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W-convexity on the Isbell-convex hull of an asymmetrically normed real vector space

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Abstract. Künzi and Yilzid introduced the concept of convexity structures in the sense of Takahashi in quasi-pseudometric spaces [7]. In this article, we continue the study of this theory, introducing the concept of W-convexity for real-valued pair of functions defined on an asymmetrically normed real vector space. Moreover, we show that all minimal pairs of functions defined on an asymmetrically normed real vector space equipped with a convex structure which is W-convex whenever W is translation-invariant.

1. Introduction

Künzi and Yildiz in [7] initiated the study on convex structures in the sense of Takahashi in T_0-quasi-metric spaces. They considered a T_0-quasi-metric space (X, q) equipped with a Takahashi convexity structure (briefly TCS). They defined a Takahashi convex structure W on a T_0-quasi-metric space (X, q) as a map from $X \times X \times [0, 1]$ to X (that is, $W(x, y, \lambda)$ is defined for all $(x, y, \lambda) \in X \times X \times [0, 1]$) satisfying the following conditions:

$$q(v, W(x, y, \lambda)) \leq \lambda q(v, x) + (1 - \lambda)q(v, y) \tag{1}$$

and

$$q^t(v, W(x, y, \lambda)) \leq \lambda q^t(v, x) + (1 - \lambda)q^t(v, y) \tag{2}$$

whenever $v \in X$.

In [12], Takahashi introduced the concept of convexity structure on a metric space. He called a metric space equipped with a convexity structure as a convex metric space. Furthermore, he studied properties of these spaces and formulated some fixed point theorems for maps which do not increase distances, the so-called nonexpansive maps, on convex metric spaces. We point out that many authors studied the convex metric spaces in Takahashi sense (see [1, 11, 13]).

In [3], Conradie et al. presented an explicit description of the algebraic and vector lattice operations on the Isbell-convex hull of an asymmetrically normed real vector space. Let us mention that earlier on before

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[3], Olela Otafudu in [9] studied further properties of the Isbell-convex hull of an asymmetrically normed real vector space and he obtained for instance that any member of the Isbell-convex hull is convex as a pair of positive real valued and convex functions.

In this article, we introduce the concept of W-convex (or convex with respect to W) function pair on an asymmetrically normed real vector space equipped with a Takahashi convexity structure W. We prove for instance that a minimal function pair on an asymmetrically normed real vector space equipped with a Takahashi convexity structure W is convex with respect to W (or W-convex) whenever W is a *t*ranslation-invariant (see Definition 3.5 (a)) and satisfies the *h*omogeneity condition (see Definition 3.5 (b)). In addition, we prove that the Chebychev center of a nonempty doubly closed convex bounded subset of a convex T_0-quasi-metric space is doubly closed convex bounded subset too under some conditions. Moreover, we generalize some well-known fixed point theorems due to Takahashi for nonexpansive maps in convex T_0-quasi-metric spaces.

2. Preliminaries

This section recalls the most important definitions and preliminary results that we shall use in this paper.

Definition 2.1. *Let X be a set and let $q : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then, q is called a* quasi-pseudometric *on X if*
(a) $q(x, x) = 0$ whenever $x \in X$,
(b) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.
We shall say that q is a T_0-quasi-metric *provided that q satisfies the following condition: for each $x, y \in X$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$.*

For any quasi-pseudometric q on a set X, we define an other quasi-pseudometric q^t by $q^t(x, y) = q(y, x)$ whenever $x, y \in X$. We shall call q^t the *conjugate quasi-pseudometric* or *dual-pseudometric* of q. As usual, a quasi-pseudometric q on X such that $q = q^t$ is called a *pseudometric*. Note that for any (T_0-)quasi-pseudometric q, we can associate the (pseudo) metric q^s of q defined by $q^s = \max\{q, q^t\} = q \vee q^t$.

For $a, b \in \mathbb{R}$ we shall put $a \dot{-} b = \max\{a - b, 0\}$. If we equip $\mathbb{R}$ with $u(a, b) = a \dot{-} b$, then $(\mathbb{R}, u)$ is a T_0-quasi-metric space that we call the *standard T_0-quasi-metric* of $\mathbb{R}$. Note that the metric u^s of u is the usual metric on $\mathbb{R}$ where $u^s(a, b) = |a - b|$ whenever $a, b \in \mathbb{R}$.

For a given quasi-pseudometric space (X, q), the set $B_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$ with $\epsilon > 0$ and $x \in X$ is called an *open ϵ-ball* at x. The collection of all open balls yields a base for a topology $\tau(q)$ and the topology $\tau(q)$ is called the *topology induced by q* on X. Similarly the set $C_q(x, \epsilon) = \{y \in X : q(x, y) \leq \epsilon\}$ where $\epsilon > 0$ and $x \in X$ is called an *ϵ-closed ball* at x. Note that $C_q(x, \epsilon)$ is $\tau(q^t)$-closed, but not $\tau(q)$-closed in general.

Furthermore, suppose that (X, q) is a T_0-quasi-metric space. For any nonempty bounded subset A of X, we set

$$r_x(A)_q := \sup\{q(x, y) : y \in A\} \text{ where } x \in X$$

and

$$r_x(A)_{q^{-1}} := \sup\{q(y, x) : y \in A\} \text{ where } x \in X.$$

Furthermore, we define $r_x(A)$ by $r_x(A) := r_x(A)_q \vee r_x(A)_{q^{-1}}$ where $x \in X$ and

$$r(A) := \inf\{r_x(A) : x \in A\}.$$

We call

$$diam(A) = \sup\{q(x, y) : x \in A, y \in A\}$$

the *diameter* of A. The set $C(A) := \{x \in A : r_x(A) = r(A)\}$ is called the *Chebychev center* of A (see [5]). For more details about properties of $r_x(A), r(A)$ and $diam(A)$ we refer the reader to [10].

Remark 2.2. *Note that for a given T_0-quasi-metric space (X,q) and a bounded subset A of X, the values of $diam(A)$, $r_x(A)$ and $r(A)$ do not change whenever are defined for the symmetrize space (X,q^s) instead of (X,q). Moreover, the Chebychev center of the symmetrize metric space (X,q^s) is the same of the Chebychev center of (X,q).*

The following example can be compared to [7, Example 8].

Example 2.3. *Let (X,q) be a T_0-quasi-metric space. For any $x,y \in X$ with $x \neq y$ and $q(x,y)+q(y,x) \neq 0$, the function $u_{q(x,y),q(y,x)} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by*

$$u_{q(x,y),q(y,x)}(\lambda,\lambda^{'}) = \begin{cases} (\lambda-\lambda^{'})q(x,y) & \text{if} \quad \lambda \geq \lambda^{'} \\ (\lambda^{'}-\lambda)q(y,x) & \text{if} \quad \lambda < \lambda^{'} \end{cases}$$

is a T_0-quasi-metric.

Remark 2.4. *For any T_0-quasi-metric (X,q), if $q(x,y)=1$ and $q(y,x)=0$ whenever $x,y\in X$, then the T_0-quasi-metric $u_{q(x,y),q(y,x)}$ is the standard T_0-quasi-metric u on $\mathbb{R}$ defined earlier on.*

Consider a T_0-quasi-metric space (X,q). Let $\mathcal{P}_0(X)$ be the set of all nonempty subsets of X. We recall that for any given $P \in \mathcal{P}_0(X)$, $q(P,x) = \inf\{q(p,x) : p \in P\}$ and $q(x,P) = \inf\{q(x,p) : p\in P\}$ for all $x \in X$.

For any $P,Q \in \mathcal{P}_0(X)$, the so-called *Hausdorff (-Bourbaki) quasi-pseudometric* q_H on $\mathcal{P}_0(X)$ is defined by

$$q_H(P,Q) = \sup_{t\in Q} q(P,t) \vee \sup_{p\in P} q(p,Q).$$

It is well-known that q_H is an extended (if q_H may attain the value ∞, then the triangle inequality is interpreted in the obvious way) T_0-quasi-metric if we restrict the set $\mathcal{P}_0(X)$ to the nonempty subsets of P of X which satisfy $P = cl_{\tau(q)}P \cap cl_{q^{-1}}P$ (see [2, 8]).

Definition 2.5. *([7, Definition 7]) Let (X,q) be a T_0-quasi-metric space. For any subset P of X, we call $cl_{\tau(q)}P \cap cl_{q^{-1}}P$ the* double closure *of P. Moreover, if $P = cl_{\tau(q)}P\cap cl_{q^{-1}}P$, we say that P is* doubly closed*.*

Let X be a real vector space. A function $\|.| : X \longrightarrow [0,\infty)$ is called an asymmetric seminorm on X if for any $x,y \in X$ and $t \in [0,\infty)$ we have:

(a) $\|tx| = t\|x|$;

(b) $\|x+y| \leq \|x| + \|y|$.

If moreover we have

(c) $\|x| = \|-x| = 0$ if and only if $x = 0$,

then $\|.|$ is called an *asymmetric norm*, and the pair $(X, \|.|)$ is called an *asymmetrically normed space*.

If $\|.|$ is an asymmetric norm on X, then the function $|.\| : X \longrightarrow [0,\infty)$ defined by

$$|x\| = \|-x|$$

for any $x\in X$ is an asymmetric norm on X and it is called the *conjugate norm* of $\|.|$.

Moreover, the function $\|.\|$ defined by

$$\|x\| = \max\{\|x|, |x\|\}$$

for any $x \in X$ is a norm on X and it is called the *symmetrisation* of $\|.|$. It is easy to see that any asymmetric norm $\|.|$ on X induces a T_0-quasi-metric $q_{\|.|}$ defined by

$$q_{\|.|}(x,y) = \|x-y|$$

for any $x\in X$.

3. Convex structure in quasi-pseudometric space

In the follow-up, we use the terminology of [7].

Definition 3.1. *Let (X, q) be a quasi-pseudometric space. We say that (X, q) has a* Takahashi convex structure *(or* convex structure*) if there exists a mapping $W : X \times X \times [0, 1] \to X$ satisfying for all $x, y \in X$ and $\lambda \in [0, 1]$,*

$$q(z, W(x, y, \lambda)) \leq \lambda q(z, x) + (1 - \lambda)q(z, y);$$

and

$$q(W(x, y, \lambda), z) \leq \lambda q(x, z) + (1 - \lambda)q(y, z)$$

whenever $z \in X$.

If W is a convex structure on a quasi-pseudometric space (X, q), then we call the triple (X, q, W) a convex quasi-pseudometric space*.*

Example 3.2. *([7, Example 4]) Let $\mathbb{R}$ be the set of real number equipped with the standard T_0-quasi-metric space $u(x, y) = x \dot{-} y = \max\{0, x - y\}$, whenever $x, y \in \mathbb{R}$. If we define $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, then $(\mathbb{R}, u, W)$ is a convex quasi-metric space. Indeed, if $z, x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have*

$$u(z, W(x, y, \lambda)) = \max\{0, z + \lambda z - \lambda z - \lambda x - (1 - \lambda)y\}$$

which implies that

$$u(z, W(x, y, \lambda)) \leq \max\{0, \lambda(z - x)\} + \max\{0, (1 - \lambda)(z - y)\}.$$

Moreover

$$u(z, W(x, y, \lambda)) \leq \lambda u(z, x) + (1 - \lambda)u(z, y).$$

Similarly, one has

$$u(W(x, y, \lambda), z) \leq \lambda u(x, z) + (1 - \lambda)u(y, z).$$

Proposition 3.3. *(a) If (X, q, W) is a convex quasi-pseudometric space, then (X, q^t, W) is a convex quasi-pseudometric space too.*

(b) If (X, q, W) is a convex quasi-pseudometric space, then (X, q^s, W) is a convex metric space.

Proof. (a) The statement immediately follows from definition.

(b) Suppose that (X, q, W) is a convex quasi-pseudometric space. Then for all $z, x, y \in X$ and $\lambda \in [0, 1]$, we have

$$q(z, W(x, y, \lambda)) \leq \lambda q(z, x) + (1 - \lambda)q(z, y) \tag{3}$$

and

$$q(W(x, y, \lambda), z) \leq \lambda q(x, z) + (1 - \lambda)q(y, z). \tag{4}$$

Since $\lambda q(z, x) \leq \lambda q^s(z, x)$ and $(1 - \lambda)q(z, y) \leq (1 - \lambda)q^s(z, y)$, then we have

$$\lambda q(z, x) + (1 - \lambda)q(z, y) \leq \lambda q^s(z, x) + (1 - \lambda)q^s(z, y). \tag{5}$$

By analogous arguments, we have

$$\lambda q(x, z) + (1 - \lambda)q(y, z) \leq \lambda q^s(z, x) + (1 - \lambda)q^s(z, y). \tag{6}$$

So, by combining inequalities (3) and (4) we obtain

$$q^s(z, W(x, y, \lambda)) \leq \max\{\lambda q(z, x) + (1 - \lambda)q(z, y), \lambda q(x, z) + (1 - \lambda)q(y, z)\}.$$

Therefore, from inequalities (5) and (6) we conclude that

$$q^s(z, W(x, y, \lambda)) \leq \lambda q^s(z, x) + (1 - \lambda) q^s(z, y).$$

Thus, (X, q^s, W) is a convex metric space. □

In the following, we prove that if W is a convex structure on the metric space (X, q^s) of a T_0-quasi-metric space (X, q), it does not imply that W is the convex structure on the T_0-quasi-metric space (X, q).

Remark 3.4. *(compare [7, Example 6]) Note that the converse of Proposition 3.3(b) is not true.*

Proof. *Indeed, we equip* $X = [0,1] \times [-\frac{1}{4}, \frac{1}{4}]$ *with the* T_0*-quasi-metric* $q_{\|.|}$ *induced by the asymmetric norm* $\|x| = \max\{\|x^1|, \|x^2|\}$ *whenever* $x = (x^1, x^2) \in \mathbb{R}^2$*. Note that the norm of* $\|.|$ *is denoted by* $\|.\|$*. Let us consider* $x^1 = (0,0), y^1 = (1,0) \in X$*. Let a convex structure* $T(x, y, \lambda) = \lambda x + (1 - \lambda)y$ *whenever* $x, y \in X$ *and* $\lambda \in [0,1]$*. At the point* $(x, y, \lambda) \in X \times X \times [0,1]$*, we define*

$$W(x, y, \lambda) = \begin{cases} (\frac{1}{2}, -\frac{1}{4}) & \text{if} \quad (x, y, \lambda) = (x^1, y^1, \frac{1}{2}) \\ T(x, y, \lambda) & \text{if} \quad (x, y, \lambda) \neq (x^1, y^1, \frac{1}{2}). \end{cases}$$

We have that W *is a convex structure on* $(X, q^s_{\|.|})$*. We have to check only that* W *is a convex structure on* $(X, q^s_{\|.|})$ *at the point* (x, y, λ)*: Let* $z = (z^1, z^2) \in X$*. We have* $z^1 \in [0,1]$ *and thus* $\frac{1}{2} = \frac{1}{2}(\|z^1 - 0\| + \|z^1 - 1\|) \leq \frac{1}{2}\|z - x^1\| + \frac{1}{2}\|z - y^1\|$*. Moreover, we have that* $\|z - W(x^1, y^1, \frac{1}{2})\| \leq \frac{1}{2}$*. Hence* W *is a convex structure on* $(X, q^s_{\|.|})$*.*
Furthermore, W *is not a convex structure on* $(X, q_{\|.|})$ *since at* $z = (\frac{1}{2}, \frac{1}{4})$*, we have that* $\frac{1}{2}\|z - x^1| + \frac{1}{2}\|z - y^1| = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$ *but* $\|z - W(x^1, y^1, \frac{1}{2})| = \frac{1}{2}$*. Hence* $\|z - W(x^1, y^1, \frac{1}{2})| > \frac{1}{2}\|z - x^1| + \frac{1}{2}\|z - y^1|$*. Therefore,* W *is not a convex structure on* $(X, q_{\|.|})$*.*

In the light of [7, Remark 7] we have the following definitions.

Definition 3.5. *Let* (X, q, W) *be an asymmetrically real normed vector space.*

(a) The convex structure W *is called* translation-invariant *if* W *satisfies the condition*

$$W(x + z, y + z, \lambda) = W(x, y, \lambda) + z$$

for all $x, y, z \in X$ *and* $\lambda \in [0,1]$*.*

(b) We say that the convex structure W *satisfies the* homogeneity condition *if for any* $\alpha \in \mathbb{R}$ *we have*

$$W(\alpha x, \alpha y, \lambda) = \alpha W(x, y, \lambda)$$

for any $x, y \in X$ *and* $\lambda \in [0,1]$*.*

It is easy to see that the convex structure in Example 3.2 is translation-invariant and satisfies the homogeneity condition.

The following useful observation is not new and it can be found in [7, Remark 5] and [13, Proposition 1.2].

Remark 3.6. *For any convex* T_0*-quasi-metric space* (X, q, W)*, the following are true:*

(1) For any $x \in X$ *and* $\lambda \in [0,1]$*, we have* $W(x, x, \lambda) = x$*.*

(2) For any $x, y \in X$*, it follows that* $W(y, x, 0) = x$ *and* $W(y, x, 1) = y$*.*

Proposition 3.7. *(compare [12, Proposition 3]) If* (X, q, W) *is a convex* T_0*-quasi-metric space, then* $W(x, y, \lambda) \in \langle x, y \rangle_q = \{z \in X : q(x, y) = q(x, z) + q(z, y)\}$ *whenever* $x, y \in X$*.*

Proof. Since W is a convex structure on (X, q), for any $x, y \in X$ and $\lambda \in [0, 1]$, we have $W(x, y, \lambda) \in X$. Then from triangle inequality of q, we have

$$q(x, y) \leq q(x, W(x, y, \lambda)) + q(W(x, y, \lambda), y).$$

Furthermore,

$$q(x, y) \leq \lambda q(x, x) + (1 - \lambda)q(x, y) + \lambda q(x, y) + (1 - \lambda)q(y, y).$$

It follows that

$$q(x, y) \leq q(x, W(x, y, \lambda)) + q(W(x, y, \lambda), y) \leq (1 - \lambda)q(x, y) + \lambda q(x, y) = q(x, y).$$

Hence

$$q(x, y) = q(x, W(x, y, \lambda)) + q(W(x, y, \lambda), y).$$

Therefore, the assertion is proved. □

Definition 3.8. *Let (X, q, W) be a convex quasi-pseudometric space. For any $x, y \in X$, the set $\mathcal{S}[x, y] := \{W(x, y, \lambda) : \lambda \in [0, 1]\}$ is called* quasi-metric segment *with endpoints x, y.*

Remark 3.9. *If (X, q, W) is a convex T_0-quasi-metric space, then for any $x, y \in X$ with $x \neq y$, the quasi-metric interval $\langle x, y\rangle_q$ contains $\mathcal{S}[x, y]$. If $x = y$, then the quasi-metric interval which is a singleton coincides with the quasi-metric segment.*

The following definition was introduced in [7] and it can be compared with [13, Definition 1.3].

Definition 3.10. *(compare [7, Definition 3]) Let (X, q, W) be a convex T_0-quasi-metric space. We say that W is a* unique *convex structure on (X, q) if for any $w \in X$ for which there exists $(x, y, \lambda) \in X \times X \times [0, 1]$ with*

$$q(z, w) \leq \lambda q(z, x) + (1 - \lambda)q(z, y)$$

and

$$q(w, z) \leq \lambda q(x, z) + (1 - \lambda)q(y, z)$$

whenever $z \in X$, $w = W(x, y, \lambda)$.

Remark 3.11. *The uniqueness of a W convex structure on T_0-quasi-metric space (X, q) means really that W is unique convex structure on (X, q). Since for any T_0-quasi-metric space (X, q), if W is a unique convex structure on (X, q^s), then W is a unique convex structure on (X, q) (see Proposition 3.3(b)). The uniqueness of the convex structure W on a T_0-quasi-metric space has connections with the concept of strict convexity (see [7, 13]).*

For more details about the concept of strict convexity in T_0-quasi-metric spaces, we refer the reader to [7].

The proof of the following important lemma can be found in [7].

Lemma 3.12. *Let W be the unique convex structure on a T_0-quasi-metric space (X, q). Then*

$$W(W(x, y, \lambda), y, \lambda') = W(x, y, \lambda\lambda')$$

whenever $x, y \in X$ and $\lambda, \lambda' \in [0, 1]$.

Proposition 3.13. *If W is the unique convex structure on a T_0-quasi-metric space (X, q), then the map $\psi : (\mathcal{S}[x, y], q) \to ([0, q(x, y)], u_{q(x,y)q(y,x)})$ defined by $\psi(W(x, y, \lambda)) = \lambda q(x, y)$ whenever $x, y \in X$ with $x \neq y$ and $\lambda \in [0, 1]$ is an isometry embedding of $\mathcal{S}[x, y]$ into $[0, q(x, y)]$.*

Proof. We first observe that for any $x, y \in X$ and $\lambda \in [0,1]$, we have that

$$q(W(x,y,\lambda),x) = (1-\lambda)q(y,x) \quad \text{and} \quad q(y, W(x,y,\lambda) = \lambda q(y,x) \tag{7}$$

and

$$q(x, W(x,y,\lambda)) = (1-\lambda)q(x,y) \quad \text{and} \quad q(W(x,y,\lambda),y) = \lambda q(x,y). \tag{8}$$

To prove that ψ is an isometry, we need to prove that for any $\lambda, \lambda' \in [0,1]$,

$$u_{q(x,y)q(y,x)}(\psi(W(x,y,\lambda)), \psi(W(x,y,\lambda'))) = q(W(x,y,\lambda), W(x,y,\lambda'))$$

whenever $x, y \in X$.
We have two cases to prove since the case where $\lambda = \lambda'$ is obvious.
Case 1. If $\lambda < \lambda'$, then we have

$$q(W(x,y,\lambda), W(x,y,\lambda')) = q(W(x,y,\lambda'\lambda/\lambda'), W(x,y,\lambda')).$$

Moreover, from Remark 3.12, it follows that

$$q(W(x,y,\lambda), W(x,y,\lambda')) = q(W(W(x,y,\lambda'), y, \lambda/\lambda'), W(x,y,\lambda')).$$

Thus from (7), we have

$$q(W(x,y,\lambda), W(x,y,\lambda')) = (1-\lambda/\lambda')q(y, W(x,y,\lambda')) = (1-\lambda/\lambda')\lambda' q(y,x).$$

So

$$q(W(x,y,\lambda), W(x,y,\lambda')) = (\lambda' - \lambda)q(y,x) = u_{q(x,y)q(y,x)}(\psi(W(x,y,\lambda)), \psi(W(x,y,\lambda'))).$$

Case 2. If $\lambda > \lambda'$, then

$$q(W(x,y,\lambda), W(x,y,\lambda')) = q(W(x,y,\lambda), W(x,y,\lambda\lambda'/\lambda)).$$

By similar arguments, as Case 1, and by using again Lemma 3.12 and (8) we have

$$q(W(x,y,\lambda), W(x,y,\lambda')) = (\lambda' - \lambda)q(y,x) = u_{q(x,y)q(y,x)}(\psi(W(x,y,\lambda)), \psi(W(x,y,\lambda'))).$$

□

The next result can be compared to [13, Remark 1.6] for corresponding metric result.

Remark 3.14. *Let (X,q) be a T_0-quasi-metric space. If $q(x,y) = 1$ and $q(y,x) = 0$ whenever $x, y \in X$, then $\mathcal{S}[x,y]$ is homeomorphic to $[0,1]$. Since the function $h(\lambda) = W(x,y,\lambda)$ whenever $\lambda \in [0,1]$ is an isometry embedding of $[0,1]$ into $X|_{\mathcal{S}[x,y]}$ (see [7, Proposition 4]).*

4. Some fixed point theorems

The following can be compared with definition [12, p.143]

Definition 4.1. *Let (X,q,W) be a convex quasi-pseudometric space and $C \subseteq X$. We say that C is* W-convex *if $W(x,y,\lambda) \in C$ whenever $x, y \in C$ and $\lambda \in [0,1]$.*

Note that for any convex quasi-pseudometric space (X,q,W), X is W-convex and any W-convex subset C of (X,q) is naturally a convex quasi-pseudometric, where the convex quasi-metric structure on C is just the restriction of W to $C \times C \times [0,1]$.

Lemma 4.2. *([7, Proposition 5]) If (X, q, W) is a convex quasi-pseudometric space, then whenever $x, y \in X$ and $r, s \geq 0$, the closed balls $C_q(x, r)$, $C_{q^{-1}}(x, s)$ and the open balls $B_q(x, r)$, $B_{q^{-1}}(x, s)$ in X are W-convex subsets of X. Moreover $C_q(x, r) \cap C_{q^{-1}}(y, s)$ and $B_q(x, r) \cap B_{q^{-1}}(y, s)$ are W-convex subsets of X.*

Remark 4.3. *(compare [12, Proposition 1]) Consider a convex quasi-pseudometric space (X, q, W) and a collection $(C_i)_{i\in I}$ of W-convex subsets of X. It can easily be proved that $\bigcap_{i\in I} C_i$ is also a W-convex subset of X.*

Let (X, q, W) be a convex T_0-quasi-metric space. We denote by $\mathcal{CB}_0(X)$ the collection of bounded W-convex elements of $\mathcal{P}_0(X)$ and denote by $\mathcal{DCB}_0(X)$ the set of all nonempty doubly closed W-convex bounded subsets of X.

Remark 4.4. *([7, p.15]) It is true that if $P, B \in \mathcal{P}_0(X)$ are bounded, then $q_H(P, Q)$ is finite. Furthermore, if $P \in \mathcal{CB}_0(X)$, then $cl_{\tau(q)}P \cap cl_{q^{-1}}P \in \mathcal{CB}_0(X)$.*

Definition 4.5. *A convex quasi-pseudometric space (X, q, W) is said to have a* property (H) *if any family $\{C_i\}_{\in I}$ (I is assumed totally ordered) of nonempty doubly closed W-convex bounded subsets of X such that $C_j \subseteq C_i$ with $i \leq j$ has nonempty intersection.*

Remark 4.6. *If a convex quasi-pseudometric space (X, q, W) satisfies the property (H), then (X, q^s, W) satisfies the property (C) in the sense of Takahashi.*

Indeed, if a subset A of X is doubly closed then A is $\tau(q^s)$-closed. Obviously, if A is q-bounded, then A is q^s-bounded too. Therefore, any nonempty family of doubly closed W-convex bounded subsets of X is a nonempty family of $\tau(q^s)$-closed W-convex and q^s-bounded.

Proposition 4.7. *If (X, q, W) is a convex T_0-quasi-metric space which has the property (H) and $A \subseteq X$, then the Chebychev center $C(A) \in \mathcal{DCB}_0(X)$.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$. Consider $A_n(x)$, a subset of A, defined by

$$A_n(x) := \left\{ y \in A : q(x, y) \leq r(A) + \frac{1}{n} \quad \text{and} \quad q(y, x) \leq r(A) + \frac{1}{n} \right\}.$$

Then

$$C_n = \bigcap_{x\in X} A_n(x) = \bigcap_{x\in X} \left[C_q\Big(x, r(A) + \frac{1}{n}\Big) \cap C_{q^{-1}}\Big(x, r(A) + \frac{1}{n}\Big) \right]$$
$$= \bigcap_{x\in X} C_{q^s}\Big(x, r(A) + \frac{1}{n}\Big).$$

Let $z, t \in C_n$. Then for any $x \in X$, we have

$$q^s(x, t) \leq r(A) + \frac{1}{n}$$

and

$$q^s(x, z) \leq r(A) + \frac{1}{n}.$$

It follows that

$$q(t, z) \leq q(t, x) + q(x, z) \leq q^s(x, t) + q^s(x, z) \leq 2r(A) + \frac{2}{n}.$$

So C_n is bounded whenever $n \in \mathbb{N}$.
We need to prove that C_n is doubly closed and W-convex whenever $n \in \mathbb{N}$. Indeed we have $C_n \subseteq cl_{\tau(q)}C_n \cap cl_{\tau(q^{-1})}C_n$.

Let $y \in cl_{\tau(q)} C_n \cap cl_{\tau(q^{-1})} C_n$. Then we have

$$x \in B_q\Big(y, r(A) + \frac{1}{n}\Big) \cap C_n$$

and

$$x' \in B_{q^{-1}}\Big(y, r(A) + \frac{1}{n}\Big) \cap C_n.$$

Since $x \in C_n$ and $q(y,x) < r(A) + \frac{1}{n}$ it follows that

$$y \in B_q\Big(x, r(A) + \frac{1}{n}\Big) \subseteq C_q\Big(x, r(A) + \frac{1}{n}\Big).$$

Thus

$$y \in \bigcap_{x \in X} C_q\Big(x, r(A) + \frac{1}{n}\Big).$$

By similar arguments we have

$$y \in \bigcap_{x \in X} C_{q^{-1}}\Big(x, r(A) + \frac{1}{n}\Big).$$

Therefore

$$y \in \bigcap_{x \in X} C_q\Big(x, r(A) + \frac{1}{n}\Big) \cap C_{q^{-1}}\Big(x, r(A) + \frac{1}{n}\Big).$$

Hence

$$y \in \bigcap_{x \in X} C_{q^s}\Big(x, r(A) + \frac{1}{n}\Big) = C_n.$$

One sees that $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of subsets of X such that $C_{n+1} \subseteq C_n$ and C_n is W-convex as intersection of W-convex subsets of X. So $C_n \in \mathcal{DCB}_0(X)$ whenever $n \in \mathbb{N}$.
Furthermore, since (X, q, W) has property (H), it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Observe that

$$C(A) = \bigcap_{n=1}^{\infty} C_n.$$

Therefore, $C(A)$ is a doubly closed W-convex bounded subset of X. □

In [10], the concept of normal structure has been introduced for q-admissible subsets of a T_0-quasi-metric space. In the following we extend this concept in the context of convex quasi-metric spaces.

Definition 4.8. *(compare [10, Definition 3.2]) A convex T_0-quasi-metric space (X, q, W) is said to have a* normal structure *if any doubly closed W-convex bounded subset A of X, we have $r(A) < diam(A)$ and $diam(A) > 0$.*

In the next result, we prove the existence of a fixed point for a self-map on a nonempty doubly closed W-convex bounded subset with the normal structure.

Theorem 4.9. *(compare [12, Theorem 1] Let (X, q, W) be a convex T_0-quasi-metric space and K be a nonempty doubly closed W-convex bounded subset of X with the normal structure. If $T : (K, q) \to (K, q)$ is a nonexpansive map, then T has a fixed point in K.*

Proof. Let us consider a set Γ defined by

$$\Gamma = \{D \in \mathcal{DCB}_0(X) \text{ such that } T : (D, q) \to (D, q) \text{ nonexpansive map}\}.$$

It follows that $\Gamma \neq \emptyset$ since $K \in \Gamma$. We partially order Γ by $A \leq B$ if and only if $A \subseteq B$ whenever $A, B \in \Gamma$.

Observe that if $\mathcal{C} = \{C_\alpha\}_{\alpha\in\Lambda}$ is a chain in Γ, then $\bigcap_{\alpha\in\Lambda} C_\alpha \in \Gamma$. Moreover whenever $\alpha \in \Gamma$, $\bigcap_{\alpha\in\Lambda} C_\alpha \subseteq C_\alpha$, hence Γ is bounded below by $\bigcap_{\alpha\in\Lambda} C_\alpha$. By Zorn's lemma it follows that Γ has a minimal element. Let A be the minimal element of Γ. Then $A \neq \emptyset$ and $T : (A, q) \to (A, q)$ is a nonexpansive map.

We need to show that A consists of a single point. Therefore, that single point will be the fixed point. Indeed, let $x \in C(A) = \{x \in A : r_x(A) = r(A)\}$. Then by nonexpansiveness of T, we have

$$q(T(x), T(y)) \leq q(x, y) \leq r(A)$$

and

$$q(T(y), T(x)) \leq q(y, x) \leq r(A)$$

whenever $y \in A$.

It implies that $T(y) \in C_{q^s}(T(x), r(A))$. Thus $T(A) \subseteq C_{q^s}(T(x), r(A))$. Furthermore, for $x \in A$, $r_{T(x)}(A) \leq r_x(A) = r(A)$ and $r(A) \leq r_{T(x)}(A)$ since $T(x) \in A$. We have $r_{T(x)}(A) = r(A)$. So $T(x) \in C(A)$ whenever $x \in A$. Hence $T : C(A) \to C(A)$ is a nonexpansive map and by Proposition 4.7, we have that $C(A) \in \mathcal{DCB}_0(X)$. Furthermore $C(A) \in \Gamma$. We claim that $diam(A) = 0$. Suppose that $diam(A) > 0$, then since X has a normal structure, it implies that $r(A) < diam(A)$. Let $z, w \in C(A)$, then

$$q(z, w) \leq r_z(A)_q \leq r_z(A) = r(A)$$

and

$$q(w, z) \leq r_z(A)_{q^{-1}} \leq r_z(A) = r(A).$$

Hence $diam(C(A)) = \sup\{q(z, w) : z, w \in C(A)\} \leq r(A) < diam(A)$. It follows that $C(A)$ is a proper subset of A and it contradicts the minimality of A. Hence $diam(A) = 0$. □

Remark 4.10. *Note that Theorem 4.9 can be proved by using [12] and Remark 4.6. Indeed if K is a nonempty doubly closed W-convex bounded subset of X, then by Remark 4.6 we have that K is $\tau(q^s)$-closed and W-convex q^s-bounded subset of X. Furthermore, since $T : (K, q) \to (K, q)$ is a nonexpansive map. We have*

$$q(T(x), T(y)) \leq q(x, y) \textit{ whenever } x, y \in K.$$

Then

$$q(T(y), T(x)) \leq q(y, x) \textit{ whenever } x, y \in K.$$

Hence

$$q^s(T(x), T(y)) \leq q^s(x, y) \textit{ whenever } x, y \in K.$$

Thus $T : (K, q^s) \to (K, q^s)$ is a nonexpansive map and K is $\tau(q^s)$-closed and W-convex q^s-bounded subset of X with normal structure. Therefore by [12, Theorem 1] T has a fixed point in K.

The following result extends [12, Theorem 2].

Theorem 4.11. *Let W be the unique convex structure on a T_0-quasi-metric space (X, q) with the property (H). If K is a nonempty doubly closed convex bounded subset of X with the normal structure, then any commuting family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K, q) has a nonempty common fixed point set (i.e. $\bigcap_{i=1}^{n} Fix(T_i) \neq \emptyset$), where $Fix(T_i)$ denotes the set of fixed points of T_i, that is $Fix(T_i) = \{x \in K : T_i(x) = x\}$.*

Proof. Observe that if W is the unique convex structure on (X, q), then $W = W|_K$ is a unique convex structure on (K, q). Suppose $T : (K, q) \to (K, q)$ is a nonexpansive maps. Then by Theorem 4.9 there exists $x \in K$ such that $T(x) = x$. Hence $Fix(T) = \{x \in K : T(x) = x\} \neq \emptyset$ and doubly closed and bounded. For any $x, y \in Fix(T)$, we have that $W(x, y, \lambda) \in K$ since K is convex subset of X. Furthermore, let $z \in K$. Then

$$q(T(z), T(W(x, y, \lambda))) \leq q(z, W(x, y, \lambda) \leq \lambda q(z, y) + (1 - \lambda) q(z, y)$$

and

$$q(T(W(x,y,\lambda),T(z)) \leq= q(W(x,y,\lambda),z) \leq \lambda q(x,z) + (1-\lambda)q(y,z).$$

So by uniqueness of W we have $T(W(x,y,\lambda)) = W(x,y,\lambda)$ whenever $x, y \in Fix(T)$. Hence $W(x,y,\lambda) \in Fix(T)$ and therefore $Fix(T)$ is a W-convex subset of X.

Consider T_1 and T_2 two nonexpansive self-maps on (K,q) such that $T_2(T_1) = T_1(T_2)$. We have to show that $T_2(Fix(T_1)) \subseteq Fix(T_1)$. Let $x \in Fix(T_1)$. Then $T_1(x) = x$ and $T_2(x) = T_2(T_1(x)) = T_1(T_2(x))$. Hence $T_2(x) \in Fix(T_1)$.

Furthermore, we have that $T_2 : Fix(T_1) \to Fix(T_1)$ has a fixed point by Theorem 4.9 since $Fix(T_1)$ is a nonempty doubly closed convex bounded subset of (K,q). Let us say y is the fixed point of T_2. Then y is a fixed point T_1 too.
Then by induction on the family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K,q), the set of common fixed point $\bigcap_{i=1}^{n} Fix(T_i) \neq \emptyset$. □

5. W-convex function pairs and Isbell-hull

In this section, we need first to know some facts of algebraic operations on the Isbell-convex hull of an asymmetrically normed real vector space. For more details about algebraic operations on the Isbell-convex hull we refer the reader to [3].

Let $(X, \|.|)$ be an asymmetrically normed real vector space and let a pair of functions $f = (f_1, f_2)$, where $f_j : X \to -[0,\infty)$ for $j = 1,2$. The pair of functions $f = (f_1, f_2)$ is called *ample* on X if $\|x - y| \leq f_2(x) + f_1(y)$ for all $x, y \in X$. Moreover, the pair of function $f = (f_1, f_2)$ is called *minimal* if for any ample pair of functions $g = (g_1, g_2)$ on X such that $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ for all $x \in X$, then $g_1 = f_1$ and $g_2 = f_2$. The set of all minimal pairs of functions on X is denoted by $\mathcal{E}(X, \|.|)$ and it is called the *Isbell-hull* of $(X, \|.|)$. Note that the Isbell-hull of an asymmetrically normed real vector space is 1-injective and Isbell-convex in the sense of [4]. If $f = (f_1, f_2) \in \mathcal{E}(X, \|.|)$, then it is well-known that for any $x \in X$,

$$f_1(x) = \sup_{z \in X} u[\|x - z| - f_2(z)]$$

and

$$f_2(x) = \sup_{z \in X} u[\|z - x| - f_1(z)].$$

For any $z \in X$, the pair of function $f_z = (\|x - z|, \|z - x|)$ is minimal.
For $t \in \mathbb{R}$ and $f \in \mathcal{E}(X, \|.|)$, the pair of functions $f^t = (f_1^t, f_2^t)$ defined by

$$f_1^t(x) = \begin{cases} t f_1(t^{-1}x) & \text{if} \quad t > 0 \\ \|x| & \text{if} \quad t = 0 \\ |t| f_2(t^{-1}x) & \text{if} \quad t < 0 \end{cases}$$

and

$$f_2^t(x) = \begin{cases} t f_2(t^{-1}x) & \text{if} \quad t > 0 \\ \|x| & \text{if} \quad t = 0 \\ |t| f_1(t^{-1}x) & \text{if} \quad t < 0 \end{cases}$$

is minimal. Then the scalar multiplication on $\mathcal{E}(X, \|.|)$ is defined by $tf := f^t$ for all $t \in \mathbb{R}$ and $f \in \mathcal{E}(X, \|.|)$.

Furthermore, for any $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, \|.|)$, the addition $\oplus$ on $\mathcal{E}(X, \|.|)$ is defined by $f \oplus g = ((f \oplus g)_1, (f \oplus g)_2)$ where,

$$(f \oplus g)_1(x) = \sup_{z \in X} u[f_1(x - z) - g_2(z)]$$

and

$$(f \oplus g)_2(x) = \sup_{z \in X} u[f_2(x - z) - g_2(z)].$$

Definition 5.1. *Let* $(X, ||.|)$ *be an asymmetrically normed real vector space. We say that* $(X, ||.|, W)$ *is* convex *asymmetrically normed real vector space, if* W *is convex structure on the quasi-metric space* $(X, q_{||.|})$.

Definition 5.2. *(compare [1, Definition 2]) Let* $(X, ||.|, W)$ *be a convex asymmetrically normed real vector space. We call a pair of functions* $f = (f_1, f_2)$ *on* X W-convex *if for any* $x, y \in X$ *and* $\lambda \in [0, 1]$,

$$f_j(W(x, y, \lambda)) \le \lambda f_j(x) + (1 - \lambda) f_j(y) \quad \text{for} \quad j = 1, 2.$$

Example 5.3. *Let* $(X, ||.|, W)$ *be a convex asymmetrically normed real vector space. For any* $z \in X$, *the pair of functions* $f_z = (||x - z|, ||z - x|)$ *is* W-*convex.*

Indeed, for any $x, y \in X$ *and* $\lambda \in [0, \infty]$, *we have*

$$\begin{aligned} (f_z)_1(W(x, y, \lambda)) &= ||W(x, y, \lambda) - z|| \\ &\le \lambda ||x - z| + (1 - \lambda)||y - z| \text{ by definition of } W \\ &= \lambda (f_z)_1(x) + (1 - \lambda)(f_z)_1(y). \end{aligned}$$

By similar arguments one has

$$(f_z)_2(W(x, y, \lambda)) \le \lambda (f_z)_2(x) + (1 - \lambda)(f_z)_2(y).$$

Thus f_z *is* W-*convex.* □

Proposition 5.4. *Suppose that* $(X, ||.|, W)$ *is a convex asymmetrically normed real vector space. For all* $t \in \mathbb{R}$ *and* $f = (f_1, f_2) \in \mathcal{E}(X, ||.|)$,

(1) the pair of functions $f = (f_1, f_2)$ *is* W-*convex whenever* W *is translation-invariant.*

(2) the pair of functions $tf = ((tf)_1, (tf)_2)$ *is* W-*convex whenever* W *satisfies the homogeneity condition.*

Proof. Suppose that W is translation-invariant and W satisfies the homogeneity condition. Let $x, y \in X$ and $\lambda \in [0, 1]$.

(1) Let $f = (f_1, f_2)$ be a minimal pair of functions on X. To show that $f = (f_1, f_2)$ is W-convex, we only need to show that f_2 satisfies the inequality:

$$f_2(W(x, y, \lambda)) \le \lambda f_2(x) + (1 - \lambda) f_2(y) \tag{9}$$

and the proof for f_1 will follow analogously.

Suppose that f_2 does not satisfy the inequality (9). Then there exists $x_0, y_0 \in X$ and $\alpha \in [0, 1]$ such that

$$\alpha f_2(x_0) + (1 - \alpha) f_2(y_0) < f_2(W(x_0, y_0, \alpha)). \tag{10}$$

Then we set

$$g_2(z) = \begin{cases} f_2(z) & \text{if} \quad z \in X \\ f_2(x_0) + (1 - \alpha) f_2(y_0) & \textbf{if} \quad z = W(x_0, y_0, \alpha). \end{cases}$$

It follows clearly that $(f_1, g_2) < (f_1, f_2)$. Consider a point $x \in X$ with $x \neq W(x_0, y_0, \alpha)$. Then we have

$$\begin{aligned} ||W(x_0, y_0, \alpha) - x| &= ||W(x - x_0, y - y_0, \alpha)| \quad \text{by the invariant transitivity of} \quad W \\ &\le \alpha ||x_0 - x| + (1 - \alpha)||y_0 - y| \quad \text{by definition of} \quad W. \end{aligned}$$

Thus

$$||W(x_0, y_0, \alpha) - x| \leq \alpha||x_0 - x| + (1-\alpha)||y_0 - y|. \tag{11}$$

But

$$||x_0 - x| \leq f_2(x_0) + f_1(x) \quad \text{and} \quad ||y_0 - x| \leq f_2(y_0) + f_1(x). \tag{12}$$

Combining (11) and (12), it follows that

$$\begin{aligned} ||W(x_0, y_0, \alpha) - x| &\leq \alpha[f_2(x_0) + f_1(x)] + (1-\alpha)[f_2(y_0) + f_1(x)] \\ &= \alpha f_2(x_0) + (1-\alpha) f_2(y_0) + f_1(x) \\ &= g_2(z_0) + f_1(x) \end{aligned}$$

whenever $z_0 = W(x_0, y_0, \alpha)$ and $x \in X$. Hence the pair of functions (f_1, g_2) is ample and $f = (f_1, f_2) < (f_1, g_2)$. This is a contradiction with regards to the minimality of (f_1, f_2). Therefore,

$$f_2(W(x, y, \lambda)) \leq \lambda f_2(x) + (1-\lambda) f_2(y) \quad \text{for all} \quad x, y \in X \quad \text{and} \quad \lambda \in [0, 1].$$

(2) Let $t \in \mathbb{R}$. To show that tf is W-convex, we have to show that for $j = 1, 2$ the function $(tf)_j$ satisfies the inequality

$$(tf)_j(W(x, y, \lambda)) \leq \lambda(tf)_j(x) + (1-\lambda)(tf)_j(y).$$

We only prove for $j = 1$ and the proof for $j = 2$ will follow by duality.

For $t > 0$ we have

$$\begin{aligned} (tf)_1(W(x, y, \lambda)) &= f_1^t(W(x, y, \lambda)) = tf_1(t^{-1}W(x, y, \lambda)) \text{ by the definition of } \lambda f \\ &= tf_1^t(W(t^{-1}x, t^{-1}y, \lambda)) \text{ by the homogeneity of } W \\ &\leq t[\lambda f_1(t^{-1x}) + (1-\lambda) f_1(t^{-1}y)] \text{ from (1) above} \\ &= \lambda(tf)_1(x) + (1-\lambda)(tf)_1(y). \end{aligned}$$

For $t = 0$ we have

$$\begin{aligned} (tf)_1(W(x, y, \lambda)) &= f_1^t(W(x, y, \lambda)) = ||W(x, y, \lambda)| \text{ by the definition of } \lambda f \\ &\leq \lambda||x| + (1-\lambda)||y| \\ &= \lambda(tf)_1(x) + (1-\lambda)(tf)_1(y). \end{aligned}$$

For $t < 0$, it is easy to see from the definition of tf and the homogeneity of W that we have

$$\begin{aligned} (tf)_1(W(x, y, \lambda)) &= |t| f_1(t^{-1}W(x, y, \lambda)) \\ &\leq (tf)_1(x) + (1-\lambda)(tf)_1(y). \end{aligned}$$

Thus tf is W-convex pair of functions. □

Proposition 5.5. *Suppose that $(X, ||.|, W)$ is a convex asymmetrically normed real vector space. If W is translation-invariant, then the pair of functions $f \oplus g = ((f \oplus g)_1, (f \oplus g)_2)$ is W-convex for all $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, ||.|)$.*

Proof. We know that

$$(f \oplus g)_1(x) = \sup_{z \in X} u[f_1(x - z) - g_2(z)] \text{ and } (f \oplus g)_2(x) = \sup_{z \in X} u[f_2(x - z) - g_2(z)].$$

For all $x, y \in X$ and $\lambda \in [0, 1]$, we have to prove that

$$(f \oplus g)_1(W(x, y, \lambda)) \leq \lambda(f \oplus g)_1(x) + (1-\lambda)(f \oplus g)_1(y)$$

and

$$(f \oplus g)_2(W(x, y, \lambda)) \leq \lambda(f \oplus g)_2(x) + (1 - \lambda)(f \oplus g)_2(y).$$

Indeed, let $x, y \in X$ and $\lambda \in [0, 1]$. Then by the invariant transivity of W we have for some $z \in X$,

$$\begin{aligned} f_1(W(x, y, \lambda) - z) - g_2(z) &= f_1(W(x - z, y - z, \lambda)) - g_2(z) \\ &\leq \lambda f_1(x - z) + (1 - \lambda) f_1(y - z) - g_2(z) \\ &= \lambda[f_1(x - z) - g_2(z)] + (1 - \lambda)[f_1(y - z) - g_2(z)]. \end{aligned}$$

Thus for some $z \in X$,

$$f_1(W(x, y, \lambda) - z) - g_2(z) \leq \lambda u(f_1(x - z) - g_2(z)) + (1 - \lambda)u(f_1(y - z) - g_2(z)).$$

It follows that

$$f_1(W(x, y, \lambda) - z) - g_2(z) \leq \lambda \sup_{z \in X} u(f_1(x - z) - g_2(z)) + (1 - \lambda) \sup_{z \in X} u(f_1(y - z) - g_2(z)).$$

Hence,

$$\sup_{z \in X} u(f_1(W(x, y, \lambda) - z) - g_2(z)) \leq \lambda(f \oplus g)_1(x) + (1 - \lambda)(f \oplus g)_1(y).$$

Therefore,

$$(f \oplus g)_1(W(x, y, \lambda)) \leq \lambda(f \oplus g)_1(x) + (1 - \lambda)(f \oplus g)_1(y).$$

One proves that

$$(f \oplus g)_2(W(x, y, \lambda)) \leq \lambda(f \oplus g)_2(x) + (1 - \lambda)(f \oplus g)_2(y)$$

by similar arguments. □

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