Filomat 37:6 (2023), 1959–1974
https://doi.org/10.2298/FIL2306959T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note of some approximation theorems of functions on the Laguerre hypergroup

O. Tyr[a], **R. Daher**[a]

[a]*Department of Mathematics, Faculty of Sciences Aïn Chock, University Hassan II, Casablanca, Morocco*

Abstract. This paper uses some basic notions and results on the Laguerre hypergroup $\mathbb{K} = [0, +\infty) \times \mathbb{R}$ to study some problems in the theory of approximation of functions in the space $L^2_\alpha(\mathbb{K})$. Analogues of the direct Jackson theorems of approximations for the modulus of smoothness (of arbitrary order) constructed by using the generalized translation operators on $\mathbb{K}$ are proved. The Nikolskii-Stechkin inequality is also obtained. In conclusion of this work, we show that the modulus of smoothness and the K-functionals constructed from the Sobolev-type space corresponding to the Laguerre operator $\mathcal{L}_\alpha$ are equivalent.

1. Introduction

The theory of approximation is one of the fields of mathematical analysis. Its main objective is to approximate functions by other functions that are simpler and easier to study. In the classical theory of approximation of functions on $\mathbb{R}$, a central role is played by the translation operators $f \to f(x+y)$, $x, y \in \mathbb{R}$, these operators are used to define the moduli of smoothness, which are the main elements of the direct and inverse theorems of approximation theory. Some results on the approximation of functions using generalized translations can be found in [4, 6, 8, 13, 15].

The modulus of smoothness play a basic role in approximation theory. For a given a positive real number δ and a positive integer r, the classical modulus of smoothness is defined for a function $f \in L^2(\mathbb{R})$ by

$$\omega_r(f, \delta) = \sup_{0<u\leq\delta} \|\Delta^r_u f\|_2,$$

where

$$\Delta^r_u f = (T_u - I)^r f,$$

I being the unit operator in $L^2(\mathbb{R})$ and T_u stands for the usual translation operator given by $T_u f(t) = f(t+u)$.

On the other side, the study of the K-functional is a classical and important topic in interpolation theory and approximation theory. The classical K-functional introduced by Peetre in [21], is defined by

$$K_r(f, \delta) = \inf\{\|f - g\|_2 + \delta \|D^r g\|_2;\ g \in W^r_2\},$$

2020 *Mathematics Subject Classification*. Primary 33D15; Secondary 33E30, 44A20
Keywords. Laguerre hypergroup, Fourier-Laguerre transform, Generalized translation operators, Direct Jackson theorems, K-functionals, Modulus of smoothness
Received: 12 February 2022; Accepted: 26 December 2022
Communicated by Miodrag Spalević
Email addresses: otywac@hotmail.fr (O. Tyr), rjdaher024@gmail.com (R. Daher)

where W_2^r be the Sobolev space constructed by the operator $D = \frac{d}{dx}$, and

$$W_2^r := \{f \in L^2(\mathbb{R}) : \ D^j f \in L^2(\mathbb{R}), \quad j = 1, ..., r\}.$$

An outstanding result of the theory of approximation of functions on $\mathbb{R}$, which establishes the equivalence between modulus of smoothness and K-functionals, can be formulated as follows:

Theorem 1.1. *[5] There are two positive constants C_1 and C_2 such that for all $f \in L^2(\mathbb{R})$ and $\delta > 0$, we have*

$$C_1\omega_r(f,\delta) \le K_r(f,\delta^r) \le C_2\omega_r(f,\delta). \tag{1}$$

Considerable attention has been devoted to discovering generalizations to new contexts for Theorem 1.1, This theorem has been proved in the multidimensional case by Peetre [22] and Butzer-Berens [2] by using Steklov averages. Equivalences (1) for some weighted pairs (L^p, W_p^n), were given by Ditzian [9], Löfström [14], Löfström-Peeter [15], Timan [26], Platonov [23, 24], Belkina-Platonov [1], Daher-Tyr [6, 7, 27, 28], El Ouadih [10, 11] and Ditzian-Totik [8].

In our current research, we are interested in the Laguerre hypergroup $\mathbb{K} = [0, +\infty) \times \mathbb{R}$ which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group [12]. Let $\alpha \ge 0$, $\mathbb{K}$ is provided with the convolution product $*_\alpha$ generalizing the convolution product of radial functions on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. We recall that $(\mathbb{K}, *_\alpha)$ is a commutative hypergroup in the sense of Jewett with the involution the homeomorphism $(x,t) \to (x,t)^- = (x,-t)$ and the Haar measure dm_α, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)}dxdt.$$

The unity element of $(\mathbb{K}, *_\alpha)$ is given by $e = (0,0)$, i.e. $\delta_{(x,t)} *_\alpha \delta_{(0,0)} = \delta_{(0,0)} *_\alpha \delta_{(x,t)} = \delta_{(x,t)}$ for all $(x,t) \in \mathbb{K}$. The convolution product of two bounded Radon measures μ and ν on $\mathbb{K}$ is defined by

$$\langle \mu *_\alpha \nu, f\rangle = \int_{\mathbb{K}\times\mathbb{K}} \mathcal{T}^{(\alpha)}_{(x,t)} f(y,s) d\mu(x,t) d\nu(y,s),$$

where $\mathcal{T}^{(\alpha)}_{(x,t)}$, $(x,t) \in \mathbb{K}$, are the generalized translation operators on $\mathbb{K}$ given for $\alpha = 0$, by

$$\mathcal{T}^{(\alpha)}_{(x,t)} f(y,s) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\sqrt{x^2+y^2+2xy\cos\theta}, t+s+xy\sin\theta\right)d\theta,$$

and, for $\alpha > 0$, by

$$\mathcal{T}^{(\alpha)}_{(x,t)} f(y,s) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f\left(\sqrt{x^2+y^2+2xyr\cos\theta}, t+s+xyr\sin\theta\right) r(1-r^2)^{\alpha-1} dr d\theta.$$

The dual of a hypergroup is the space of all bounded continuous and multiplicative functions χ such that $\overline{\chi} = \chi$. The dual of the Laguerre hypergroup $\widehat{\mathbb{K}}$ can be topologically identified with the so-called Heisenberg fan [12], i.e., the subset embedded in $\mathbb{R}^2$ given by

$$\bigcup_{m\in\mathbb{N}} \left\{(\lambda,\mu) \in \mathbb{R}^2 : \mu = |\lambda|(2m+\alpha+1), \ \lambda \neq 0\right\} \cup \left\{(0,\mu) \in \mathbb{R}^2 : \mu \ge 0\right\}.$$

Moreover, the subset $\left\{(0,\mu) \in \mathbb{R}^2 : \mu \ge 0\right\}$ is usually disregarded, since it has zero Plancherel measure. Following [18], in this paper, we identify the dual of the Laguerre hypergroup by $\widehat{\mathbb{K}} := \mathbb{R} \times \mathbb{N}$ (see **Figure 1:**).

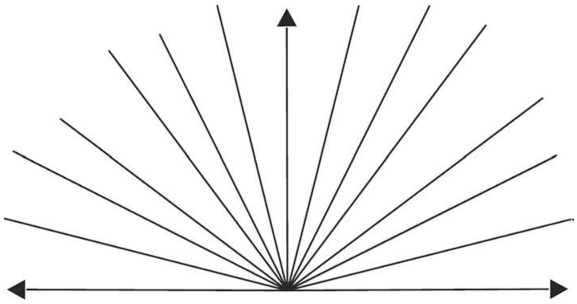

Figure 1: Heisenberg fan

The topology on $\mathbb{K}$ is given by the norm $|(x,t)| = |(x,t)|_{\mathbb{K}} = (x^4+4t^2)^{1/4}$, while we assign to $\widehat{\mathbb{K}}$ the topology generated by the quasi-semi-norm $|(\lambda,m)| = |(\lambda,m)|_{\widehat{\mathbb{K}}} = 4\kappa_m|\lambda|$, where $\kappa_m = m+\frac{\alpha+1}{2}$.

One may naturally ask what are the analogous results for the Fourier-Laguerre transform of Theorem 1.1? As far as we know, this question has not been answered yet. In this paper, we prove some approximation theorems which will help us to give a generalization of Theorem 1.1 in Laguerre hypergroups. More precisely, we use generalized translation operators to study problems of approximation of functions on $\mathbb{K}$, we prove analogues of Jackson's direct theorems for the moduli of smoothness of all orders constructed by generalized translations. We use as the approximation tool a class of functions with bounded spectrum, that is a class of functions for which their Fourier-Laguerre transform $\mathcal{F}_L$ are functions with compact support. The moduli of smoothness are shown to be equivalent to the K-functionals constructed from Sobolev-type spaces.

The outline of the content of this work is as follows: the next section contains some basic facts needed in the sequel about the harmonic analysis in the Laguerre hypergroup and its dual, while the last section deals with the main results of this paper.

2. Harmonic analysis on the Laguerre hypergroup

For the convenience of the reader, we collect here some basic results and notations in harmonic analysis related to Laguerre hypergroups and useful in the sequel. For more information about the Laguerre hypergroup, its dual we refer the reader to [3, 12, 17, 18, 25]. Throughout this paper, we use the following notations:

• $L^p_\alpha(\mathbb{K})$, $1\le p\le\infty$, the space of measurable functions on $\mathbb{K}$, satisfying

$$\|f\|_{p,m_\alpha} = \begin{cases} \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t)\right)^{1/p} < \infty & \text{if } 1\le p<\infty, \\ \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} |f(x,t)| < \infty & \text{if } p=\infty. \end{cases}$$

• $\mathcal{L}^{(\alpha)}_m$ is the Laguerre function defined on $[0,\infty)$ by

$$\mathcal{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}}\frac{L^\alpha_m(x)}{L^\alpha_m(0)}, \tag{2}$$

where L^α_m is the Laguerre polynomial of degree m and order α, given by

$$L^\alpha_m(x) = \sum_{l=0}^m (-1)^l \frac{\Gamma(m+\alpha+1)}{\Gamma(l+\alpha+1)}\frac{1}{l!(m-l)!}x^l. \tag{3}$$

• $\widehat{\mathbb{K}} := \mathbb{R} \times \mathbb{N}$ equipped with the weighted Lebesgue measure γ_α on $\widehat{\mathbb{K}}$ given by

$$\int_{\widehat{\mathbb{K}}} g(\lambda, m) d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} L_m^\alpha(0) \int_{\mathbb{R}} g(\lambda, m) |\lambda|^{\alpha+1} d\lambda.$$

• $L_\alpha^p(\widehat{\mathbb{K}})$, $1 \leq p \leq \infty$, the space of measurable functions $g : \widehat{\mathbb{K}} \to \mathbb{C}$, such that

$$\|g\|_{p,\gamma_\alpha} = \begin{cases} \left(\int_{\widehat{\mathbb{K}}} |g(\lambda, m)|^p d\gamma_\alpha(\lambda, m) \right)^{1/p} < \infty & \text{if } 1 \leq p < \infty, \\ \underset{(\lambda,m)\in\widehat{\mathbb{K}}}{\operatorname{ess\,sup}} |g(\lambda, m)| < \infty & \text{if } p = \infty. \end{cases}$$

It was shown in [18] that for all $(\lambda, m) \in \widehat{\mathbb{K}}$, the system

$$\begin{cases} \mathcal{D}_1 u(x,t) = i\lambda u(x,t), \\ \mathcal{D}_2 u(x,t) = -|(\lambda, m)| u(x,t), \\ u(0,0) = 1, \quad \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\lambda,m}$, given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t} \mathcal{L}_m^{(\alpha)}(|\lambda| x^2), \tag{4}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ be the singular partial differential operators, given by

$$\begin{cases} \mathcal{D}_1 = \dfrac{\partial}{\partial t}, \\ \mathcal{D}_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x} \dfrac{\partial}{\partial x} + x^2 \dfrac{\partial^2}{\partial t^2}, \quad (x,t) \in]0, \infty[\times \mathbb{R}, \end{cases}$$

where α is a nonnegative number.

The harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ is generated by the singular operator

$$\mathcal{L}_\alpha := \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}, \quad \alpha \geq 0.$$

For $\alpha = n - 1$, n being a positive integer, the operator $\mathcal{L}_{n-1}$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and the functions $(z,t) \mapsto \varphi_{\lambda,m}(\|z\|, t)$ are zonal spherical functions of the Gelfand pairs $(G, U(\mathbb{C}^n))$, where G is the semi-direct product of $U(\mathbb{C}^n)$ by $\mathbb{H}^n$ (see [12]). For the general case $\alpha \geq 0$, the functions $\varphi_{\lambda,m}$, $(\lambda, m) \in \widehat{\mathbb{K}}$ are characters of the Laguerre hypergroup $(\mathbb{K}, *_\alpha)$.

Some other properties of the Laguerre kernel are given in the following results:

Proposition 2.1. (*a*) *For all* $(\lambda, m) \in \widehat{\mathbb{K}}$, *the function* $\varphi_{\lambda,m}$ *is infinitely differentiable on* $\mathbb{R}^2$, *even with respect to the first variable and satisfies*

$$|\varphi_{\lambda,m}(x,t)| \leq 1, \quad \forall (x,t) \in \mathbb{K}. \tag{5}$$

(*b*) *For all* $(\lambda, m) \in \widehat{\mathbb{K}}$, *the kernel* $\varphi_{\lambda,m}$ *verifies the following product formula*

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = \mathcal{T}_{(x,t)}^{(\alpha)} \varphi_{\lambda,m}(y,s), \quad (x,t),(y,s) \in \mathbb{K}. \tag{6}$$

Proof. See [18]. □

Lemma 2.2. *for all $x > 0$ and $t \in \mathbb{R}$, we have*

$$\lim_{|\lambda,m|\to+\infty} \varphi_{\lambda,m}(x,t) = 0. \tag{7}$$

Proof. See [16, Lemma 4.3]. □

With the help of the following relations (2), (3) and (4), one can deduce the properties below (see also [16]):

$$\frac{|\lambda,m|x^2}{||\varphi_{\lambda,m}(x,t)|-1|} \to 4\alpha+4 \quad \text{as} \quad |\lambda,m|x^2 \to 0. \tag{8}$$

The behavior in 0 of the characters $\varphi_{\lambda,m}(x,t)$ could be deduced as follows:

$$|\varphi_{\lambda,m}(x,t)-1|^2 = |\lambda t|^2 + \frac{|\lambda,m|^2x^4}{16(\alpha+1)^2} + o(|\lambda|^2|x,t|^4).$$

In consequence, there exist $C > 0$ and $\eta > 0$ such that for all $(x,t) \in \mathbb{K}$,

$$|\lambda||x,t|^2 < \eta \Rightarrow |\varphi_{\lambda,m}(x,t)-1|^2 \leq C|\lambda,m|^2|x,t|^4. \tag{9}$$

We now recall the definition of the Fourier-Laguerre transform and its main properties (see [17, 18])

Definition 2.3. (*i*) *The Fourier-Laguerre transform $\mathcal{F}_L$ of a function f in $L^1_\alpha(\mathbb{K})$ is given by*

$$\mathcal{F}_L(f)(\lambda,m) = \int_{\mathbb{K}} f(x,t)\varphi_{-\lambda,m}(x,t)dm_\alpha(x,t) \quad \textit{for all} \ (\lambda,m) \in \widehat{\mathbb{K}}.$$

(*ii*) *The inverse of the Fourier-Laguerre transform $\mathcal{F}_L^{-1}$ of a function g in $L^1_\alpha(\widehat{\mathbb{K}})$ is defined by*

$$\mathcal{F}_L^{-1}(g)(x,t) = \int_{\widehat{\mathbb{K}}} g(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) \quad \textit{for all} \ (x,t) \in \mathbb{K}. \tag{10}$$

Proposition 2.4. *Let f be in $L^1_\alpha(\mathbb{K})$. Then*

(*i*) *For all $m \in \mathbb{N}$, the function $\lambda \mapsto \mathcal{F}_L(f)(\lambda,m)$ is continuous on $\mathbb{R}$.*

(*ii*) *The function $\mathcal{F}_L(f)$ is bounded on $\widehat{\mathbb{K}}$ and satisfies*

$$\|\mathcal{F}_L(f)\|_{\infty,\gamma_\alpha} \leq \|f\|_{1,m_\alpha}.$$

Proof. See [18, Proposition II.5]. □

Proposition 2.5. *For all $(\lambda,m) \in \widehat{\mathbb{K}}$ and $f \in L^2_\alpha(\mathbb{K})$, we have*

$$\mathcal{F}_L(\mathcal{L}_\alpha f)(\lambda,m) = -|\lambda,m|\mathcal{F}_L(f)(\lambda,m). \tag{11}$$

Proof. See [18, Proposition II.8]. □

In their article [18], M.M. Nessibi and K. Trimèche determined and proved that the Fourier-Laguerre transform $\mathcal{F}_L$ extends to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$ onto $L^2_\alpha(\widehat{\mathbb{K}})$ and we have the following Plancherel's formula

$$\int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) = \int_{\mathbb{K}} |f(x,t)|^2 dm_\alpha(x,t). \tag{12}$$

It is well known from [17, 18] that the translation $\mathcal{T}^{(\alpha)}_{(x,t)}$ is linear operator from $L^p_\alpha(\mathbb{K})$, $1 \le p \le \infty$ onto itself and satisfies

$$\|\mathcal{T}^{(\alpha)}_{(x,t)} f\|_{p,m_\alpha} \le \|f\|_{p,m_\alpha}. \tag{13}$$

It also verifies, as a consequence of the product formula (6), the relation

$$\mathcal{F}_L(\mathcal{T}^{(\alpha)}_{(x,t)} f)(\lambda, m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_L(f)(\lambda, m). \tag{14}$$

3. Main result

In this section, we give the main results of this work. To deal with these results, we will need some important definitions and lemmas that will allow us to achieve the objectives.
Throughout this paper, we fix $\alpha \ge 0$ and for $h > 0$, we will denote by

$$(x,t)_h := (hx, h^2 t),$$

the dilation of $(x,t) \in \mathbb{K}$. Clearly, the dilations are compatible with the structure of the hypergroup.

For every $f \in L^2_\alpha(\mathbb{K})$, we define the differences $\Delta^k_{(x,t)_h} f$ of order k, $k = 1, 2, ...,$ with step $h > 0$ by:

$$\Delta^1_{(x,t)_h} f = \Delta_{(x,t)_h} f := \mathcal{T}^{(\alpha)}_{(x,t)_h} f - f,$$
$$\Delta^k_{(x,t)_h} f = \Delta_{(x,t)_h}(\Delta^{k-1}_{(x,t)_h} f) \quad \text{for} \quad k \ge 2.$$

Also, we can write that

$$\Delta^k_{(x,t)_h} f = (\mathcal{T}^{(\alpha)}_{(x,t)_h} - \mathcal{I})^k f = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} \mathcal{T}^{(\alpha)\nu}_{(x,t)_h} f, \tag{15}$$

where $\mathcal{I}$ is the identity operator in $L^2_\alpha(\mathbb{K})$.

Definition 3.1. *For every natural number k, the modulus of smoothness $\Omega^{(\alpha)}_k$ of order k is defined in the space $L^2_\alpha(\mathbb{K})$ by the following relation:*

$$\Omega^{(\alpha)}_k(f, \delta)_2 = \sup_{0<h\le\delta} \|\Delta^k_{(x,t)_h} f\|_{2,m_\alpha},$$

where $(x,t) \in \mathbb{K}$ and $\delta > 0$.

The Sobolev space $W^k_{2,\alpha}(\mathbb{K})$ on Laguerre hypergroup $\mathbb{K}$ is defined by

$$W^k_{2,\alpha}(\mathbb{K}) := \{f \in L^2_\alpha(\mathbb{K}) : \ \mathcal{L}^j_\alpha f \in L^2_\alpha(\mathbb{K}), \ \ j = 1, 2, ..., k\},$$

where

$$\mathcal{L}^0_\alpha f = f, \ \ \mathcal{L}^j_\alpha f = \mathcal{L}_\alpha(\mathcal{L}^{j-1}_\alpha f), \ \ j = 1, 2, ..., k.$$

Let us define the K-functional constructed by the pair $(L^2_\alpha(\mathbb{K}); W^k_{2,\alpha}(\mathbb{K}))$

Definition 3.2. *For all $\delta > 0$, the K-functional for the pair $(L^2_\alpha(\mathbb{K}); W^k_{2,\alpha}(\mathbb{K}))$ is defined by*

$$K\left(f, \delta; L^2_\alpha(\mathbb{K}); W^k_{2,\alpha}(\mathbb{K})\right) = \inf\left\{\|f - g\|_{2,m_\alpha} + \delta\|\mathcal{L}^k_\alpha g\|_{2,m_\alpha} : g \in W^k_{2,\alpha}(\mathbb{K})\right\},$$

where $f \in L^2_\alpha(\mathbb{K})$.

For brevity, we denote

$$K_k^{(\alpha)}(f,\delta)_2 = K\left(f,\delta; L^2_\alpha(\mathbb{K}); W^k_{2,\alpha}(\mathbb{K})\right).$$

In the following of this paper, let $c, c_1, c_2, c_3, ...,$ denote positive constants that are, generally speaking, different in different places and can depend on k, m, α and other inessential parameters.

We now give some interesting lemmas that will help us in the rest of this paper.

Lemma 3.3. *For $f \in W^k_{2,\alpha}(\mathbb{K})$, we have*

$$\mathcal{F}_L(\mathcal{L}^s_\alpha f)(\lambda, m) = (-1)^s |\lambda, m|^s \mathcal{F}_L(f)(\lambda, m), \tag{16}$$

for all $s = 0, 1, 2, ..., k$.

Proof. The proof follows immediately by recurrence from equation (11). □

Lemma 3.4. *Let $h > 0$ and $(x,t) \in \mathbb{K}$. The inequality*

$$|1 - \varphi_{\lambda,m}((x,t)_h)| \geq c \tag{17}$$

is true with $h|\lambda, m| \geq 1$, where $c > 0$ is a certain constant.

Proof. Let $h > 0$, from relation (4), we have

$$\varphi_{\lambda,m}((x,t)_h) = \varphi_{\lambda,m}(hx, h^2 t) = \varphi_{h\lambda,m}(\sqrt{h}x, ht).$$

Therefore, in view of Lemma 2.2, we get

$$\lim_{h|\lambda,m| \to +\infty} \varphi_{\lambda,m}((x,t)_h) = 0.$$

Consequently, a number $A > 0$ exists such that with $h|\lambda, m| \geq A$, the inequality

$$|\varphi_{\lambda,m}((x,t)_h)| \leq \frac{1}{2}$$

is true. Let

$$a = \min_{1 \leq h|\lambda,m| \leq A} |1 - \varphi_{\lambda,m}((x,t)_h)|.$$

With $h|\lambda, m| \geq 1$, we get the inequality

$$|1 - \varphi_{\lambda,m}((x,t)_h)| \geq c,$$

where $c = \min(a, \frac{1}{2})$. □

Lemma 3.5. *Let $f \in L^2_\alpha(\mathbb{K})$, $(x,t) \in \mathbb{K}$ and $h > 0$. Then*

$$||\Delta^k_{(x,t)_h} f||^2_{2,m_\alpha} = \int_{\widehat{\mathbb{K}}} |1 - \varphi_{\lambda,m}((x,t)_h)|^{2k} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m), \tag{18}$$

where $k = 0, 1, 2, ...$

Proof. By relation (15) and an iteration of relation (14), we obtain

$$\begin{aligned}\mathcal{F}_L(\Delta^k_{(x,t)_h} f)(\lambda, m) &= \sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu}\mathcal{F}_L(\mathcal{T}^{(\alpha)\nu}_{(x,t)_h} f)(\lambda, m) \\ &= \left(\sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu}\varphi^{\nu}_{\lambda,m}((x,t)_h)\right)\mathcal{F}_L(f)(\lambda, m) \\ &= (\varphi_{\lambda,m}((x,t)_h) - 1)^k\mathcal{F}_L(f)(\lambda, m). \qquad (19)\end{aligned}$$

Now, by Plancherel formula (12), we have (18). □

Remark 3.6. *Lets $f \in W^k_{2,\alpha}(\mathbb{K})$ and $h > 0$, it follows from Lemmas 3.3 and 3.5 that*

$$\|\Delta^k_{(x,t)_h}(\mathcal{L}^s_\alpha f)\|^2_{2,m_\alpha} = \int_{\widehat{\mathbb{K}}} |\lambda, m|^{2s} |1 - \varphi_{\lambda,m}((x,t)_h)|^{2k} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m), \qquad (20)$$

where $s = 0, 1, ..., k$.

Lemma 3.7. *The modulus of smoothness $\Omega^{(\alpha)}_k(f, \delta)_2$, $k = 1, 2, ...$, $\delta > 0$ possesses the following properties.*

(a) $\Omega^{(\alpha)}_k(f, \delta)_2$ *is a non-decreasing function of* δ.

(b) $\Omega^{(\alpha)}_k(f, \delta)_2$ *is a continuous function of* δ *and* $\Omega^{(\alpha)}_k(f, \delta)_2 \to 0$ *as* $\delta \to 0$.

(c) $\Omega^{(\alpha)}_k(f \pm g, \delta)_2 \leq \Omega^{(\alpha)}_k(f, \delta)_2 + \Omega^{(\alpha)}_k(g, \delta)_2$, $f, g \in L^2_\alpha(\mathbb{K})$.

(d) $\Omega^{(\alpha)}_k(f, \delta_1 + \delta_2)_2 \leq \Omega^{(\alpha)}_k(f, \delta_1)_2 + \Omega^{(\alpha)}_k(f, \delta_2)_2$, $\delta_1, \delta_2 > 0$, $f \in L^2_\alpha(\mathbb{K})$.

(e) $\Omega^{(\alpha)}_k(f, \delta)_2 \leq 2^k \|f\|_{2,m_\alpha}$.

(f) *If* $l \leq k$, *then* $\Omega^{(\alpha)}_k(f, \delta)_2 \leq 2^{k-l}\Omega^{(\alpha)}_l(f, \delta)_2$.

(j) *If* $f \in W^k_{2,\alpha}(\mathbb{K})$, *then*

$$\Omega^{(\alpha)}_k(f, \delta)_2 \leq c_1\delta^{2k}\|\mathcal{L}^k_\alpha f\|_{2,m_\alpha}, \qquad (21)$$

where c_1 *is a constant.*

(h) *If* $f \in W^k_{2,\alpha}(\mathbb{K})$ *and* $l > k$, *then*

$$\Omega^{(\alpha)}_l(f, \delta)_2 \leq c_2\delta^{2k}\Omega^{(\alpha)}_{l-k}(\mathcal{L}^k_\alpha f, \delta)_2, \qquad (22)$$

where c_2 *is a constant.*

Proof. Properties (a), (c) and (d) follow from the definition of the modulus of smoothness $\Omega^{(\alpha)}_k(f, \delta)_2$. Property ($b$) holds because the function $\mathcal{T}^{(\alpha)}_{(x,t)_h} f$ depends continuously on h in the space $L^2_\alpha(\mathbb{K})$ ($\|\mathcal{T}^{(\alpha)}_{(x,t)_h} f - f\|_{2,m_\alpha} \to 0$ *as* $h \to 0^+$). Properties (e) and (f) follow from the fact that

$$\|\mathcal{T}^{(\alpha)}_{(x,t)_h} f\|_{2,m_\alpha} \leq 2\|f\|_{2,m_\alpha} \quad \text{for all } f \in L^2_\alpha(\mathbb{K}).$$

If $f \in W^k_{2,\alpha}(\mathbb{K})$, then by relation (18), we have

$$\|\Delta^k_{(x,t)_h} f\|^2_{2,m_\alpha} = \int_{\widehat{\mathbb{K}}} |1 - \varphi_{\lambda,m}((x,t)_h)|^{2k} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) = \mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 = \int_{|\lambda,m|<\frac{4\kappa_m\eta}{h^2|x,t|^2}} |1-\varphi_{\lambda,m}((x,t)_h)|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)$$

and

$$\mathcal{J}_2 = \int_{|\lambda,m|\geq\frac{4\kappa_m\eta}{h^2|x,t|^2}} |1-\varphi_{\lambda,m}((x,t)_h)|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

Estimate the summands $\mathcal{J}_1$ and $\mathcal{J}_2$: From relations (9), (16) and Plancherel formula (12), we get

$$\begin{aligned}
\mathcal{J}_1 &= \int_{|\lambda,m|<\frac{4\kappa_m\eta}{h^2|x,t|^2}} |1-\varphi_{\lambda,m}((x,t)_h)|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&\leq C^k h^{4k}|x,t|^{4k}\int_{|\lambda,m|<\frac{4\kappa_m\eta}{h^2|x,t|^2}} |\lambda,m|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&\leq C^k h^{4k}|x,t|^{4k}\int_{\widehat{\mathbb{K}}} |\lambda,m|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&= C^k h^{4k}|x,t|^{4k}\int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(\mathcal{L}_\alpha^k f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&= C^k h^{4k}|x,t|^{4k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}^2.
\end{aligned} \tag{23}$$

To estimate $\mathcal{J}_2$, we use the inequality (5), relation (16) and Plancherel formula (12) and we get

$$\begin{aligned}
\mathcal{J}_2 &= \int_{|\lambda,m|\geq\frac{4\kappa_m\eta}{h^2|x,t|^2}} |1-\varphi_{\lambda,m}((x,t)_h)|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&\leq 2^{2k}\int_{|\lambda,m|\geq\frac{4\kappa_m\eta}{h^2|x,t|^2}} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&\leq \frac{2^{2k}h^{4k}|x,t|^{4k}}{(4\kappa_m\eta)^{2k}}\int_{|\lambda,m|\geq\frac{4\kappa_m\eta}{h^2|x,t|^2}} |\lambda,m|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&\leq \frac{2^{2k}h^{4k}|x,t|^{4k}}{(4\kappa_m\eta)^{2k}}\int_{\widehat{\mathbb{K}}} |\lambda,m|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&= \frac{2^{2k}h^{4k}|x,t|^{4k}}{(4\kappa_m\eta)^{2k}}\int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(\mathcal{L}_\alpha^k f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\
&= \frac{2^{2k}h^{4k}|x,t|^{4k}}{(4\kappa_m\eta)^{2k}}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}^2.
\end{aligned} \tag{24}$$

Therefore, combining the relations (23) and (24), we get

$$\|\Delta_{(x,t)_h}^k f\|_{2,m_\alpha}^2 \leq \left(C^k + \frac{1}{(2\kappa_m\eta)^{2k}}\right)|x,t|^{4k}h^{4k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}^2.$$

Consequently,

$$\|\Delta_{(x,t)_h}^k f\|_{2,m_\alpha} \leq \left(C^k + \frac{1}{(2\kappa_m\eta)^{2k}}\right)^{1/2}|x,t|^{2k}h^{2k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}.$$

Calculating the supremum with respect to all $h\in]0,\delta]$, we obtain

$$\Omega_k^{(\alpha)}(f,\delta)_2 \leq c_1\delta^{2k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}.$$

Then, the property (j) is well verified.
If $f \in W^k_{2,\alpha}(\mathbb{K})$ and $l > k$, it is not hard to check that

$$\|\Delta^l_{(x,t)_h} f\|_{2,m_\alpha} \leq c_1 \delta^{2k} \|\Delta^{l-k}_{(x,t)_h}(\mathcal{L}^k_\alpha f)\|_{2,m_\alpha}.$$

The above formula yields the proof of property (h). □

Definition 3.8. *For any function $f \in L^2_\alpha(\mathbb{K})$ and any number $\sigma > 0$, we define the function*

$$\begin{aligned} \mathcal{P}_\sigma(f)(x,t) &:= \int_{|\lambda,m|\leq\sigma} \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) \\ &= \mathcal{F}_L^{-1}\left(\mathcal{F}_L f(\lambda,m)\chi_\sigma(\lambda,m)\right), \end{aligned}$$

where $\chi_\sigma(\lambda,m)$ is the function defined by $\chi_\sigma(\lambda,m) = 1$ for $|\lambda,m| \leq \sigma$ and 0 otherwise and $\mathcal{F}_L^{-1}$ is the inverse Fourier-Laguerre transform.

One can easily prove that the function $\mathcal{P}_\sigma(f)$ is infinitely differentiable and belongs to all classes $W^k_{2,\alpha}(\mathbb{K})$, $k \in \mathbb{N}$.

Definition 3.9. *A function $f \in L^2_\alpha(\mathbb{K})$ is called a function with bounded spectrum of order $\sigma > 0$ if*

$$\mathcal{F}_L f(\lambda,m) = 0 \quad \text{for} \quad |\lambda,m| > \sigma.$$

The set of all such functions is denoted by $\mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$.

The best approximation of a function $f \in L^2_\alpha(\mathbb{K})$ by the functions in $\mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$ is defined by

$$E^{(\alpha)}_\sigma(f)_2 := \inf_{g\in\mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})} \|f-g\|_{2,m_\alpha}.$$

Lemma 3.10. *The following assertions hold:*

(i) For every function $f \in L^2_\alpha(\mathbb{K})$, $\mathcal{P}_\sigma(f) \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$.

(ii) For every function $\psi \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$, $\mathcal{P}_\sigma(\psi) = \psi$.

(iii) If $f \in L^2_\alpha(\mathbb{K})$, then

$$\|\mathcal{P}_\sigma(f)\|_{2,m_\alpha} \leq \|f\|_{2,m_\alpha}, \tag{25}$$

$$\|f - \mathcal{P}_\sigma(f)\|_{2,m_\alpha} \leq c_3 E^{(\alpha)}_\sigma(f)_2. \tag{26}$$

Proof. (i) Since

$$\mathcal{F}_L(\mathcal{P}_\sigma(f))(\lambda,m) = \chi_\sigma(\lambda,m)\mathcal{F}_L(f)(\lambda,m) = 0$$

for $|\lambda,m| > \sigma$, then we have $\mathcal{P}_\sigma(f) \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$.
(ii) If $\psi \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$, hence $\mathcal{F}_L(\psi)(\lambda,m) = 0$ for $|\lambda,m| > \sigma$ and since $\chi_\sigma(\lambda,m) = 1$ for $|\lambda,m| \leq \sigma$. Thus, by using the inversion formula (10) we get

$$\begin{aligned} \mathcal{P}_\sigma(\psi)(x,t) &= \int_{\widehat{\mathbb{K}}} \mathcal{F}_L(\mathcal{P}_\sigma(\psi))(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) \\ &= \int_{|\lambda,m|\leq\sigma} \mathcal{F}_L(\psi)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) \\ &= \int_{\widehat{\mathbb{K}}} \mathcal{F}_L(\psi)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m) = \psi(x,t). \end{aligned}$$

(iii) Suppose that $f \in L^2_\alpha(\mathbb{K})$. By the Plancherel formula (12) and Definition 3.8, we have

$$\begin{aligned}
\|\mathcal{P}_\sigma(f)\|^2_{2,m_\alpha} &= \|\mathcal{F}_L(\mathcal{P}_\sigma(f))\|^2_{2,\gamma_\alpha} \\
&= \|\chi_\sigma(\lambda,m)\mathcal{F}_L(f)(\lambda,m)\|^2_{2,\gamma_\alpha} \\
&= \int_{|\lambda,m|\leq\sigma} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\
&\leq \int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\
&= \|f\|^2_{2,m_\alpha}.
\end{aligned}$$

On the other hand, suppose that $f \in L^2_\alpha(\mathbb{K})$. Take an arbitrary function $\psi \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$ such that

$$\|f-\psi\|_{2,m_\alpha} \leq 2E^{(\alpha)}_\sigma(f)_2.$$

By the equality $\mathcal{P}_\sigma(\psi) = \psi$ and inequality (25), we have

$$\begin{aligned}
\|f-\mathcal{P}_\sigma(f)\|_{2,m_\alpha} &= \|f-\psi+\mathcal{P}_\sigma(\psi-f)\|_{2,m_\alpha} \\
&\leq \|f-\psi\|_{2,m_\alpha} + \|f-\psi\|_{2,m_\alpha} \\
&\leq 4E^{(\alpha)}_\sigma(f)_{2,m_\alpha},
\end{aligned}$$

which proves (26). □

Lemma 3.11. *If $f \in L^2_\alpha(\mathbb{K})$ and $\sigma > 0$. Then*

(i) $\mathcal{P}_\sigma(f) \in C^\infty(\mathbb{K})$ and we have

$$\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))(x,t) = (-1)^k \int_{|\lambda,m|\leq\sigma} |\lambda,m|^k \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \tag{27}$$

for all $(x,t) \in \mathbb{K}$ and $k = 0,1,...$

(ii) For all $k = 0,1,...$, $\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f)) \in L^2_\alpha(\mathbb{K})$ and

$$\mathcal{F}_L(\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))(\lambda,m) = (-1)^k|\lambda,m|^k\mathcal{F}_L(f)(\lambda,m)\chi_\sigma(\lambda,m). \tag{28}$$

Proof. The fact that $\mathcal{P}_\sigma(f) \in C^\infty(\mathbb{K})$ follows from a derivation under the integral sign. Identity (27) follows easily from (16). Assertion (ii) is a consequence of (27). □

The following theorems are analogues of Jackson's direct theorems in the classical approximation theory (see [20, Chapter 5]):

Theorem 3.12. *If $f \in L^2_\alpha(\mathbb{K})$, then the following inequality holds for any $\sigma > 0$:*

$$E^{(\alpha)}_\sigma(f)_2 \leq c_4\Omega^{(\alpha)}_k\left(f,\frac{1}{\sigma}\right)_2, \tag{29}$$

where $k \in \mathbb{N}$ and c_4 is a positive constant.

Proof. Suppose that $f \in L^2_\alpha(\mathbb{K})$. The Plancherel formula (12) gives that

$$\begin{aligned}\|f - \mathcal{P}_\sigma(f)\|^2_{2,m_\alpha} &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(f - \mathcal{P}_\sigma(f))(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\ &= \int_{\widehat{\mathbb{K}}} |1 - \chi_\sigma(\lambda)|^2 |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\ &= \int_{|\lambda,m|\geq\sigma} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m).\end{aligned}$$

In view of Lemma 3.4, we get

$$|1 - \varphi_{\lambda,m}((x,t)_{1/\sigma})| \geq c \quad \text{for } |\lambda, m| \geq \sigma.$$

Therefore, from (19) and the Plancherel formula (12), we deduce that

$$\begin{aligned}\|f - \mathcal{P}_\sigma(f)\|^2_{2,m_\alpha} &\leq c^{-2k} \int_{|\lambda,m|\geq\sigma} |1 - \varphi_{\lambda,m}((x,t)_{1/\sigma})|^{2k} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\ &= c^{-2k} \int_{|\lambda,m|\geq\sigma} |\mathcal{F}_L(\Delta^k_{(x,t)_{1/\sigma}} f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\ &\leq c^{-2k} \int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(\Delta^k_{(x,t)_{1/\sigma}} f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\ &= c^{-2k} \|\mathcal{F}_L(\Delta^k_{(x,t)_{1/\sigma}} f)\|^2_{2,\gamma_\alpha}\\ &= c^{-2k} \|\Delta^k_{(x,t)_{1/\sigma}} f\|^2_{2,m_\alpha}.\end{aligned}$$

Therefore, as $\mathcal{P}_\sigma(f) \in \mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})$, we get

$$\begin{aligned}E^{(\alpha)}_\sigma(f)_2 &= \inf_{g\in\mathcal{B}^{(\alpha)}_\sigma(\mathbb{K})} \|f - g\|_{2,m_\alpha} \leq \|f - \mathcal{P}_\sigma(f)\|_{2,m_\alpha}\\ &\leq c^{-k} \|\Delta^k_{(x,t)_{1/\sigma}} f\|_{2,m_\alpha} \leq c^{-k} \Omega^{(\alpha)}_k \left(f, \frac{1}{\sigma}\right)_2.\end{aligned}$$

Then Theorem 3.12 is proved with $c_4 = c^{-k}$. □

Theorem 3.13. *Assume that $f, \mathcal{L}_\alpha f, ..., \mathcal{L}^d_\alpha f$, $d \in \mathbb{N}$, belong to $L^2_\alpha(\mathbb{K})$, where $\mathcal{L}_\alpha$ is the Laguerre operator. Then*

$$E^{(\alpha)}_\sigma(f)_2 \leq c_5 \sigma^{-2d} \Omega^{(\alpha)}_k \left(\mathcal{L}^d_\alpha f, \frac{1}{\sigma}\right)_2, \tag{30}$$

where c_5 is a positive constant.

Proof. Replacing k by $k + d$ in the previous theorem, we get

$$E^{(\alpha)}_\sigma(f)_2 \leq c_6 \Omega^{(\alpha)}_{k+d} \left(f, \frac{1}{\sigma}\right)_2, \quad \sigma > 0, \tag{31}$$

where c_6 is a constant. It follows from the property of the modulus of smoothness (22) that

$$\Omega^{(\alpha)}_{k+d} \left(f, \frac{1}{\sigma}\right)_2 \leq c_2 \sigma^{-2k} \Omega^{(\alpha)}_k \left(\mathcal{L}^d_\alpha f, \frac{1}{\sigma}\right)_2. \tag{32}$$

Now (30) follows from (31) and (32). □

Now, we will prove Nikolskii-Stechkin inequality [19] for Laguerre hypergroup.

Lemma 3.14. *For any $f \in L^2_\alpha(\mathbb{K})$ and $\sigma > 0$, we have*

$$\|\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))\|_{2,m_\alpha} \leq c_7\sigma^{2k}\|\Delta^k_{(x,t)_{1/\sigma}}f\|_{2,m_\alpha} \tag{33}$$

for all $k = 0, 1, ...$

Proof. From (16), (18) and the Plancherel formula (12), we deduce that

$$\begin{aligned}
&\|\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))\|^2_{2,m_\alpha} = \int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f)))(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\
&= \int_{|\lambda,m|\leq\sigma} |\lambda, m|^{2k}|\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\
&= \int_{\widehat{\mathbb{K}}} \frac{|\lambda, m|^{2k}\chi(\lambda, m)}{|1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}} |1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}|\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m).
\end{aligned}$$

We note that

$$\begin{aligned}
\sup_{(\lambda,m)\in\widehat{\mathbb{K}}} \frac{|\lambda, m|^{2k}\chi(\lambda, m)}{|1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}} &= \frac{\sigma^{4k}}{x^{4k}} \sup_{(\lambda,m)\in\widehat{\mathbb{K}}} \frac{(|\lambda, m|\frac{x^2}{\sigma^2})^{2k}\chi(\lambda, m)}{|1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}}\\
&= \frac{\sigma^{4k}}{x^{4k}} \sup_{|\lambda,m|\leq\sigma} \frac{(|\lambda, m|\frac{x^2}{\sigma^2})^{2k}}{|1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}}\\
&\leq \frac{\sigma^{4k}}{x^{4k}} \sup_{|\lambda,m|\leq\sigma} \frac{(|\lambda, m|\frac{x^2}{\sigma^2})^{2k}}{|1 - |\varphi_{\lambda,m}(\frac{x}{\sigma}, \frac{t}{\sigma^2})||^{4k}}\\
&= \frac{c_8}{x^{4k}}\sigma^{4k},
\end{aligned}$$

where

$$c_8 = \sup_{|\lambda,m|\leq\sigma} \frac{(|\lambda, m|\frac{x^2}{\sigma^2})^{2k}}{|1 - |\varphi_{\lambda,m}(\frac{x}{\sigma}, \frac{t}{\sigma^2})||^{2k}}.$$

Note that if $|\lambda, m| \to 0$, then by relation (8), we conclude that

$$\frac{(|\lambda, m|\frac{x^2}{\sigma^2})^{2k}}{|1 - |\varphi_{\lambda,m}(\frac{x}{\sigma}, \frac{t}{\sigma^2})||^{2k}} \to 4^{2k}(\alpha + 1)^{2k}.$$

Hence c_8 must be finite.
Therefore

$$\begin{aligned}
\|\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))\|^2_{2,m_\alpha} &\leq \frac{c_8}{x^{4k}}\sigma^{4k} \int_{\widehat{\mathbb{K}}} |1 - \varphi_{\lambda,m}((x, t)_{1/\sigma})|^{2k}|\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\
&= \frac{c_8}{x^{4k}}\sigma^{4k} \int_{\widehat{\mathbb{K}}} |\mathcal{F}_L(\Delta^k_{(x,t)_{1/\sigma}}f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)\\
&= \frac{c_8}{x^{4k}}\sigma^{4k}\|\Delta^k_{(x,t)_{1/\sigma}}f\|^2_{2,m_\alpha}.
\end{aligned}$$

Thus

$$\|\mathcal{L}^k_\alpha(\mathcal{P}_\sigma(f))\|_{2,m_\alpha} \leq c_7\sigma^{2k}\|\Delta^k_{(x,t)_{1/\sigma}}f\|_{2,m_\alpha},$$

and this proves (33). □

As noted in Lemma 3.14 that $\mathcal{P}_\sigma(\psi) = \psi$ for any $\psi \in \mathcal{B}_\sigma^{(\alpha)}(\mathbb{K})$, the following corollary is immediate.

Corollary 3.15. *There is a positive constant c_7 such that*

$$\|\mathcal{L}_\alpha^k(\psi)\|_{2,m_\alpha} \leq c_7\sigma^{2k}\|\Delta^k_{(x,t)_{1/\sigma}}\psi\|_{2,m_\alpha} \tag{34}$$

for any $\psi \in \mathcal{B}_\sigma^{(\alpha)}(\mathbb{K})$, $k \in \mathbb{N}$ and $\sigma > 0$.

The following corollary follows from the definition of modulus of smoothness.

Corollary 3.16. *The inequality*

$$\|\mathcal{L}_\alpha^k(\mathcal{P}_\sigma(f))\|_{2,m_\alpha} \leq c_7\sigma^{2k}\Omega_k^{(\alpha)}\left(f, \frac{1}{\sigma}\right)_2 \tag{35}$$

holds for any $f \in L^2_\alpha(\mathbb{K})$, $m \in \mathbb{N}$ and $\sigma > 0$.

Our main purpose will be shown here. We will prove in the following theorem that the K-functional for the pair $(L^2_\alpha(\mathbb{K}); W^k_{2,\alpha}(\mathbb{K}))$ and modulus of smoothness generated by Laguerre translation operators are equivalent.

Theorem 3.17. *for any $f \in L^2_\alpha(\mathbb{K})$ and $\delta > 0$, we have*

$$\Omega_k^{(\alpha)}(f,\delta)_2 \sim K_k^{(\alpha)}(f,\delta^{2k})_2.$$

i.e., there are two positive constants C_1 and C_2 such that

$$C_1\Omega_k^{(\alpha)}(f,\delta)_2 \leq K_k^{(\alpha)}(f,\delta^{2k})_2 \leq C_2\Omega_k^{(\alpha)}(f,\delta)_2, \tag{36}$$

Proof. Take $g \in W^k_{2,\alpha}(\mathbb{K})$. Now by using the properties of modulus of continuity $\Omega_k^{(\alpha)}(f,\delta)_2$, we get (see Lemma 3.7)

$$\begin{aligned}
\Omega_k^{(\alpha)}(f,\delta)_2 &\leq \Omega_k^{(\alpha)}(f-g,\delta)_2 + \Omega_k^{(\alpha)}(g,\delta)_2 \\
&\leq 2^m\|f-g\|_{2,m_\alpha} + c_1\delta^{2k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha} \\
&\leq c_9\left(\|f-g\|_{2,m_\alpha} + \delta^{2k}\|\mathcal{L}_\alpha^k f\|_{2,m_\alpha}\right),
\end{aligned}$$

where $c_9 = \max\{2^m, c_1\}$. Taking the infimum over all $g \in W^k_{2,\alpha}(\mathbb{K})$, which proves the left-hand inequality in (36).

Now we prove the right-hand inequality in (36). Since $\mathcal{P}_\sigma(f) \in W^k_{2,\alpha}(\mathbb{K})$, by the definition of K-functional we obtain

$$K_k^{(\alpha)}(f,\delta^{2k})_2 \leq \|f-\mathcal{P}_\sigma(f)\|_{2,m_\alpha} + \delta^{2k}\|\mathcal{L}_\alpha^k(\mathcal{P}_\sigma(f))\|_{2,m_\alpha}. \tag{37}$$

It follows from relations (26), Corollary 3.16 and Theorem 3.12 that

$$\begin{aligned}
K_k^{(\alpha)}(f,\delta^{2k})_2 &\leq c_3E_\sigma^{(\alpha)}(f)_2 + c_7(\sigma\delta)^{2k}\Omega_k^{(\alpha)}\left(f,\frac{1}{\sigma}\right)_2 \\
&\leq c_3c_4\Omega_k^{(\alpha)}\left(f,\frac{1}{\sigma}\right)_2 + c_7(\sigma\delta)^{2k}\Omega_k^{(\alpha)}\left(f,\frac{1}{\sigma}\right)_2 \\
&= (c_3c_4 + c_7(\sigma\delta)^{2k})\Omega_k^{(\alpha)}\left(f,\frac{1}{\sigma}\right)_2.
\end{aligned}$$

Since σ is an arbitrary positive value, choosing $\delta = \frac{1}{\sigma}$, we obtain

$$K_k^{(\alpha)}(f, \delta^{2k})_2 \leq C_2 \Omega_k^{(\alpha)}(f, \delta)_2,$$

where $C_2 = c_3 c_4 + c_7$. This concludes the proof. □

As a consequence of Theorem 3.17, we obtain another property of the modulus of smoothness.

Theorem 3.18. *There is a constant c_{10} such that the inequality*

$$\Omega_k^{(\alpha)}(f, \nu\delta)_2 \leq c_{10} \max\{1, \delta^{2k}\} \Omega_k^{(\alpha)}(f, \nu)_2 \tag{38}$$

holds for any $f \in W_{2,\alpha}^k(\mathbb{K})$ and $\nu > 0$.

Proof. If $0 < \delta \leq 1$, then the definition of K-functional yields

$$\begin{aligned} K_k^{(\alpha)}(f, (\nu\delta)^{2k})_2 &= \inf\left\{\|f - g\|_{2,m_\alpha} + (\nu\delta)^{2k} \|\mathcal{L}_\alpha^k g\|_{2,m_\alpha} : g \in W_{2,\alpha}^k(\mathbb{K})\right\} \\ &\leq \inf\left\{\|f - g\|_{2,m_\alpha} + \nu^{2k} \|\mathcal{L}_\alpha^k g\|_{2,m_\alpha} : g \in W_{2,\alpha}^k(\mathbb{K})\right\} \\ &= K_k^{(\alpha)}(f, \nu^{2k})_2. \end{aligned}$$

While for $\delta \geq 1$, we have

$$\begin{aligned} K_k^{(\alpha)}(f, (\nu\delta)^{2k})_2 &\leq \inf\left\{\delta^{2k}\|f - g\|_{2,m_\alpha} + \delta^{2k}\nu^{2k} \|\mathcal{L}_\alpha^k g\|_{2,m_\alpha} : g \in W_{2,\alpha}^k(\mathbb{K})\right\} \\ &= \delta^{2k} \inf\left\{\|f - g\|_{2,m_\alpha} + \nu^{2k} \|\mathcal{L}_\alpha^k g\|_{2,m_\alpha} : g \in W_{2,\alpha}^k(\mathbb{K})\right\} \\ &= \delta^{2k} K_k^{(\alpha)}(f, \nu^{2k})_2. \end{aligned}$$

From this and Theorem 3.17, (38) follows. □

References

[1] E. S. Belkina, S. S. Platonov, Equivalence of K-functionals and modulus of smoothness constructed by generalized dunkl translations, Izv. Vyssh. Uchebn. Zaved. Mat. 315(8) (2008) 3–15.

[2] H. Berens, P. L. Buter, Semigroups of operators and approximation, Vol. 145, Springer, Berlin, 1967.

[3] W. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, de Gruyter Stud. Math. Vol. 20, Berlin/New York, 1994.

[4] P.L. Butzer, R.L. Stens, M. Wehrens, Higher order moduli of continuity based on the Jacobi translation operator and best approximation, C. R. Math. Rep. Acad. Sci. Canada 2(2) (1980) 83–88.

[5] P.L. Butzer, H. Behrens, Semi-Groups of Operators and Approximation, Springer, Berlin, Heidelbarg, New York 1967.

[6] R. Daher, O. Tyr, Equivalence of K-functionals and modulus of smoothness generated by a generalized Jacobi-Dunkl transform on the real line, Rend. Circ. Mat. Palermo, II. Ser 70 (2021) 687–698.

[7] R. Daher, O. Tyr, Modulus of Smoothness and Theorems Concerning Approximation in the Space $L_{q,\alpha}^2(\mathbb{R}_q)$ with Power Weight, Mediterr. J. Math. 18(69) (2021).

[8] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, Berlin, 1987.

[9] Z. Ditzian, On interpolation of $L_p[a, b]$ and weihgted Sobolev spaces, Pacific J. Math. 90 (1980) 307–323.

[10] S. El Ouadih, An equivalence theorem for a K-functional constructed by Beltrami-Laplace operator on symmetric spaces, J. Pseudo-Difer. Oper. Appl. 11 (2020) 1951–1962.

[11] S. El Ouadih, R. Daher, O. Tyr, F. Saadi, Equivalence of K-functionals and moduli of smoothness generated by the Beltrami-Laplace operator on the spaces $\mathcal{S}^{(p,q)}(\sigma^{m-1})$, Rend. Circ. Mat. Palermo, II. Ser 71 (2022) 445–458.

[12] J. Faraut, K. Harzallah, Deux cours d'Analyse Harmonique. In: Ecole d'été d'Analyse Harmonique de Tunis. Birkhäuser, Boston, 1984.

[13] I. P. Li, C. M. Su, V. I. Ivanov, Some problems of approximation theory in the spaces L_p on the line with power weight. Mat. Zametki 90(3) (2011) 362–383. English transl., Math. Notes 90:3 (2011), 344–364.

[14] J. Löfström, Interpolation of weighted spaces of diferentiable functions on $\mathbb{R}^d$. Ann. Math. Pura. Appl. 132 (1982) 189–214.

[15] J. Löfström, J. Peetre, Approximation theorems connected with generalized translations, Math. Ann. 181 (1969) 255–268.

[16] S. Negzaoui, Lipschitz Conditions in Laguerre Hypergroup, Mediterr. J. Math. 14(191) (2017) 1-12.

[17] M. M. Nessibi, M. Sifi, Laguerre hypergroup and limit theorem. In: Komrakov, B.P., Krasilshchink, I.S., Litvinov, G.L., Sossink, A.B. (eds.) Lie Groups and Lie Algebras Their Representations, Generalizations and Applications, pp. 133–145. Kluwer Academic Publishers, Dordrecht (1998).

[18] M. M. Nessibi, K. Trimèche, Inversion of the Radon Transform on the Laguerre hypergroup by using generalized wavelets, J. Math. Anal. Appl. 208, (1997) 337–363.
[19] S. M. Nikol'skii, A generalization of an inequality of S. N. Bernstein. Dokl. Akad. Nauk. SSSR 60(9) (1948) 1507–1510.
[20] S. M. Nikol'skii, Approximation of Functions in Several Variables and Embedding Theorems, Nauka, Moscow, 1977.
[21] J. Peetre, A theory of interpolation of normed spaces. Notes de Universidade de Brasilia, pp. 88. Brasilia 1963.
[22] J. Peetre, Espaces d'interpolation et théorème de Soboleff, Ann. Inst. Fourier (Grenoble) 16 (1966) 279–317.
[23] S. S. Platonov, Fourier–Jacobi harmonic analysis and approximation of functions. Izv. RAN. Ser. Mat. (78)1 (2014) 106–153.
[24] S. S. Platonov, Fourier-Jacobi harmonic analysis and some problems of approximation of functions on the half-axis in L_2 metric: Jackson's type direct theorems, Integral Transform Spec. Funct. 30(4) (2019) 264–281.
[25] K. Stempak, Mean summability methods for Laguerre series, Trans. AMS. 322(2) (1990) 129–147.
[26] A. F. Timan, Theory of Approximation of Functions of a Real Variable. Fizmatgiz, Moscow, 1960.(English transl., Pergamon Press, Oxford-New York (1963)).
[27] O. Tyr, R. Daher, Discrete Jacobi–Dunkl Transform and Approximation Theorems, Mediterr. J. Math. 19(224) (2022) 1-17.
[28] O. Tyr, R. Daher, S. El Ouadih, O. El Fourchi, On the Jackson-type inequalities in approximation theory connected to the q-Dunkl operators in the weighted space $L^2_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$, Bol. Soc. Mat. Mex. 27(51) (2021) 1-21.