2 \le A_1,\ \|z'_{p_k}\|_2 \le A_2,\ \|z_{p_k}\|_{U_k} \le A_3,\ \|z'_{p_k}\|_{U_k} \le A_4,\ p = 1, 2, ..., n. \tag{3.18}$$

where A_1, A_2, A_3 and A_4 are constants defined in Theorem 3.1.

Proof In order to apply Lemma 2.3, for each $k \in \mathbb{N}$, we consider the following equation:

$$\frac{dz_p(t)}{dt} = -\lambda d_p(t)z_p(t) + \lambda \sum_{q=1}^{n} a_{pq}(t) f_q(z_q(t)) + \lambda \sum_{q=1}^{n} b_{pq}(t) f_q(z_q(t - \tau_{pq}(t))) + \lambda H_{p_k}(t),\ \lambda \in (0, 1]. \tag{3.19}$$

Let $\Omega_1 \subset U_k$ represent the set of all $2kT$-periodic solutions to (3.19). Since $(0, 1) \subset (0, 1]$, then we can see that $\Omega_1 \subset \Gamma$, where Γ is defined by Theorem 3.1. If $z_p \in \Omega_1$, then by applying Theorem 3.1, we can have

$$\|z_p\|_{U_k} \le A_3,\ \ \|z'_p\|_{U_k} \le A_4.$$

Let

$$\Omega_2 = \Big\{z_p : z_p \in \ker L, QNz_p = 0\Big\}.$$

If $z_p \in \Omega_2$, then $z_p = c_p \in \mathbb{R}$ and

$$QNz_p = \frac{1}{2kT}\int_{-kT}^{kT}\Bigg(- d_p(\xi)c_p + \sum_{q=1}^{n} a_{pq}(t) f_q(c_q) + \sum_{q=1}^{n} b_{pq}(t) f_q(c_q) + H_{p_k}(\xi)\Bigg)d\xi,$$

i.e.,

$$\Delta(c_p) := \frac{1}{2kT}\int_{0}^{2kT}\Bigg(- d_p(\xi)c_p + \sum_{q=1}^{n} a_{pq}(t) f_q(c_q) + \sum_{q=1}^{n} b_{pq}(t) f_q(c_q) + H_{p_k}(\xi)\Bigg)d\xi = 0.$$

Now, if we set

$$\Omega = \Big\{z_p : z_p \in U_k,\ \|z_p\|_0 \le A_3,\ \|z'_p\|_0 \le A_4 + 1\Big\},$$

then $\Omega_1 \cup \Omega_2 \subset \Omega$. So the condition (*i*) and (*ii*) of Lemma 2.3 are satisfied.

In order to verifying the condition (*iii*) in Lemma 2.3, we define

$$H(z_p, \mu) : (\Omega \cap \mathbb{R}) \times [0, 1] \to \mathbb{R},\ H(z_p, \mu) = -\mu z_p + (1 - \mu)\Delta(z_p),$$

where

$$\Delta(z_p) := \frac{1}{2kT}\int_{0}^{2kT}\Big(- d_p(s)z_p(s) + \sum_{q=1}^{n} a_{pq}(s) f_q(z_q(s)) + \sum_{q=1}^{n} b_{pq}(s) f_q(z_q(s)) + H_{p_k}(s)\Big)ds.$$

From assumption (H1)-(H3), it is easy to see that

$$H(z_p, \mu) \ne 0,\ \ \forall (z_p, \mu) \in [\partial(\Omega \cap \mathbb{R})] \times [0, 1].$$

Hence,

$$d_B(\Delta, \Omega \cap \mathbb{R}, 0) = \deg(H(z_p, 0), \Omega \cap \mathbb{R}, 0) = \deg(H(z_p, 1), \Omega \cap \mathbb{R}, 0) \ne 0.$$

Therefore, by applying Lemma 2.3, we can see that (3.19) has a $2kT$-periodic solution $z_{p_k} \in \overline{\Omega}$. Obviously, $z_{p_k}(t)$ is a $2kT$-periodic solution to (1.3) for the case of $\lambda \ne 1$, so $z_{p_k} \in \Gamma$. Thus, by using Theorem 3.1, we get

$$\|z_{p_k}\|_2 \le A_1,\ \|z'_{p_k}\|_2 \le A_2,\ \|z_{p_k}\|_{U_k} \le A_3,\ \|z'_{p_k}\|_{U_k} \le A_4, p = 1, 2, ..., n.$$

Theorem 3.3. *Assume conditions (H1), (H2) and (H4) hold, then* (1.2) *has at least one* $2T$*-periodic solution.*

Proof From the assumption (H3), we see that $H_p \in \mathbb{C}$ is a continuous $2T$-periodic function, $H_p(t) \neq 0$ for all $t \neq 0$, $T > 0$ is a given constant, and

$$\left(\int_{-T}^{T} |H_p(t)|^2 dt\right)^2 + \varepsilon_0^{1/2} \sup_{t\in\mathbb{R}} |H_p(t)| < +\infty.$$

So by using Theorem 3.2, we know that (1.1) has at least one $2T$-periodic solution. Thus, (1.2) has at least one $2T$-periodic solution.

Theorem 3.4. *Assume that* $f(0) = 0$, $\lim_{|t|\to+\infty} H_p(t) = 0$ *and conditions (H1) holds, then* (1.2) *has at least one homoclinic solution.*

Proof From Theorem 3.2, we see that for each $k \in \mathbb{N}$, there exists a $2kT$-periodic solution $z_{p_k}(t)$ to (1.3). So for every $k \in \mathbb{N}$, $z_{p_k}(t)$ is satisfied

$$\frac{dz_{p_k}(t)}{dt} = -d_p(t)z_{p_k}(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_{q_k}(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_{q_k}(t-\tau_{pq}(t))) + H_p(t). \tag{3.20}$$

Furthermore, it follows from Theorem 3.2 that

$$\|z_{p_k}\|_{U_k} \leq A_4.$$

Then by (3.20) and (H1), we have

$$\|z'_{p_k}\|_{U_k} \leq \max_{1\leq p\leq n}\left\{\overline{d}_p A_4 + \sum_{q=1}^{n} \overline{a}_{pq} M_q + \sum_{q=1}^{n} \overline{b}_{pq} M_q + B\right\} := \rho,\ p = 1, 2, \cdots, n.$$

Clearly, ρ is a constant independent of k. By applying Lemma 2.2, we can see that there exists a $z_{p_0} \in \mathbb{C}$ and a subsequence $\{z_{p_{k_j}}\}$ of $\{z_{p_k}\}$ such that for each interval $[c, d] \in \mathbb{R}$, $z_{p_{k_j}}(t) \to z_{p_0}(t)$ and $z'_{p_{k_j}}(t) \to z'_{p_0}(t)$ uniformly on $[c, d]$. For all $a, b \in \mathbb{R}$ with $a < b$, there must exist a positive integer j_0 such that for $j > j_0$, $\left[a - \|\tau_{pq}\|_0,\ b + \|\tau_{pq}\|_0\right] \subset [-k_jT, k_jT - \varepsilon_0]$. So for $t \in \left[a - \|\tau_{pq}\|_0, b + \|\tau_{pq}\|_0\right]$, it follows from (1.4) and (3.20) that

$$\frac{dz_{p_{k_j}}(t)}{dt} = -d_p(t)z_{p_{k_j}}(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_{q_{k_j}}(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_{q_{k_j}}(t-\tau_{pq}(t))) + H_p(t). \tag{3.21}$$

In view of $z_{p_{k_j}}(s) \to z_{p_0}(s)$, $z_{p_{k_j}}(s - \tau_{pq}(s)) \to z_{p_0}(s)$ uniformly on $[a, b]$, and by (3.21), we see that

$$\begin{aligned} z'_{p_{k_j}}(t) = \frac{dz_{p_{k_j}}(t)}{dt} &= -d_p(t)z_{p_{k_j}}(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_{q_{k_j}}(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_{q_{k_j}}(t-\tau_{pq}(t))) + H_p(t) \\ &\to -d_p(t)z_{p_0}(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_{p_0}(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_{p_0}(t-\tau_{pq}(t))) + H_p(t) := \varpi(t), \\ &\quad \text{uniformly on } [a, b] \end{aligned}$$

which together with the fact that $z'_{p_{k_j}}(t)$ is the continuous differential for $z_{p_{k_j}}(t)$ on (a, b) for every $j > j_0$, and $z'_{p_{k_j}}(t) \to \varpi(t)$ uniformly on $[a, b]$, then we can obtain $\varpi(t) = \frac{d}{dt}[z_{p_0}(t)]$ on (a, b). Since a, b are arbitrary and $a < b$, we get $\varpi(t) = \frac{d}{dt}[z_{p_0}(t)]$, $t \in \mathbb{R}$, that means $z_{p_0}(t)$ is a solution of (1.1) .

In the following, we will prove that $z_{p_0}(t) \to 0$ and $z'_{p_0}(t) \to 0$ as $|t| \to +\infty$.
Since

$$\int_{-\infty}^{+\infty} (|z_{p_0}(t)|^2 + |z'_{p_0}(t)|^2)dt = \lim_{n\to+\infty} \int_{-nT}^{nT} (|z_{p_0}(t)|^2 + |z'_{p_0}(t)|^2)dt = \lim_{n\to+\infty} \lim_{j\to+\infty} \int_{-nT}^{nT} (|z_{p_{k_j}}(t)|^2 + |z'_{p_{k_j}}(t)|^2)dt,$$

clearly, for every $n \in \mathbb{N}$ if $k_j > n$, then it follows from Theorem 3.2 that ,

$$\int_{-nT}^{nT} (|z_{p_{k_j}}(t)|^2 + |z'_{p_{k_j}}(t)|^2)dt \le \int_{-k_jT}^{k_jT} (|z_{p_{k_j}}(t)|^2 + |z'_{p_{k_j}}(t)|^2)dt \le A_1^2 + A_2^2.$$

Let $n \to +\infty$ and $j \to +\infty$, then we have

$$\int_{-\infty}^{+\infty} (|z_{p_0}(t)|^2 + |z'_{p_0}(t)|^2)dt \le A_1^2 + A_2^2,$$

and as $r \to +\infty$, then we get

$$\int_{|t|\ge r} (|z_{p_0}(t)|^2 + |z'_{p_0}(t)|^2)dt \to 0. \tag{3.22}$$

By applying the Lemma 2.1, we obtain

$$\begin{aligned} |z_{p_0}(t)| \le& (2T)^{-\frac{1}{2}} \Big(\int_{t-T}^{t+T} |z_{p_0}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \Big(\int_{t-T}^{t+T} |z'_{p_0}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \\ \le& \Big[(2T)^{-\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \Big] \cdot \Big[\int_{t-T}^{t+T} (|z_{p_0}(\xi)|^2 + |z'_{p_0}(\xi)|^2) d\xi \Big]^{\frac{1}{2}} \to 0, \qquad as\ \ |t| \to +\infty. \end{aligned} \tag{3.23}$$

Finally, we will prove that

$$|z'_{p_0}(t)| \to 0, \ \text{ as } \ |t| \to +\infty. \tag{3.24}$$

From $f(0) = 0$ and $\lim_{|t|\to+\infty} H_p(t) = 0$, it follows that

$$z'_{p_0}(t) = -d_p(t)z_{p_0}(t) + \sum_{q=1}^{n} a_{pq}(t) f_q(z_{q_0}(t)) + \sum_{q=1}^{n} b_{pq}(t) f_q(z_{q_0}(t - \tau_{pq}(t))) + H_p(t) \to 0, \ \ as\ \ |t| \to +\infty.$$

Therefore, (3.24) holds. So there exist a homoclinic solution for (1.1).

Furthermore, since $z_0(t) = (z_{1_0}(t), z_{2_0}(t), ..., z_{n_0}(t))^\top$, then from (3.23), we can see that $z_0(t) \to 0 = (0, 0, ..., 0)^\top$ as $|t| \to +\infty$. Similar, by (3.24), we can have $z'_0(t) \to 0 = (0, 0, ..., 0)^\top$ as $|t| \to +\infty$. Thus, (1.2) has at least one homoclinic solution. Therefore, the prove of Theorem 3.3 is completed.

Remark 3.5. *Although, the existence of homoclinic solutions of the real systems have been widely studied, see, to name a few, [12], [15], [18], [24], there is no result on homoclinic solutions of the complex-valued neural networks. As we all known, the problem of existence of homoclinic solutions is one of the most important problems in qualitative theory of differential systems. Theorem 3.3 and Theorem 3.4 have provided the new results on homoclinic solutions of the complex-valued neural networks. They can be regarded as the first results on such problem.*

4. Asymptotic behaviours of solution $z(t) = 0$

In order to obtain globally asymptotic stability, let $H_p(t) = 0$ and $f_p(0) = 0$. Then (1.1) is changed into

$$\frac{dz_p(t)}{dt} = -d_p(t)z_p(t) + \sum_{q=1}^{n} a_{pq}(t) f_q(z_q(t)) + \sum_{q=1}^{n} b_{pq}(t) f_q(z_q(t - \tau_{pq}(t))), \tag{4.1}$$

and $z = \mathbf{0}$ is the equilibriun point of (4.1).

Theorem 4.1. *Assume that the conditions of Theorem 3.1 are satisfied, further that*

(H5) there exists a positive constant L_q, and $\eta > 0$ such that

$$|f_q(x) - f_q(y)| \leq L_q|x - y|, \ \ \forall x, y \in \mathbb{C}, \ q = 1, 2, \cdots, n,$$

and

$$x_q f_q(x_q) < -\eta|x_q|^2, \ \ q = 1, 2, \cdots, n.$$

(H6) Let $\xi_p > 0$, where

$$\xi_p = \lim_{t\to+\infty} \inf\Big[2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p - \gamma_p(t)\Big]$$

where $\alpha_p = \sum_{q=1,p\neq q}^{n} \overline{a}_{pq}L_q$, $\beta_p = \sum_{q=1}^{n} \overline{b}_{pq}L_q$, $\gamma_p(t) := \sum_{q=1}^{n}\Big(\overline{a}_{pq}L_q + \overline{b}_{pq}L_q\omega_{pq}(t)\Big)$, $\omega_{pq}(t) = \frac{1}{1-\tau'_{pq}(\widetilde{\tau}_{pq}(t))}$, $\widetilde{\tau}_{pq}(t)$ is inverse function of *$t - \tau_{pq}(t)$.*

Then (4.1) *has unique periodic wave solution $z^*(t) = (z_1^*(t), z_2^*(t), ..., z_n^*(t))^\top$ which is globally asymptotic stable.*

Proof Suppose $x(t)$ be any solution of (4.1). Let

$$V_p(t) = (z_p(t))^2, \ p = 1, 2, \cdots, n.$$

Derivation of it along the solution of (4.1) gives

$$\begin{aligned}
V'_p(t) =& 2z_p(t)z'_p(t) = 2z_p(t)\Big[- d_p(t)z_p(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(z_q(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(z_q(t - \tau_{pq}(t)))\Big] \\
=& - 2d_p z_p^2(t) + 2z_p(t)\sum_{q=1}^{n} a_{pq}(t)f_q(z_q(t)) + 2z_p(t)\sum_{q=1}^{n} b_{pq}(t)f_q(z_q(t - \tau_{pq}(t))) \\
\leq& - 2\underline{d}_p z_p^2(t) - 2\underline{a}_{pp}\eta|z_p(t)|^2 + 2|z_p(t)|\sum_{q=1,p\neq q}^{n} \overline{a}_{pq}L_q|z_q(t)| + 2|z_p(t)|\sum_{q=1}^{n} \overline{b}_{pq}L_q|z_q(t - \tau_{pq}(t))| \\
\leq& - 2\underline{d}_p z_p^2(t) - 2\underline{a}_{pp}\eta|z_p(t)|^2 + \alpha_p|z_p(t)|^2 + \sum_{q=1,p\neq q}^{n} \overline{a}_{pq}L_q|z_q(t)|^2 + \beta_p|z_p(t)|^2 + \sum_{q=1}^{n} \overline{b}_{pq}L_q|z_q(t - \tau_{pq}(t))|^2 \\
=& - \Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p\Big)|z_p(t)|^2 + \sum_{\substack{q=1,\\ p\neq q}}^{n} \overline{a}_{pq}L_q|z_q(t)|^2 + \sum_{q=1}^{n} \overline{b}_{pq}L_q|z_q(t - \tau_{pq}(t))|^2.
\end{aligned}$$

Define

$$V_{\tau_{pq}}(t) = \sum_{q=1}^{n} \overline{b}_{pq}L_q \int_{t-\tau_{pq}(t)}^{t} \omega_{pq}(s)z_q^2(s)ds.$$

Then we have

$$V'_{\tau_{pq}}(t) = \sum_{q=1}^{n} \overline{b}_{pq}L_q\Big[\omega_{pq}(t)z_q^2(t) - z_p^2(t - \tau_{pq}(t))\Big].$$

Choose the Lyapunov function for (4.1) in the following form:

$$V(t) = \sum_{p=1}^{n} \Big[V_p(t) + V_{\tau_{pq}}(t)\Big].$$

Derivating it along the solution of (4.1) gives

$$\begin{aligned}
V'(t) \leq &\sum_{p=1}^{n}\Big\{-\Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p\Big)|z_p(t)|^2 + \sum_{q=1,p\neq q}^{n}\overline{a}_{pq}L_q|z_q(t)|^2 + \sum_{q=1}^{n}\overline{b}_{pq}L_q|z_q(t-\tau_{pq}(t))|^2 \\
&+ \sum_{q=1}^{n}\overline{b}_{pq}L_q\Big[\omega_{pq}(t)z_q^2(t) - z_p^2(t-\tau_{pq}(t))\Big]\Big\} \\
= &\sum_{p=1}^{n}\Big\{-\Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p\Big)|z_p(t)|^2 + \sum_{q=1,p\neq q}^{n}\overline{a}_{pq}L_q|z_q(t)|^2 + \sum_{q=1}^{n}\overline{b}_{pq}L_q|z_q(t-\tau_{pq}(t))|^2 \\
&+ \sum_{q=1}^{n}\overline{b}_{pq}L_q\omega_{pq}(t)z_q^2(t) - \sum_{q=1}^{n}\overline{b}_{pq}L_q z_p^2(t-\tau_{pq}(t))\Big\} \\
= &\sum_{p=1}^{n}\Big\{-\Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p\Big)|z_p(t)|^2 + \sum_{q=1,p\neq q}^{n}\overline{a}_{pq}L_q|z_q(t)|^2 + \sum_{q=1}^{n}\overline{b}_{pq}L_q\omega_{pq}(t)z_q^2(t)\Big\} \\
= &-\sum_{p=1}^{n}\Big\{\Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p\Big)|z_p(t)|^2 - \sum_{q=1}^{n}\Big(\overline{a}_{pq}L_q + \overline{b}_{pq}L_q\omega_{pq}(t)\Big)|z_q(t)|^2\Big\} \\
\leq &-\sum_{p=1}^{n}\Big(2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p - \gamma_p(t)\Big)|z_p(t)|^2.
\end{aligned} \tag{4.2}$$

Assumption $(H6)$ yields, for any $\varepsilon > 0$ and $\xi_p - \varepsilon > 0$, there exists a positive constant T (enough large) such that

$$2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p - \gamma_p(t) \geq \xi_p - \varepsilon, \text{ for all } t > T,$$

which together with (4.2) gives

$$V'(t) \leq -\sum_{p=1}^{n}(\xi_p - \varepsilon)|z_p(t)|^2 < 0, \text{ for all } t > T. \tag{4.3}$$

Integrating both sides of (4.3) from T to $+\infty$ gives

$$V(t) + \int_T^{+\infty}\sum_{p=1}^{n}(\xi_p - \varepsilon)|z_p(s)|^2 ds \leq V(0).$$

By applying the Barbalat's Lemma [1], we can have

$$\lim_{t\to+\infty}\sum_{p=1}^{n}|z_p(t)| = 0.$$

Therefore, the prove of Theorem 4.1 is completed.

5. An illustrative example

Example 5.1. *Consider the following two-neuron complex-valued recurrent neural networks with time-varying delays:*

$$\dot{z}(t) = -D(t)z(t) + A(t)f(z(t)) + B(t)f(z(t-\tau(t))) + H(t), \tag{5.1}$$

where

$$D(t) = \begin{pmatrix} \frac{2}{3} + e^{-t},\ 0 \\ 0,\ \frac{2}{3} + e^{-t} \end{pmatrix},\ A(t) = \begin{pmatrix} \frac{1}{5} - \frac{i}{5}\sin t, \frac{2}{5} - \frac{i}{5}\cos t \\ \frac{2}{5} - \frac{i}{5}\cos t, \frac{1}{5} - \frac{i}{5}\sin t \end{pmatrix}\ B(t) = \begin{pmatrix} \frac{1}{6} - \frac{i}{6}\cos t, \frac{1}{6} - \frac{i}{6}\sin t \\ \frac{1}{6} - \frac{i}{6}\sin t, \frac{1}{6} - \frac{i}{6}\cos t \end{pmatrix},$$

$$H(t) = \begin{pmatrix} -\frac{1}{3} + 2i\sin t \\ -\frac{1}{3} + 3i\cos t \end{pmatrix},\ \tau(t) = 2t,\ \ f_q(z_q) = -\frac{1}{2}x_q + \frac{i}{2}y_q,$$

where $z_q = x_q + iy_q \in \mathbb{C}$ and $p, q = 1, 2$. Then we can see that the conditions (H1)-(H4) are satisfied. Thus, by applying Theorem 3.1-3.4, (5.1) has at least one periodic solution and one homoclinic solution .

Furthermore, for any $x_q = x_q^R + ix_q^I \in \mathbb{C}$, $y_q = y_q^R + iy_q^I \in \mathbb{C}$, and $q = 1, 2$, we have

$$|f_q(x_q) - f_q(y_q)| = \left| -\frac{1}{2}x_q^R + \frac{i}{2}x_q^I + \frac{1}{2}y_q^R - \frac{i}{2}y_q^I \right| \le \frac{1}{2}\sqrt{|x_q^R - y_q^R|^2 + |x_q^I - y_q^I|^2} = \frac{1}{2}|x_q - y_q|,$$

and

$$f_q(z_q)z_q = \Big(-\frac{1}{2}x_q + \frac{i}{2}y_q\Big)\cdot(x_q + iy_q) = -\frac{1}{2}\big(|x_q|^2 + |y_q|^2\big) = -\frac{1}{2}|z_q|^2.$$

Then, we have $L_q = \frac{1}{2}$ and $\eta = \frac{1}{2}$, which implies that (H5) holds. Moreover, we can get

$$\underline{d}_p = \frac{2}{3},\ \bar{a}_{pp} = \frac{\sqrt{2}}{5},\ \bar{a}_{pq} = \frac{\sqrt{5}}{5},\ \bar{b}_{pq} = \frac{\sqrt{2}}{6}, \alpha_p = \sum_{q=1, p\neq q}^{2} \bar{a}_{pq}L_q = \frac{\sqrt{5}}{10},\ \beta_p = \sum_{q=1}^{2} \bar{b}_{pq}L_q = \frac{\sqrt{2}}{12},$$

$$\omega_{pq}(t) = -1,\ \gamma_p(t) := \sum_{q=1}^{2}\left(\bar{a}_{pq}L_q + \bar{b}_{pq}L_q\omega_{pq}(t)\right) = \frac{\sqrt{5}}{10} - \frac{\sqrt{2}}{12},$$

thus we have

$$\xi_p = \liminf_{t\to+\infty}\left[2\underline{d}_p + 2\underline{a}_{pp}\eta - \alpha_p - \beta_p - \gamma_p(t)\right] = \liminf_{t\to+\infty}\left[\frac{4}{3} + \frac{\sqrt{2}}{5} - \frac{\sqrt{5}}{10} - \frac{\sqrt{2}}{12} - \frac{\sqrt{5}}{10} + \frac{\sqrt{2}}{12}\right]$$
$$= \frac{4}{3} + \frac{\sqrt{2}}{5} - \frac{\sqrt{5}}{5} > 0.$$

Therefore, it follows from Theorem 4.1 that the solution of (5.1) is globally asymptotic stable.

Remark 5.2. *Though [5], [17], [21], [22], [28] the complex-valued neural networks with time-varying delays are studied, the methods using in [5], [17], [21], [22], [28] to obtain the periodic solutions are Matrix measure method, different approaches or Brouwer's fixed point theorem, which are different from the methods using in this paper. In this paper, we use an extension of Mawhin's continuation theorem. Moreover, the problem of existence of homoclinic solution is not touched in the references and hence the results there cannot directly be applied to* (5.1) *either.*

6. Conclusion

In this paper, we are concerned with the complex-valued neural networks with time-varying delays. The results on the existence of at least one homoclinic solution and periodic solution have been completely established by means of an extension of Mawhin's continuation theorem and an approximation technique, the global exponential stability of the solutions are further obtained by applying the Lyapunov function. It must be mentioned that it is the first time to discuss the existence of homoclinic solutions for the complex-valued neural networks with time-varying delays and the methods of obtaining the periodic solutions in this paper are different from the corresponding ones in the literature. So, the results established in the present paper are essentially new and can improve and extend previous works.

The proposed results in the paper can be applied to the impulsive or stochastic complex-valued neural networks and further consider the fixed-time stability of the homoclinic solutions and periodic solutions, some related papers can be referred, such as [3], [10], [11]. These are the further researches.

Conflict of interest statement

The authors declare that they have no conflict of interest.

Acknowledgements

The authors thank the editor and reviewers for their insightful suggestions which improved this work significantly.

References

[1] I. Barbalat, Systems d'equations differential d'oscillationsn onlinearities, Revue Roumaine de Mathématique Pures et Appliquées 4 (1959) 267–270.

[2] M. Bohner, V. S. H. Rao, S. Sanyal, Global stability of complex-valued neural networks on time scales, Differential Equations and Dynamical Systems 19 (2011) 3–11.

[3] Z. W. Cai, L. H. Huang, Z. Y. Wang, Finite-/fixed-time stability of nonautonomous functional differential inclusion: lyapunov approach involving indefinite derivative, IEEE Transactions on Neural Networks and Learning Systems (2021) doi: 10.1109/TNNLS.2021.3083396.

[4] R. Ceylan, M. Ceylan, Y. Ozbay, S. Kara, Fuzzy clustering complex-valued neural network to diagnose cirrhosis disease, Expert Systems with Applications 38 (2011) 9744–9751.

[5] W. Q. Gong, J. L. Liang, J. D. Cao, Matrix measure method for global exponential stability of complex-valued recurrent neural networks with time-varying delays, Neural Networks 70 (2015) 81–89.

[6] A. Hirose, Complex-Valued Neural Networks: Theories and Applications, World Scientific Publishing Co. In.c, River Edge, NJ, 2003.

[7] A. Hirose, Recent progress in applications of complex-valued neural networks, in: Proceedings of 10th International Conference on Artificial Intelligence and Soft Computing II, 2010, pp.42–46.

[8] J. Hu, J. Wang, Global stability of complex-valued recurrent neural networks with time-delays, IEEE Transactions on Neural Networks and Learning Systems 23 (2012) 853–865.

[9] G. Huseyin, A novel diagnosis system for Parkinson's disease using complex-valued artificial neural network with k-means clustering feature weighting method, Neural Computing and Applications 28 (2017) 1657–1666.

[10] F. C. Kong, Q. X. Zhu, T. W. Huang, New fixed-time stability lemmas and applications to the discontinuous fuzzy inertial neural networks, IEEE Transactions on Fuzzy Systems 29 (2021) 3711–3722.

[11] F. C. Kong, Q. X. Zhu, New fixed-time synchronization control of discontinuous inertial neural networks via indefinite Lyapunov-Krasovskii functional method, International Journal of Robust and Nonlinear Control 31 (2021) 471–495.

[12] M. Lzydorek, J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems. Journal of Differential Equations 219 (2005) 375–389.

[13] X. Liu, K. Fang, B. Liu, A synthesis method based on stability analysis for complex-valued Hopfield neural network, in: Proceeding of 7th Asian Control Conference, Hongkong China, (2009) 1245–1250.

[14] S. P. Lu, Homoclinic solutions for a class of second-order p-Laplacian differential systems with delay, Nonlinear Analysis: Real World Applications 12 (2011) 780–788.

[15] S.P. Lu, F.C. Kong, Homoclinic solutions for n-dimensional prescribed mean curvature p-Laplacian equations, Bound Value Problem 2015 (2015) 105.

[16] X. W. Liu, T. P. Chen, Global exponential stability for complex-valued recurrent neural networks with asynchronous time delays, IEEE Transactions on Neural Networks and Learning Systems 27 (2016) 593–606.

[17] J. Pan, X. Liu, W. Xie, Exponential stability of a class of complex-valued neural networks with time-varying delays, Neurocomputing 164 (2015) 293–299.

[18] P. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proceedings of the Royal Society of Edinburgh, Section: A Mathematics 114 (1990) 33–38.

[19] R. Rakkiyappan, G. Velmurugan, X. Li, Complete stability analysis of complex-valued neural networks with time delays and impulses, Neural Processing Letters 41 (2015) 435–468.

[20] K. Subramanian, P. Muthukumar, Global asymptotic stability of complex-valued neural networks with additive time-varying delays, Cognitive Neurodynamics 11 (2017) 293–306.

[21] Q. K. Song, Stability analysis of complex-valued neural networks with probabilistic time-varying delays, Neurocomputing 159 (2015) 96–104.

[22] Q. K. Song, Z. Zhao, Y. Liu, Impulsive effects on stability of discrete-time complex-valued neural networks with both discrete and distributed time-varying delays, Neurocomputing 168 (2015) 1044–1050.

[23] Q. K. Song, Q. Q. Yu, F. E. Alsaadi, Dynamics of complex-valued neural networks with variable coefficients and proportional delays, Neurocomputing 275 (2018) 2762–2768.

[24] X. H. Tang, L. Xiao, Homoclinic solutions for ordinary p-Laplacian systems with a coercive potential, Nonlinear Analysis 71 (2009) 1124–1322.

[25] Z. Y. Wang, J. D. Cao, Z.W. Cai, L. H. Huang, Periodicity and finite-time periodic synchronization of discontinuous complex-valued neural networks, Neural Networks 119 (2019) 249–260.

[26] Z. Y. Wang, X. Z. Liu, Exponential stability of impulsive complex-valued neural networks with time delay, Mathematics and Computers in Simulation 156 (2019) 143–157.

[27] X. Xu, J. Zhang, J. Shi, Exponential stability of complex-valued neural networks with mixed delays, Neurocomputing 128 (2014) 483–490.

[28] Y. Zhang, Z. Li, K. Li, Complex-valued Zhang neural network for online complex-valued time varying matrix inversion, Applied Mathematics and Computation 217 (2011) 10066–10073.

[29] Z. Q. Zhang, D. L. Hao, Global asymptotic stability for complex-valued neural networks with time-varying delays via new Lyapunov functionals and complex-valued inequalities, Neural Processing Letters 48 (2018) 995–1017.