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On c-sober spaces and ω^*-well-filtered spaces

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Abstract. Based on countably irreducible version of Topological Rudin's Lemma, we give some characterizations of c-sober spaces and ω^*-well-filtered spaces. In particular, we prove that a topological space is c-sober iff its Smyth power space is c-sober and a c-sober space is an ω^*-well-filtered space. We also show that a locally compact ω^*-well-filtered P-space is c-sober and a T_0 space X is c-sober iff the one-point compactification of X is c-sober.

1. Introduction

In non-Hausdorff topology and domain theory, the d-spaces, sober spaces and well-filtered spaces form three of the most important class (see [1, 3–6, 9–18]). In the past few years, the research on sober spaces and well-filtered spaces has got some breakthrough progress (see [14]). In order to study some aspects of well-filtered spaces concerning various countability properties, Xu, Shen, Xi and Zhao introduced two new types of spaces – ω-well-filtered spaces and ω^*-well-filtered spaces ([10, 11]), both of which generalize well-filtered spaces, and the authors obtained many interesting results. For instance, a first countable T_0 space X is sober iff X is an ω-well-filtered d-space; every first-countable ω^*-well-filtered d-space is sober.

In the past two decades, some variants, or more specifically, generalizations, of sobriety such as bounded sobriety and k-bounded sobriety are introduced and studied. In [15], we introduced the concept of countably sober (c-sober for short) spaces to give some characterizations of countably approximating lattices [7] from topology structure perpective. In such spaces, every countably irreducible closed set is the closure of a unique singleton, where a set is countably irreducible simply means it cannot be covered by countably many closed sets unless one of the closed already covers it. C-sober spaces enjoy many pleasing properties similar to sober spaces (see [15, 16]). In [16], we established the dual equivalent between the category of complete lattices ordered generated by their countably prime elements and the category of c-sober P-spaces, where a P-space is a space in which the countable intersection of open sets is open [2, 8].

In this paper, We further study the properties of ω^*-well-filtered spaces and c-sober spaces. It is well-known that every sober space is a well-filtered space, and a locally compact well-filtered space is sober. Recently, Lawson and Xi [6], Xu, Shen et al. [9, 10] proved every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem. It is a natural question whether there are some links between

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c-sober spaces and ω^*-well-filtered spaces. Following Xu, Shen, Xi and Zhao's methods [9–11, 13], we give some new characterizations of c-sober spaces and ω^*-well-filtered spaces. We obtain countably irreducible version of Topological Rudin's Lemma, and prove that a topological space is c-sober iff its Smyth power space is c-sober and a c-sober space is an ω^*-well-filtered space. We also show that a locally compact ω^*-well-filtered P-space is c-sober and a topological space X is c-sober iff the one-point compactification of X is c-sober.

2. Preliminary

We refer to [1] for the standard definitions and notations of order theory and domain theory, and to [3] for the topology.

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow \{x\}$ and $\uparrow x$ for $\uparrow \{x\}$. Define $A^{\uparrow} = \{x \in P : x \text{ is an upper bound of } A \text{ in } P\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). For a nonempty set B of P, let $max(B) = \{b \in B : b \text{ is a maximal element of } B\}$ and $min(B) = \{b \in B : b \text{ is a minimal element of } B\}$. For a set X, $|X|$ will denote the cardinality of X. Let $\mathbb{N}$ denote the set of all natural numbers with the usual order and $\omega = |\mathbb{N}|$.

A nonempty subset D of a poset P is *directed* (resp., *countably directed*) if every nonempty finite (resp., countable) subset of D have an upper bound in D. A subset $I \subseteq P$ is called an *ideal* of P if I is a directed lower set. Dually, we define the notion of *filters*. A poset P is called a *directed complete poset* (resp., *countably directed complete poset*), or *dcpo* (resp., *cdcpo*) for short, if for any directed (countably directed) subset $D \subseteq P$, $\bigvee D$ exists in P. In [11], *cdcpo* is written as ω^*-*dcpo*.

For a T_0 space X and $A \subseteq X$, the closure of A in X is denoted by $\mathrm{cl}_X A$ or simply by $\overline{A}$ if there no confusion. We use $\leq_X$ to repsent the *specialization order* on X, that is, $x \leq_X y$ iff $x \in \overline{\{y\}}$. In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other explanation. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X.

A nonempty subset A of X is *irreducible* if for any $F_1, F_2 \in \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A space X is called *sober*, if for every irreducible closed set F, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$. We denote the set of all irreducible (resp., irreducible closed) subsets of space X by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$).

Definition 2.1. ([15, 16]) *Let X be a topological space and $F \subseteq X$.*
(1) *F is called countably irreducible if F is nonempty and if for any countable family $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{C}(X)$, $F \subseteq \bigcup_{i \in \mathbb{N}} B_i$ implies that $F \subseteq B_i$ for some $i \in \mathbb{N}$.*
(2) *X is called a countably sober space, or c-sober space for short, if for every countably irreducible closed set F, there exists a unique $a \in X$ such that $F = \overline{\{a\}}$.*

We denote the set of all countably irreducible (resp., irreducible closed) subsets of space X by $\mathsf{CIrr}(X)$ (resp., $\mathsf{CIrr}_c(X)$). Since $\mathsf{CIrr}_c(X) \subseteq \mathsf{Irr}_c(X)$, sober spaces are c-sober spaces and the converse is not true. Let X be an infinite countable set endowed with cofinite topology. Then X is a c-sober but not a sober space.

Lemma 2.2. *Let X and Y be two spaces.*
(1) *If A is a countably directed subset of X, then $A \in \mathsf{CIrr}(X)$.*
(2) *If $A \in \mathsf{CIrr}(X)$, then $\mathrm{cl}_X A \in \mathsf{CIrr}_c(X)$.*
(3) *If Y is a subspace of X and $A \subseteq Y$, then $A \in \mathsf{CIrr}(Y)$ iff $A \in \mathsf{CIrr}(X)$.*
(4) *If $f : X \to Y$ is continuous and $A \in \mathsf{CIrr}(X)$, then $f(A) \in \mathsf{CIrr}(Y)$.*

Remark 2.3. *Let X be an uncountably infinite set endowed with the co-countable topology (the empty set and the complements of countable subsets of X are open). Let A be a countably infinite subset of X. Then $\mathrm{cl}_X A = X \in \mathsf{CIrr}_c(X)$ but $A \notin \mathsf{CIrr}(X)$.*

For any topological space X, $\mathcal{G} \subseteq 2^X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \cap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The symbols $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$ respectively, if there is no ambiguous. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base, and the resulting space is denoted by $P_S(\mathcal{G})$.

A subset A of a space X is called saturated if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. X is called well-filtered if it is T_0, and for any open set U and filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The space $P_S(\mathsf{K}(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X. It is easy to verify that the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$). The canonical mapping $\xi_X(= x \mapsto \uparrow x) : X \to P_S(X)$, is an order and topological embedding.

Remark 2.4. ([9, 13]) *Let X be a T_0 space and $\mathcal{A} \subseteq \mathsf{K}(X)$. Then $\bigcap \mathcal{A} = \bigcap \mathrm{cl}_{P_S(X)} \mathcal{A}$.*

The proof of the following proposition is similar to that of [1, Exercise V-4.4], and we omit it.

Proposition 2.5. *Let X be a T_0 space. Then*

(1) *$P_H(\mathsf{CIrr}_c(X))$ is a c-sober space.*

(2) *The mapping $\eta_X : X \to P_H(\mathsf{CIrr}_c(X))$ given by $\eta_X(x) = \overline{\{x\}}$, is an order and topological embedding.*

(3) *If Y is a c-sober space and $f : X \to Y$ is a continuous mapping, then there exists a unique continuous mapping $f^* : P_H(\mathsf{CIrr}_c(X)) \to Y$ such that $f^* \circ \eta_X = f$.*

We call the space $P_H(\mathsf{CIrr}_c(X))$, shortly denoted X^{cs}, with the mapping η_X the *c-sobrification* of X.

Rudin's Lemma plays a crucial role in domain theory (see [1, 3, 4, 9–14, 14]). In 2013, Heckmann and Keimel [4] established the following topological variant of Rudin's Lemma.

Lemma 2.6. (Topological Rudin's Lemma) *Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that meets all members of $\mathcal{A}$.*

In [10] and [11], Xu, Shen, Xi and Zhao introduced the following two kinds of countable version of well-filtered spaces.

Definition 2.7. ([10]) *A T_0 space X is called ω-well-filtered, if for any countable filtered family $\{K_i : i < \omega\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that*

$$\bigcap_{i<\omega} K_i \subseteq U \Rightarrow \exists i_0 < \omega, K_{i_0} \subseteq U.$$

Let X be a set and $\mathcal{A} \subseteq 2^X$. $\mathcal{A}$ is called a countably filtered family if $\mathcal{A}$ is a countably directed subset of the poset $(2^X, \supseteq)$, which means for any countable subfamily $\mathcal{F} \subseteq \mathcal{A}$, there exists an $A \in \mathcal{A}$ such that $A \subseteq B$ for each $B \in \mathcal{F}$.

Definition 2.8. ([11]) *A T_0 space X is called ω^*-well-filtered, if for any countably filtered family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it satisfies that*

$$\bigcap_{i \in I} K_i \subseteq U \Rightarrow \exists i_0 \in I, K_{i_0} \subseteq U.$$

3. *C*-sober spaces

In this section, we formulate and prove some equational characterizations of c-sober spaces. First of all, we give countably irreducible version of Topological Rudin's Lemma, which plays a vital role in characterzing c-sober spaces and ω^*-well-filtered spaces.

Lemma 3.1. *Let X be a topological space and $\mathcal{A}$ a countably irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal countably irreducible closed subset A that meets all members of $\mathcal{A}$.*

Proof. Let $\mathcal{C} = \{B \subseteq C : B \text{ is closed and } B \cap A \neq \emptyset \text{ for each } A \in \mathcal{A}\}$. Then $C \in \mathcal{C} \neq \emptyset$. Since all members of $\mathcal{A}$ are compact, $\mathcal{C}$ is closed under filtered intersection. By the order-dual of Zorn's Lemma, $\mathcal{C}$ contains a minimal element A. Now we show that A is countably irreducible.

Let $A \subseteq \bigcup_{i\in\mathbb{N}} B_i$, where $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{C}(X)$. Then $A = \bigcup_{i\in\mathbb{N}}(A \cap B_i)$. For any $K \in \mathcal{A}$, since $K \cap A \neq \emptyset$, there is some $i \in \mathbb{N}$ such that $K \cap A \cap B_i \neq \emptyset$, and whence $K \in \Diamond(A \cap B_i)$. Thus $\mathcal{A} \subseteq \bigcup_{i\in\mathbb{N}} \Diamond(A \cap B_i)$. Since $\mathcal{A}$ is a countably irreducible subsets of the space $P_S(X)$ and the sets $\Diamond(A \cap B_i)$ are closed in $P_S(X)$, $\mathcal{A} \subseteq \Diamond(A \cap B_j)$ for some $j \in \mathbb{N}$. Thus $A \cap B_j \in \mathcal{C}$. By minimality of A in $\mathcal{C}$, $A = A \cap B_j \subseteq B_j$. Therefore A is countably irreducible. □

Corollary 3.2. *Let X be a T_0 space. If $\mathcal{A} \in \mathsf{CIrr}_c(P_S(X))$, then there exists a family $\{A_i : i \in I\}$ of minimal countably irreducible closed sets such that $\mathcal{A} = \bigcap_{i\in I} \Diamond A_i$.*

Proposition 3.3. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a c-sober space.*

(2) *For any $A \in \mathsf{CIrr}(X)$, $\overline{A} \cap \bigcap\limits_{a\in A} \uparrow a \neq \emptyset$.*

(3) *For any $A \in \mathsf{CIrr}_c(X)$, $A \cap \bigcap\limits_{a\in A} \uparrow a \neq \emptyset$.*

(4) *For any $A \in \mathsf{CIrr}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap\limits_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.*

(5) *For any $A \in \mathsf{CIrr}_c(X)$ and $U \in \mathcal{O}(X)$, $\bigcap\limits_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.*

Proof. The proof is similar to that of [9, Proposition 5.7]. □

Theorem 3.4. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a c-sober space.*

(2) *For any $\mathcal{A} \in \mathsf{CIrr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*

(3) *For any $\mathcal{A} \in \mathsf{CIrr}_c(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*

(4) *$P_S(X)$ is a c-sober space.*

Proof. (1) ⇒ (2): Let $\mathcal{A} \in \mathsf{CIrr}(P_S(X))$ and $U \in \mathcal{O}(X)$ with $\bigcap \mathcal{A} \subseteq U$. If $K \nsubseteq U$ for all $K \in \mathcal{A}$, then $K \cap (X \setminus U) \neq \emptyset$. By Lemma 3.1, there exists a minimal countably irreducible closed set $A \subseteq X \setminus U$ such that A meets all members of $\mathcal{A}$. Since X is c-sober, there exists an $a \in X$ such that $A = \overline{\{a\}}$. It follows from $A \subseteq X \setminus U$ that $a \notin U$. On the other hand, since $K \cap A = K \cap \overline{\{a\}} \neq \emptyset$ for all $K \in \mathcal{A}$, $a \in K$, and whence $a \in \bigcap \mathcal{A} \subseteq U$, this is a contradiction.

(2) ⇒ (3): Trivial.

(3) ⇒ (4): Suppose $\mathcal{A} \in \mathsf{CIrr}_c(P_S(X))$ and let $H = \bigcap \mathcal{A}$. By condition (3), $H \neq \emptyset$. Now we prove the following:

(i) $H \in \mathsf{K}(X)$.

Let $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ be a directed family with $\bigcap \mathcal{A} = H \subseteq \bigcup_{i\in I} U_i$. By condition (3), there exists some $K \in \mathcal{A}$ such that $K \subseteq \bigcup_{i\in I} U_i$. Since K is compact, there is an $i \in I$ such that $K \subseteq U_i$. Thus $H = \bigcap \mathcal{A} \subseteq U_i$.

(ii) $\mathcal{A} = \mathrm{cl}_{P_S(X)}\{H\}$.

Since $\mathcal{A} = \mathrm{cl}_{P_S(X)}\mathcal{A}$, we need only to prove that $\mathcal{A} \cap \Box U \neq \emptyset$ if and only if $\{H\} \cap \Box U \neq \emptyset$ for any $U \in \mathcal{O}(X)$. In fact,

$$\begin{aligned} \mathcal{A} \cap \Box U \neq \emptyset &\Leftrightarrow \exists\, K \in \mathcal{A} \text{ such that } K \subseteq U \\ &\Leftrightarrow \bigcap \mathcal{A} = H \subseteq U \quad \text{(By condition (3))} \\ &\Leftrightarrow \{H\} \cap \Box U \neq \emptyset. \end{aligned}$$

(4) ⇒ (1): For any $A \in \mathsf{CIrr}_c(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap_{a\in A} \uparrow a \subseteq U$. Then $\xi_X(A) \in \mathsf{CIrr}(P_S(X))$. If $K \in \bigcap_{a\in A} \uparrow_{\mathsf{K}(X)} \xi_X(a)$, then $K \subseteq \uparrow a$ for all $a \in A$, and whence $K \subseteq \bigcap_{a\in A} \uparrow a \subseteq U$. Thus $\bigcap_{a\in A} \uparrow_{\mathsf{K}(X)} \xi_X(a) \subseteq \Box U$. Since $P_S(X)$ is sober, by the equivalence of (1) and (4) in Proposition 3.3, there exists some $a \in A$ such that $\uparrow_{\mathsf{K}(X)} \xi_X(a) \subseteq \Box U$. Hence $a \in U$. By Proposition 3.3 again, X is c-sober. □

Theorem 3.5. *Let X be a T_0 space. Then the following conditions are equivalent:*

(1) *X is c-sober.*

(2) *For every continuous mapping $f : X \to Y$ from X to a T_0 space Y and any $\mathcal{A} \in \mathsf{CIrr}(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*

(3) *For every continuous mapping $f : X \to Y$ from X to a T_0 space Y and any $\mathcal{A} \in \mathsf{CIrr}_c(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*

(4) *For every continuous mapping $f : X \to Y$ from X to a c-sober space Y and any $\mathcal{A} \in \mathsf{CIrr}(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*

(5) *For every continuous mapping $f : X \to Y$ from X to a c-sober space Y and any $\mathcal{A} \in \mathsf{CIrr}_c(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*

Proof. (1) $\Rightarrow$ (2): It need only to check $\bigcap_{K\in\mathcal{A}} \uparrow f(K) \subseteq \uparrow f(\bigcap \mathcal{A})$. Let $y \in \bigcap_{K\in\mathcal{A}} \uparrow f(K)$. Then for each $K \in \mathcal{A}$, $f(K) \cap \overline{\{y\}} \neq \emptyset$, equivalently, $K \cap f^{-1}(\overline{\{y\}}) \neq \emptyset$. Since X is c-sober, we can show $f^{-1}(\overline{\{y\}}) \cap \bigcap \mathcal{A} \neq \emptyset$. In fact, if $f^{-1}(\overline{\{y\}}) \cap \bigcap \mathcal{A} = \emptyset$, then $\bigcap \mathcal{A} \subseteq X \setminus f^{-1}(\overline{\{y\}})$. By Theorem 3.4, $K \subseteq X \setminus f^{-1}(\overline{\{y\}})$ for some $K \in \mathcal{A}$. Thus $K \cap f^{-1}(\overline{\{y\}}) = \emptyset$, a contradiction.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): Let $\eta_X : X \to X^{cs}(= P_H(\mathsf{CIrr}_c(X)))$ be the topological embedding from X into its c-sobrification and $\xi_X : X \to P_S(X)$ the canonical topological embedding from X into the Smyth power space of X. Let $A \in \mathsf{CIrr}_c(X)$. Then $\mathrm{cl}_{P_S(X)}\xi_X(A) = \Diamond_{\mathsf{K}(X)}A \in \mathsf{CIrr}_c(P_S(X))$. Thus

$$\begin{aligned}\uparrow_{\mathsf{CIrr}_c(X)} \eta_X(\textstyle\bigcap \xi_X(A)) &= \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(\textstyle\bigcap \mathrm{cl}_{P_S(X)}\xi_X(A)) \quad \text{(By Remark 2.4)}\\ &= \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(\textstyle\bigcap \Diamond_{\mathsf{K}(X)}A)\\ &= \textstyle\bigcap_{K\in\Diamond_{\mathsf{K}(X)}A} \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(K) \quad \text{(By condition (5))}\end{aligned}$$

Since $\uparrow_{\mathsf{CIrr}_c(X)} \eta_X(\bigcap \xi_X(A)) = \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(A^{\uparrow})$ and $\bigcap_{K\in\Diamond_{\mathsf{K}(X)}A} \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(K) = \uparrow_{\mathsf{CIrr}_c(X)} A$, $A \in \uparrow_{\mathsf{CIrr}_c(X)} A = \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(A^{\uparrow})$. Therefore, there is some $x \in A^{\uparrow}$ such that $\overline{\{x\}} \subseteq A$, and consequently, $A = \overline{\{x\}}$. Thus X is c-sober. □

Theorem 3.6. *The following conditions are equivalent for a T_0 space X:*

(1) *X is c-sober.*

(2) *For any $(A, K) \in \mathsf{CIrr}_c(X) \times \mathsf{K}(X)$, $\Diamond_{\mathsf{K}(X)}A$ is an ideal of $(\mathsf{K}(X), \sqsubseteq)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is c-sober and $(A, K) \in \mathsf{CIrr}_c(X) \times \mathsf{K}(X)$. Then there exists an $x \in X$ such that $A = \overline{\{x\}}$, and hence $max(A) = \{x\} \neq \emptyset$. Note that $\Diamond_{\mathsf{K}(X)}A = \{K \in \mathsf{K}(X) : K \cap \downarrow x \neq \emptyset\} = \{K \in \mathsf{K}(X) : \uparrow x \subseteq K\} = \{K \in \mathsf{K}(X) : K \sqsubseteq \uparrow x\} = \downarrow_{\mathsf{K}(X)} \uparrow x$, $\Diamond_{\mathsf{K}(X)}A$ is an ideal of $(\mathsf{K}(X), \sqsubseteq)$. Now we show that $\downarrow (A\cap K) = \downarrow (\downarrow x \cap A) \in \mathcal{C}(X)$. If $\downarrow x \cap K = \emptyset$, that is, $x \notin K$, then $\downarrow (\downarrow x \cap K) = \emptyset$; if $x \in K$, then $\downarrow (\downarrow x \cap K) = \downarrow x \in \mathcal{C}(X)$.

(2) $\Rightarrow$ (1): Let $\mathcal{A} \in \mathsf{CIrr}(P_S(X))$ and $U \in \mathcal{O}(X)$ with $\bigcap \mathcal{A} \subseteq U$. If $K \nsubseteq U$ for each $K \in \mathcal{A}$, then by Lemma 3.1, $X \setminus U$ contains a minimal countably irreducible closed subset A with $\mathcal{A} \subseteq \Diamond_{\mathsf{K}(X)}A$. For any $K \in \mathcal{A}$, we can show that $\downarrow (A \cap K)$ meets all members of $\mathcal{A}$. In fact, let $K' \in \mathcal{A}$, then there exists a $K'' \in \Diamond_{\mathsf{K}(X)}A$ such that $K'' \subseteq K \cap K'$ since $\Diamond_{\mathsf{K}(X)}A$ is directed. Thus $\emptyset \neq K'' \cap A \subseteq K \cap K' \cap A \subseteq \downarrow (K \cap A) \cap K'$. Note that $\downarrow (A \cap K) \in \mathcal{C}(X)$ by condition (2), it follows from the minimality of A that $\downarrow (A \cap K) = A$ for all $K \in \mathcal{A}$. Select an $x \in max(A)$. Then $x \in \downarrow (A \cap K)$ for each $K \in \mathcal{A}$, and consequently, there exists an $a_k \in A \cap K$ such that $x \leq a_k$. Then $x = a_k$ since $x \in max(A)$. Therefore $x \in \bigcap \mathcal{A} \subseteq U \subseteq X \setminus A$, a contradiction. By Theorem 3.4, X is c-sober. □

Let X be topological space, set $X^* = X \cup \{\infty\}$ with the topology whose members are the open subsets of X and all subsets U of X^* such that $X^* \setminus U$ is a closed compact subset of X. The space X^* is called the *one-point compactification* [3] of X. It is well-known the following properties hold:

(i) $\mathcal{C}(X^*) = \{C \cup \{\infty\} : C \in \mathcal{C}(X)\} \cup \{E \subseteq X : X$ is closed and compact in $(X, \mathcal{O}(X))\}$.

(ii) X^* is a T_0 space if and only if X is a T_0 space.

Theorem 3.7. *Let X be a topological space. Then the following conditions are equivalent:*

(1) *X is a c-sober space.*

(2) *The space X^*, which is the one-point compactification of X, is a c-sober space.*

Proof. (1) ⇒ (2): Suppose $A \in \mathsf{CIrr}_c(X^*)$. We now distinguish the following three cases:

Case 1. $\infty \notin A$. Then A is a closed compact subset of X. It is easy to show that $A \in \mathsf{CIrr}_c(X)$. In fact, for any $\{C_i : i \in \mathbb{N}\} \subseteq C(X)$, if $A \subseteq \bigcup_{i\in\mathbb{N}} C_i$, then $A = \bigcup_{i\in\mathbb{N}}(A \cap C_i)$. Note that $A \cap C_i$ is a closed and compact subset of X, $A \cap C_i \in C(X^*)$ for every $i \in \mathbb{N}$. It follows from $A \in \mathsf{CIrr}_c(X^*)$ that there exists an $i \in \mathbb{N}$ such that $A \subseteq A \cap C_i \subseteq C_i$. Thus $A \in \mathsf{CIrr}_c(X)$. Since X is a c-sober space, there is an $x \in X$ such that $A = \mathrm{cl}_X\{x\}$. Since $\infty \in X^* \setminus A \in O(X^*)$ and $x \notin X^* \setminus A$, $\infty \notin \mathrm{cl}_{X^*}\{x\}$. Therefore, $A = \mathrm{cl}_X\{x\} = (\mathrm{cl}_{X^*}\{x\}) \cap X = \mathrm{cl}_{X^*}\{x\}$.

Case 2. $A = \{\infty\}$. Trivial.

Case 3. $A = A_1 \cup \{\infty\}$, where A_1 is a nonempty closed subset of X. Since $A \in \mathsf{CIrr}_c(X^*)$, A_1 is not a compact subset of X. Now we prove that $A_1 \in \mathsf{CIrr}_c(X)$. For any $\{C_i : i \in \mathbb{N}\} \subseteq C(X)$, if $A_1 \subseteq \bigcup_{i\in\mathbb{N}} C_i$, then $A \subseteq \bigcup_{i\in\mathbb{N}}(C_i \cup \{\infty\})$. Note that $C_i \cup \{\infty\} \in C(X^*)$ for each $i \in \mathbb{N}$ and $A \in \mathsf{CIrr}_c(X^*)$, there exists an $i \in \mathbb{N}$ such that $A = A_1 \cup \{\infty\} \subseteq C_i \cup \{\infty\}$. Then $A_1 \subseteq C_i$, proving that $A_1 \in \mathsf{CIrr}_c(X)$. Since X is a c-sober space, there exists an $x \in X$ such that $A_1 = \mathrm{cl}_X\{x\}$. Hence $A = \mathrm{cl}_X\{x\} \cup \{\infty\} = ((\mathrm{cl}_{X^*}\{x\}) \cap X) \cup \{\infty\} = \mathrm{cl}_{X^*}\{x\} \cup \{\infty\}$. Now we show that $\infty \in \mathrm{cl}_{X^*}\{x\}$. We only need to prove that $x \notin E$ for any closed compact subset E of X. Suppose there exists a closed compact subset E of X such that $x \in E$, then $A_1 = \mathrm{cl}_X\{x\} \subseteq E$. Thus A_1 is a compact subset of X. This is a contradiction.

(2) ⇒ (1): Suppose $A \in \mathsf{CIrr}_c(X)$. Then $\mathrm{cl}_{X^*}A \in \mathsf{CIrr}_c(X^*)$. Since X^* is a c-sober space, there exists an $x \in X^*$ such that $\mathrm{cl}_{X^*}A = \mathrm{cl}_{X^*}\{x\}$. It is obvious that $x \neq \infty$. Thus $A = \mathrm{cl}_X A = (\mathrm{cl}_{X^*}A) \cap X = (\mathrm{cl}_{X^*}\{x\}) \cap X = \mathrm{cl}_X\{x\}$. Therefore X is a c-sober space.

□

4. ω^*-well-filtered spaces

In this section, we formulate some characterizations of ω^*-well-filtered spaces and establish some connections between c-sober spaces and ω^*-well-filtered spaces.

Theorem 4.1. *Let X be a T_0 space. Consider the following conditions:*

(1) *X is c-sober.*

(2) *X is ω^*-well-filtered.*

Then (1) ⇒ (2), *and if X is a locally compact P-space, then* (1) *and* (2) *are equivalent.*

Proof. (1) ⇒ (2): Suppose that $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is a countably filtered family, $U \in O(X)$, and $\bigcap_{i\in I} K_i \subseteq U$. Then $\{K_i : i \in I\}$ is countably directed in poset $(\mathsf{K}(X), \leq_{P_S(X)})$, and whence $\{K_i : i \in I\} \in \mathsf{CIrr}(P_S(X))$. By Theorem 3.4, there exists an $i \in I$ such that $K_i \subseteq U$. Thus X is ω^*-well-filtered.

(2) ⇒ (1): Suppose X is a locally compact ω^*-well-filtered P-space and $A \in \mathsf{CIrr}_c(X)$. Let $\mathcal{K}_A = \{K \in \mathsf{K}(X) : A \cap \mathrm{int}K \neq \emptyset\}$. Select an $a \in A$. Since X is locally compact, there exists $K \in \mathsf{K}(X)$ such that $a \in \mathrm{int}K$. Then $a \in A \cap \mathrm{int}K$, and whence $K \in \mathcal{K}_A \neq \emptyset$. We claim that $\mathcal{K}_A$ is countably filtered. Let $\{K_i : i \in \mathbb{N}\} \subseteq \mathcal{K}_A$. Then for each $i \in \mathbb{N}$, $A \cap \mathrm{int}K_i \neq \emptyset$. Since $A \in \mathsf{CIrr}_c(X)$, $A \cap \bigcap_{i\in\mathbb{N}} \mathrm{int}K_i \neq \emptyset$. Select an $x \in A \cap \bigcap_{i\in\mathbb{N}} \mathrm{int}K_i$. Since X is a P-space, $\bigcap_{i\in\mathbb{N}} \mathrm{int}K_i \in O(X)$. By the local compactness of X, there is a $K^* \in \mathsf{K}(X)$ such that $x \in \mathrm{int}K^* \subseteq K^* \subseteq \bigcap_{i\in\mathbb{N}} \mathrm{int}K_i$, and whence $K^* \in \mathcal{K}_A$ and $K^* \subseteq \bigcap_{i\in\mathbb{N}} K_i$. Therefore $\mathcal{K}_A$ is countably filtered.

Note that $A \cap K \neq \emptyset$ for each $K \in \mathcal{K}_A$ and $A \in C(X)$, we have $\bigcap_{K\in\mathcal{K}_A}(K \cap A) \neq \emptyset$ since X is an ω^*-well-filtered space. Let $a \in \bigcap_{K\in\mathcal{K}_A}(K \cap A)$. Then $\overline{\{a\}} = \downarrow a \subseteq A$. If there is an $x \in A \setminus \downarrow a$, then $x \in X \setminus \downarrow a$. Since X is locally compact, there exists a $K \in \mathsf{K}(X)$ such that $x \in \mathrm{int}K \subseteq K \subseteq X \setminus \downarrow a$. Thus $K \in \mathcal{K}_A$ and $a \notin K$, a contradiction. Therefore $A = \overline{\{a\}}$. □

Example 4.2. (1) *Let X be a countably infinite set and X_{cof} the space equipped with the co-finite topology. Then X_{cof} is a locally compact c-sober space, but X_{cof} is not a well-filtered space.*

(2) *Let X be a uncountable set and X_{coc} the space equipped with the co-countable topology. Then X_{coc} is a well-filtered P-space, and hence an ω^*-well-filtered space, but X_{coc} is not a c-sober space.*

Theorem 4.3. *For a T_0 topological space X, the following conditions are equivalent:*

(1) *X is ω^*-well-filtered.*

(2) *For every continuous mapping $f : X \to Y$ from X to a T_0 space and a countably filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f(\bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow f(K)$.*

(3) *For every continuous mapping $f : X \to Y$ from X to a c-sober space and a countably filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f(\bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow f(K)$.*

Proof. (1) ⇒ (2): Let $\mathcal{K} \subseteq \mathsf{K}(X)$ be a countably filtered family. It need only to check $\bigcap_{K\in\mathcal{K}} \uparrow f(K) \subseteq \uparrow f(\bigcap \mathcal{K})$. Let $y \in \bigcap_{K\in\mathcal{K}} \uparrow f(K)$. Then for each $K \in \mathcal{K}$, $\overline{\{y\}} \cap f(K) \neq \emptyset$, that is, $K \cap f^{-1}(\overline{\{y\}}) \neq \emptyset$. We can show that $f^{-1}(\overline{\{y\}}) \cap \bigcap \mathcal{K} \neq \emptyset$. In fact, suppose $f^{-1}(\overline{\{y\}}) \cap \bigcap \mathcal{K} = \emptyset$, then $\bigcap \mathcal{K} \subseteq X \setminus f^{-1}(\overline{\{y\}})$. Since X is ω^*-well-filtered, there exists some $K \in \mathcal{K}$ such that $K \subseteq X \setminus f^{-1}(\overline{\{y\}})$, a contradiction. Thus $\overline{\{y\}} \cap f(\bigcap \mathcal{K}) \neq \emptyset$. Therefore $y \in \uparrow f(\bigcap \mathcal{K})$, and consequently $\bigcap_{K\in\mathcal{K}} \uparrow f(K) \subseteq \uparrow f(\bigcap \mathcal{K})$.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $\eta_X : X \to X^{cs} (= P_H(\mathsf{CIrr}_c(X)))$ be the topological embedding from X into its c-sobrification. Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is countably filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K \not\subseteq U$ for all $K \in \mathcal{K}$, then by Lemma 3.1, $X \setminus U$ contains a minimal countably irreducible closed subset A that still meets all members of $\mathcal{K}$. By condition (3), $\bigcap_{K\in\mathcal{K}} \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(K) = \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(\bigcap \mathcal{K}) \subseteq \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(U) = \Diamond_{\mathsf{CIrr}_c(X)} U$. For every $K \in \mathcal{K}$, since $A \cap K \neq \emptyset$, $A \in \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(K)$. Thus $A \in \bigcap_{K\in\mathcal{K}} \uparrow_{\mathsf{CIrr}_c(X)} \eta_X(K) = \Diamond_{\mathsf{CIrr}_c(X)} U$, and this means $A \cap U \neq \emptyset$, a contradiction. Therefore X is ω^*-well-filtered. □

Lemma 4.4. ([1]) *For a nonempty family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigvee_{i\in I} K_i$ exists in $\mathsf{K}(X)$ iff $\bigcap_{i\in I} K_i \in \mathsf{K}(X)$. In this case $\bigvee_{i\in I} K_i = \bigcap_{i\in I} K_i$.*

Theorem 4.5. *For a T_0 topological space X, the following conditions are equivalent:*

(1) *X is an ω^*-well-filtered space.*

(2) *$\mathsf{K}(X)$ is a cdcpo, and $\uparrow (A \cap \bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow (A \cap K)$ for any countably filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ and $A \in \mathcal{C}(X)$.*

(3) *$\mathsf{K}(X)$ is a cdcpo, and $\uparrow (A \cap \bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow (A \cap K)$ for any countably filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ and $A \in \mathsf{CIrr}_c(X)$.*

Proof. (1) ⇒ (2): Suppose $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is countably directed, then $\bigcap_{i\in I} K_i \in \mathsf{K}(X)$ since X is an ω^*-well-filtered space. Thus $\bigvee_{\mathsf{K}(X)}\{K_i : i \in I\} = \bigcap\{K_i : i \in I\}$, and whence $\mathsf{K}(X)$ is a cdcpo. Suppose $\mathcal{K} \subseteq \mathsf{K}(X)$ is a countably filtered family and $A \in \mathcal{C}(X)$. It need only to check $\bigcap_{K\in\mathcal{K}} \uparrow (A \cap K) \subseteq \uparrow (A \cap \bigcap \mathcal{K})$. Let $x \in \bigcap_{K\in\mathcal{K}} \uparrow (A \cap K)$. Then for each $K \in \mathcal{K}$, $\downarrow x \cap A \cap K \neq \emptyset$. It follows from the ω^*-well-filteredness of X that $\bigcap_{K\in\mathcal{K}}(\downarrow x \cap A \cap K) \neq \emptyset$. Thus $x \in \bigcap_{K\in\mathcal{K}} \uparrow (A \cap K)$. Therefore $\bigcap_{K\in\mathcal{K}} \uparrow (A \cap K) \subseteq \uparrow (A \cap \bigcap \mathcal{K})$.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is countably filtered, $U \in \mathcal{O}(X)$ and $\bigcap \mathcal{K} \subseteq U$. If $K \not\subseteq U$ for every $K \in \mathcal{K}$, then by Lemma 3.1, $X \setminus U$ contains a minimal countably irreducible closed subset A that still meets all members of $\mathcal{K}$. Let $\mathcal{K}^* = \{\uparrow (K \cap A) : K \in \mathcal{K}\}$. Then $\mathcal{K}^* \subseteq \mathsf{K}(X)$ and $\mathcal{K}^*$ is countably filtered. Since $\mathsf{K}(X)$ is a cdcpo, by Lemma 4.4, $\emptyset \neq \bigcap \mathcal{K}^* = \bigcap_{K\in\mathcal{K}} \uparrow (K \cap A) \in \mathsf{K}(X)$. By condition (3), $\uparrow (A \cap \bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}}(\uparrow A \cap K) \neq \emptyset$. On the other hand, $\uparrow (A \cap \bigcap \mathcal{K}) \subseteq \uparrow (A \cap U) = \emptyset$ since $A \subseteq X \setminus U$, a contradiction. Therefore X is an ω^*-well-filtered space. □

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