Filomat 37:6 (2023), 1941–1958
https://doi.org/10.2298/FIL2306941C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by λ-Bernstein type operators on triangular domain

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Abstract. In this paper, a new type of λ-Bernstein operators $\left(B^w_{m,\lambda}g\right)(w,z)$ and $\left(B^z_{n,\lambda}g\right)(w,z)$, their Products $\left(P_{mn,\lambda}g\right)(w,z)$, $\left(Q_{nm,\lambda}g\right)(w,z)$, and their Boolean sums $\left(S_{mn,\lambda}g\right)(w,z)$, $\left(T_{nm,\lambda}g\right)(w,z)$ are constructed on triangle $\mathcal{R}_h$ with parameter $\lambda \in [-1,1]$. Convergence theorem for Lipschitz type continuous functions and a Voronovskaja-type asymptotic formula are studied for these operators. Remainder terms for error evaluation by using the modulus of continuity are discussed. Graphical representations are added to demonstrate the consistency of theoretical findings for the operators approximating functions on the triangular domain. Also, we show that the parameter λ will provide flexibility in approximation; in some cases, the approximation will be better than its classical analogue.

1. Introduction

In 1912, Bernstein [9] introduced the sequence of polynomials (Bernstein operators) to provide constructive proof of Weierstrass Approximation Theorem [39] to approximate any continuous function defined on $[0,1]$ as follows:

$$B_\mu(g;w) = \sum_{r=0}^{\mu} g\left(\frac{r}{\mu}\right) b_{\mu,r}(w), \tag{1}$$

where $w \in [0,1]$, $\mu = 1,2,\cdots$, and $b_{\mu,r}(w)$ are Bernstein basis functions which are defined as:

$$b_{\mu,r}(w) = \binom{\mu}{r} w^r (1-w)^{\mu-r}, \quad r = 0,1,2,\cdots,\mu. \tag{2}$$

These Bernstein basis functions play a significant role in approximation by constructing operators and shape designing via Bézier curves in Computer Aided Geometric Design (CAGD). Later on, various generalizations of this sequence of classical Bernstein operators are constructed and studied, such as Lupaş Bernstein

2020 *Mathematics Subject Classification*. primary 41A35; secondary 41A36, 41A80.
Keywords. λ-Bernstein operators; Triangular domain; Product and Boolean sum operator; Modulus of continuity; Error evaluation.
Received: 13 December 2021; Accepted: 06 December 2022
Communicated by Miodrag Spalevic
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operators (rational) [24], and their post quantum analogue by Khalid and Lobiyal [23], Phillips Bernstein operators (polynomials) [33] and their Post quantum analogue by Mursaleen et al. in [31]. For other works related to Bernstein type operators and their basis functions, one can refer to [3, 11, 12, 15, 22, 25, 27–30, 34–36]. Similarly, for works related to approximation by operators, Bézeir curves and error evaluation, one can read [2, 14, 16, 18–21, 26, 32, 37, 38].

2. Essential preliminaries and review of results of [40] and [13]

In 2010, Ye et al. [40] constructed new Bézeir bases with shape parameter $\lambda \in [-1, 1]$, which are defined as follows:

$$\tilde{b}_{\mu,r}(\lambda; w) = \begin{cases} b_{\mu,0}(w) - \frac{\lambda}{\mu+1} b_{\mu+1,1}(w), & r = 0, \\ b_{\mu,r}(w) + \lambda\Big(\frac{\mu-2r+1}{\mu^2-1} b_{\mu+1,r}(w) - \frac{\mu-2r-1}{\mu^2-1} b_{\mu+1,r+1}(w)\Big), & 1 \le r \le \mu - 1, \\ b_{\mu,\mu}(w) - \frac{\lambda}{\mu+1} b_{\mu+1,\mu}(w), & r = \mu. \end{cases} \tag{3}$$

When $\lambda = 0$, these basis functions (3) turn into classical Bernstein basis functions (2).

Using these basis functions (3), Cai et al. [13] defined λ-Bernstein operators to approximate continuous functions. In 1973, R. E. Barnhill et al. [4] studied smooth interpolation on triangles. For other related works by R. E. Barnhill et al. regarding the study of polynomial interpolation on boundary data on triangles and error bounds for smooth interpolation on triangles, one can refer [5–8]. In 2009, P. Blaga and G. Coman [10] constructed the Bernstein-type operators on triangle $\mathcal{R}_h$ to approximate any real valued function g as:

$$\Big(B_m^w g\Big)(w, z) = \sum_{i=0}^{m} p_{m,i}(w, z) g\Big(\frac{i}{m}(h - z), z\Big), \tag{4}$$

where

$$p_{m,i}(w, z) = \frac{\binom{m}{i} w^i (h - w - z)^{m-i}}{(h - z)^m}, \quad 0 \le w + z \le h, \tag{5}$$

respectively,

$$\Big(B_n^z g\Big)(w, z) = \sum_{j=0}^{n} q_{n,j}(w, z) g\Big(w, \frac{j}{n}(h - w)\Big), \tag{6}$$

with

$$q_{n,j}(w, z) = \frac{\binom{n}{j} z^j (h - w - z)^{n-j}}{(h - w)^n}, \quad 0 \le w + z \le h, \tag{7}$$

where

$$\mathcal{R}_h = \{(w, z) \in \mathbb{R}^2 \mid w \ge 0, z \ge 0, w + z \le h\}, \quad \text{for} \quad h > 0.$$

Motivated by the work mentioned above and the fact that parameter λ provides flexibility in approximation, we construct λ-Bernstein operators on a triangle and discuss their properties.

The paper is arranged as follows: In section 3, we introduce operators $\Big(B_{m,\lambda}^w g\Big)(w, z)$ and $\Big(B_{m,\lambda}^z g\Big)(w, z)$ on triangular domain. In section 4, some auxiliary results have been derived, and their moments and central moments are estimated. In Section 5, we study approximation properties and compute the remainder term for error bound as well as derive a convergence theorem for the Lipschitz continuous functions. Also, we discuss a Voronovskaja-type asymptotic formula. In Section 6, we define Product operators and study the remainder theorem for these operators. In Section 7, we define Boolean sum operators, investigate their properties and discuss the remainder theorem in light of Boolean sum operators. In Section 8, we present some graphs to show the convergence of $\left(B^{w}_{m,\lambda}g\right)(w,z)$ *to* $g(w,z)$ and $\left(B^{z}_{n,\lambda}g\right)(w,z)$ *to* $g(w,z)$ with different values of parameter λ.

3. Construction of univariate operators on triangular domain

Consider a real valued function g defined on a triangle $\mathcal{R}_h$. Through the point $(w,z) \in \mathcal{R}_h$, consider the lines which are parallel to coordinate axes and intersect the edges $\Gamma_i,\ i = 1,2,3,$ of the triangle at the points $(0,z)$ and $(h-z,z)$, respectively, $(w,0)$ and $(w,h-w)$ [10, Figure 1].

Let $\Delta^{w}_{m} = \left\{i\frac{h-z}{m}, i = 0,1,\ldots,m\right\}$ and $\Delta^{z}_{n} = \left\{j\frac{h-w}{n}, i = 0,1,\ldots,n\right\}$ be uniform partitions of the intervals $[0,h-z]$ and $[0,h-w]$, respectively. We introduce the new λ-Bernstein type operators, namely, $\left(B^{w}_{m,\lambda}g\right)(w,z)$ and $\left(B^{z}_{n,\lambda}g\right)(w,z)$ on triangular domain as:

$$\left(B^{w}_{m,\lambda}g\right)(w,z) = \begin{cases} \sum_{i=0}^{m} \tilde{p}_{m,i}(\lambda;w,z)g\left(\frac{i}{m}(h-z),z\right), & (w,z) \in \mathcal{R}_h \setminus (0,h), \\ g(0,h), & (0,h) \in \mathcal{R}_h, \end{cases} \tag{8}$$

and

$$\left(B^{z}_{n,\lambda}g\right)(w,z) = \begin{cases} \sum_{j=0}^{n} \tilde{q}_{n,j}(\lambda;w,z)g\left(w,\frac{j}{n}(h-w)\right), & (w,z) \in \mathcal{R}_h \setminus (h,0), \\ g(h,0), & (h,0) \in \mathcal{R}_h, \end{cases} \tag{9}$$

where

$$\tilde{p}_{m,i}(\lambda;w,z) = \begin{cases} p_{m,0}(w,z) - \frac{\lambda}{m+1}p_{m+1,1}(w,z), & i = 0, \\ p_{m,i}(w,z) + \lambda\left(\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z) - \frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)\right), & 1 \le i \le m-1, \\ p_{m,m}(w,z) - \frac{\lambda}{m+1}p_{m+1,m}(w,z), & i = m, \end{cases} \tag{10}$$

respectively,

$$\tilde{q}_{n,j}(\lambda;w,z) = \begin{cases} q_{n,0}(w,z) - \frac{\lambda}{n+1}q_{n+1,1}(w,z), & j = 0, \\ q_{n,j}(w,z) + \lambda\left(\frac{n-2j+1}{n^2-1}q_{n+1,j}(w,z) - \frac{n-2j-1}{n^2-1}q_{n+1,j+1}(w,z)\right), & 1 \le j \le n-1, \\ q_{n,n}(w,z) - \frac{\lambda}{n+1}q_{n+1,n}(w,z), & j = n, \end{cases} \tag{11}$$

where $\tilde{p}_{m,i}(\lambda;w,z)$ and $\tilde{q}_{n,j}(\lambda;w,z)$ are defined only on $\mathcal{R}_h \setminus (0,h)$ and $\mathcal{R}_h \setminus (h,0)$, respectively, and $p_{m,i}(w,z)$ $(i = 0,1,\ldots,m)$ and $q_{n,j}(w,z)$ $(j = 0,1,\ldots,n)$ are defined in equation (5) and (7) and $\lambda \in [-1,1]$.

4. Some preliminary results

Theorem 4.1. *Let us consider a real valued function g on $\mathcal{R}_h$, then*

$$(i)\ \left(B^{w}_{m,\lambda}g\right)(w,z) = g(w,z) \text{ on } \Gamma_2 \cup \Gamma_3; \tag{12}$$

$$(ii)\ \left(B^{w}_{m,\lambda}e_{00}\right)(w,z) = 1, \tag{13}$$

$$(iii)\ \left(B^{w}_{m,\lambda}e_{10}\right)(w,z) = w + \frac{\lambda}{m(m-1)}\left(h-z-2w + \frac{w^{m+1}-(h-w-z)^{m+1}}{(h-z)^{m}}\right), \tag{14}$$

$$(iv)\left(B^{w}_{m,\lambda}e_{20}\right)(w,z) = w^2 + \frac{w(h-w-z)}{m} + \lambda\left[\frac{1}{m(m-1)}\left(2w(h-z)-4w^2+\frac{2w^{m+1}}{(h-z)^{m-1}}\right)\right.$$
$$\left. + \frac{1}{m^2(m-1)}\left(\frac{w^{m+1}+(h-z-w)^{m+1}}{(h-z)^{m-1}} - (h-z)^2\right)\right], \tag{15}$$

$$(v)\ \left(B^{w}_{m,\lambda}e_{30}\right)(w,z) = w^3 + \frac{3w^2(h-z-w)}{m} + \frac{2w^3-3w^2(h-z)+w(h-z)^2}{m^2}$$
$$+\lambda\left[\frac{1}{m^2}\left(-6w^3+\frac{6w^{m+1}}{(h-z)^{m-2}}\right)+\frac{1}{m(m-1)}\left(3w^2(h-z)-\frac{3w^{m+1}}{(h-z)^{m-2}}\right)\right.$$
$$-\frac{1}{m^2(m-1)}\left(-9w^2(h-z)+\frac{9w^{m+1}}{(h-z)^{m-2}}\right)+\frac{1}{m^3(m-1)}\left(-4w(h-z)^2\right.$$
$$\left.\left.+\frac{4w^{m+1}}{(h-z)^{m-2}}\right)+\frac{m+3}{m^3(m^2-1)}\left((h-z)^3-\frac{w^{m+1}}{(h-z)^{m-2}}-\frac{(h-z-w)^{m+1}}{(h-z)^{m-2}}\right)\right], \tag{16}$$

$$(vi)\ \left(B^{w}_{m,\lambda}e_{40}\right)(w,z) = w^4 + \frac{6w^3(h-z-w)}{m} + \frac{7w^2(h-z)^2-18w^3(h-z)+11w^4}{m^2}$$
$$+\frac{w(h-z)^3-7w^2(h-z)^2+12w^3(h-z)-6w^4}{m^3}$$
$$+\lambda\left[\frac{1}{m^2}\left(6w^2(h-z)^2-2w^3(h-z)-8w^4+\frac{4w^{m+1}}{(h-z)^{m-3}}\right)\right.$$
$$+\frac{1}{m^3}\left(-w^2(h-z)^2-32w^3(h-z)-16w^4+\frac{17w^{m+1}}{(h-z)^{m-3}}\right)$$
$$+\frac{1}{m^4}\left(w(h-z)^3-\frac{w^{m+1}}{(h-z)^{m-3}}\right)+\frac{1}{m^2(m-1)}\left(7w^2(h-z)^2-\frac{7w^{m+1}}{(h-z)^{m-3}}\right)$$
$$-\frac{1}{m^3(m-1)}\left(w(h-z)^3-23w^2(h-z)^2+\frac{22w^{m+1}}{(h-z)^{m-3}}\right)$$
$$\left.+\frac{1}{m^4(m-1)}\left(\frac{(h-z-w)^{m+1}}{(h-z)^{m-3}}-(h-z)^3(h-z-w)\right)\right], \tag{17}$$

where $e_{ij}(w,z) = w^i z^j$.

Proof. Using the equations $\tilde{p}_{m,i}(\lambda;0,z) = \begin{cases} 1, & \text{for } i=0, \\ 0, & \text{for } i>0, \end{cases}$ and $\tilde{p}_{m,i}(\lambda;h-z,z) = \begin{cases} 1, & \text{for } i=m, \\ 0, & \text{for } i<m, \end{cases}$ we get the interpolation property *(i)*.

From (5), we get $\sum_{i=0}^{m} p_{m,i}(\lambda;w,z) = 1$, and with the help of this identity we can easily prove (13).

$$\left(B^{w}_{m,\lambda}e_{10}\right)(w,z) = \sum_{i=0}^{m}\frac{i}{m}(h-z)p_{m,i}(\lambda;w,z)$$
$$= (h-z)\sum_{i=0}^{m-1}\frac{i}{m}\left[p_{m,i}(w,z)+\lambda\left(\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z)-\frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)\right)\right]$$

$$+ (h-z)\Big(p_{m,m}(w,z) - \frac{\lambda}{m+1}p_{m+1,m}(w,z)\Big)$$
$$= (h-z)\sum_{i=0}^{m}\frac{i}{m}p_{m,i}(w,z) + \lambda(h-z)\Big(\sum_{i=0}^{m}\frac{i}{m}\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z)$$
$$-\sum_{i=1}^{m-1}\frac{i}{m}\frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)\Big).$$

It is to note that the classical Bernstein operators (4) on triangle preserve linear functions. Let us call the two parts of the last equation shown in bracket as $\triangle_1(m;w,z)$ and $\triangle_2(m;w,z)$. We have

$$\Big(B^{w}_{m,\lambda}e_{10}\Big)(w,z) = w + \lambda(h-z)\Big(\triangle_1(m;w,z) + \triangle_2(m;w,z)\Big). \tag{18}$$

Now, we compute $\triangle_1(m;w,z)$, and $\triangle_2(m;w,z)$.

$$\begin{aligned}\triangle_1(m;w,z) &= \sum_{i=0}^{m}\frac{i}{m}\frac{m-2i+1}{m^2-1}p_{m+1,i}\,(w,z)\\
&= \frac{1}{m-1}\sum_{i=0}^{m}\frac{i}{m}p_{m+1,i}(w,z) - \frac{2}{m^2-1}\sum_{i=0}^{m}\frac{i^2}{m}p_{m+1,i}(w,z)\\
&= \frac{(m+1)}{m(m-1)}\frac{w}{(h-z)}\sum_{i=0}^{m-1}p_{m,i}(w,z) - \frac{2}{m-1}\frac{w^2}{(h-z)^2}\sum_{i=0}^{m-2}p_{m-1,i}(w,z) - \frac{2}{m(m-1)}\frac{w}{(h-z)}\sum_{i=0}^{m-1}p_{m,i}(w,z)\\
&= \frac{(m+1)}{m(m-1)}\frac{w}{(h-z)}\Big(1-\frac{w^m}{(h-z)^m}\Big) - \frac{2}{m-1}\frac{w^2}{(h-z)^2}\Big(1-\frac{w^{m-1}}{(h-z)^{m-1}}\Big) - \frac{2}{m(m-1)}\frac{w}{(h-z)}\Big(1-\frac{w^m}{(h-z)^m}\Big)\\
&= \frac{w}{m(h-z)} - \frac{2w^2}{(m-1)(h-z)^2} + \frac{w^{m+1}}{m(h-z)^{m+1}} + \frac{2w^{m+1}}{m(m-1)(h-z)^{m+1}},\end{aligned} \tag{19}$$

and

$$\begin{aligned}\triangle_2(m;w,z) &= -\sum_{i=0}^{m-1}\frac{i}{m}\frac{m-2i-1}{m^2-1}p_{m+1,i+1}\,(w,z)\\
&= -\frac{w}{m(h-z)}\sum_{i=1}^{m-1}p_{m,i}(w,z) + \frac{1}{m(m+1)}\sum_{i=1}^{m-1}p_{m+1,i+1}(w,z) + \frac{2w^2}{(m-1)(h-z)^2}\sum_{i=0}^{m-2}p_{m-1,i}(w,z)\\
&- \frac{2w}{m(m-1)(h-z)}\sum_{i=1}^{m-1}p_{m,i}(w,z) + \frac{2}{m(m^2-1)}\sum_{i=1}^{m-1}p_{m+1,i+1}(w,z)\\
&= -\frac{w}{m(h-z)}\Big(1-\frac{(h-z-w)^m}{(h-z)^m}-\frac{w^m}{(h-z)^m}\Big) + \frac{1}{m(m+1)}\Big(1-\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}}\\
&- \frac{(m+1)w(h-z-w)^m}{(h-z)^{m+1}} - \frac{w^{m+1}}{(h-z)^{m+1}}\Big) + \frac{2w^2}{(m-1)(h-z)^2}\Big(1-\frac{w^{m-1}}{(h-z)^{m-1}}\Big)\\
&- \frac{2w}{m(m-1)(h-z)}\Big(1-\frac{(h-z-w)^m}{(h-z)^m}-\frac{w^m}{(h-z)^m}\Big)\\
&+ \frac{2}{m(m^2-1)}\Big(1-\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}} - \frac{(m+1)w(h-z-w)^m}{(h-z)^{m+1}} - \frac{w^{m+1}}{(h-z)^{m+1}}\Big)\\
&= \frac{1}{m-1}\Big(\frac{2w^2}{(h-z)^2} - \frac{w}{(h-z)} - \frac{w^{m+1}}{(h-z)^{m+1}}\Big) + \frac{1}{m(m-1)}\Big(1-\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}} - \frac{w}{(h-z)}\Big).\end{aligned} \tag{20}$$

Combining (18), (19) and (20), we have

$$\left(B^{w}_{m,\lambda}e_{10}\right)(w,z) = w + \frac{1}{m(m-1)}\left(h - z - 2w + \frac{w^{m+1}}{(h-z)^m} - \frac{(h-z-w)^{m+1}}{(h-z)^m}\right)\lambda. \tag{21}$$

Hence, (14) is proved. Finally by (8), we have

$$\begin{aligned}\left(B^{w}_{m,\lambda}e_{20}\right)(w,z) &= \sum_{i=0}^{m}\frac{i^2}{m^2}(h-z)^2 p_{m,i}(\lambda;w,z)\\ &= (h-z)^2\sum_{i=0}^{m}\frac{i^2}{m^2}\left[p_{m,i}(w,z) + \lambda\left(\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z) - \frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)\right)\right]\\ &+ (h-z)^2\left(p_{m,m}(w,z) - \frac{\lambda}{m+1}p_{m+1,m}(w,z)\right)\\ &= (h-z)^2\sum_{i=0}^{m}\frac{i^2}{m^2}p_{m,i}(w,z) + \lambda(h-z)^2\left(\sum_{i=0}^{m}\frac{i^2}{m^2}\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z)\right.\\ &\left.-\sum_{i=1}^{m-1}\frac{i^2}{m^2}\frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)\right).\end{aligned}$$

By (4), we get $\left(B^{w}_{m}e_{20}\right)(w,z) = (h-z)^2\sum_{i=0}^{m}\frac{i^2}{m^2}p_{m,i}(w,z) = w^2 + \frac{w(h-w-z)}{m}$ and let us call the two parts of the last equation shown in the bracket as $\triangle_3(m;w,z)$ and $\triangle_4(m;w,z)$, we have

$$\left(B^{w}_{m,\lambda}e_{20}\right)(w,z) = w^2 + \frac{w(h-w-z)}{m} + \lambda(h-z)^2\left(\triangle_3(m;w,z) + \triangle_4(m;w,z)\right). \tag{22}$$

Now, we have

$$\begin{aligned}\triangle_3(m;w,z) &= \sum_{i=0}^{m}\frac{i^2}{m^2}\frac{m-2i+1}{m^2-1}p_{m+1,i}(w,z)\\ &= \frac{1}{m-1}\sum_{i=0}^{m}\frac{i^2}{m^2}p_{m+1,i}(w,z) - \frac{2}{m^2-1}\sum_{i=0}^{m}\frac{i^3}{m^2}p_{m+1,i}(w,z)\\ &= \frac{(m+1)w^2}{m(m-1)(h-z)^2}\sum_{i=0}^{m-2}p_{m-1,i}(w,z) + \frac{(m+1)w}{m^2(m-1)(h-z)}\sum_{i=0}^{m-1}p_{m,i}(w,z)\\ &- \frac{2w^3}{m(h-z)^3}\sum_{i=0}^{m-3}p_{m-2,i}(w,z) - \frac{6w^2}{m(m-1)(h-z)^2}\sum_{i=0}^{m-2}p_{m-1,i}(w,z) - \frac{2w}{m^2(m-1)(h-z)}\sum_{i=0}^{m-1}p_{m,i}(w,z)\\ &= \frac{(m+1)w^2}{m(m-1)(h-z)^2}\left(1 - \frac{w^{m-1}}{(h-z)^{m-1}}\right) + \frac{(m+1)w}{m^2(m-1)(h-z)}\left(1 - \frac{w^m}{(h-z)^m}\right)\\ &- \frac{2w^3}{m(h-z)^3}\left(1 - \frac{w^{m-2}}{(h-z)^{m-2}}\right) - \frac{6w^2}{m(m-1)(h-z)^2}\left(1 - \frac{w^{m-1}}{(h-z)^{m-1}}\right) - \frac{2w}{m^2(m-1)(h-z)}\left(1 - \frac{w^m}{(h-z)^m}\right)\\ &= \frac{1}{m}\left(\frac{2w^{m+1}}{(h-z)^{m+1}} - \frac{2w^3}{(h-z)^3}\right) + \frac{1}{m-1}\left(\frac{w^2}{(h-z)^2} - \frac{w^{m+1}}{(h-z)^{m+1}}\right)\\ &+ \frac{1}{m(m-1)}\left(\frac{w}{(h-z)} - \frac{5w^2}{(h-z)^2} + \frac{4w^{m+1}}{(h-z)^{m+1}}\right) + \frac{1}{m^2(m-1)}\left(\frac{w^{m+1}}{(h-z)^{m+1}} - \frac{w}{(h-z)}\right).\end{aligned} \tag{23}$$

On the other hand

$$\triangle_4(m;w,z) = -\sum_{i=1}^{m-1}\frac{i^2}{m^2}\frac{m-2i-1}{m^2-1}p_{m+1,i+1}(w,z)$$

$$
\begin{aligned}
&= -\frac{1}{m+1}\sum_{i=0}^{m-1}\frac{i^2}{m^2}p_{m+1,i+1}(w,z) + \frac{2}{m^2-1}\sum_{i=0}^{m-1}\frac{i^3}{m^2}p_{m+1,i+1}(w,z)\\
&= -\frac{w^2}{m(h-z)^2}\sum_{i=0}^{m-2}p_{m-1,i}(w,z) + \frac{w}{m^2(h-z)}\sum_{i=1}^{m-1}p_{m,i}(w,z)\\
&- \frac{1}{m^2(m+1)}\sum_{i=1}^{m-1}p_{m+1,i+1}(w,z) + \frac{2w^3}{m(h-z)^3}\sum_{i=0}^{m-3}p_{m-2,i}(w,z)\\
&+ \frac{2w}{m^2(m-1)(h-z)}\sum_{i=1}^{m-1}p_{m,i}(w,z) - \frac{2}{m^2(m^2-1)}\sum_{i=1}^{m-1}p_{m+1,i+1}(w,z)\\
&= -\frac{w^2}{m(h-z)^2}\Big(1-\frac{w^{m-1}}{(h-z)^{m-1}}\Big) + \frac{w}{m^2(h-z)}\Big(1-\frac{(h-z-w)^m}{(h-z)^m}-\frac{w^m}{(h-z)^m}\Big)\\
&- \frac{1}{m^2(m+1)}\Big(1-\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}}-\frac{(m+1)w(h-z-w)^m}{(h-z)^{m+1}}-\frac{w^{m+1}}{(h-z)^{m+1}}\Big)\\
&+ \frac{2w^3}{m(h-z)^3}\Big(1-\frac{w^{m-2}}{(h-z)^{m-2}}\Big) + \frac{2w}{m^2(m-1)(h-z)}\Big(1-\frac{(h-z-w)^m}{(h-z)^m}-\frac{w^m}{(h-z)^m}\Big)\\
&- \frac{2}{m^2(m^2-1)}\Big(1-\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}}-\frac{(m+1)w(h-z-w)^m}{(h-z)^{m+1}}-\frac{w^{m+1}}{(h-z)^{m+1}}\Big)\\
&= \frac{1}{m}\Big(\frac{2w^3}{(h-z)^3}-\frac{w^2}{(h-z)^2}\Big) + \frac{w}{m(m-1)(h-z)} + \frac{1}{m^2(m-1)}\Big(\frac{w}{(h-z)}+\frac{(h-z-w)^{m+1}}{(h-z)^{m+1}}-1\Big)\\
&- \frac{w^{m+1}}{(m-1)(h-z)^{m+1}}.
\end{aligned} \tag{24}
$$

Combining (22),(23) and (24), we obtain

$$
\begin{aligned}
\Big(B^w_{m,\lambda}e_{20}\Big)(w,z) &= w^2 + \frac{w(h-w-z)}{m} + \lambda\Bigg[\frac{1}{m(m-1)}\Big(2w(h-z)-4w^2+\frac{2w^{m+1}}{(h-z)^{m-1}}\Big)\\
&+ \frac{1}{m^2(m-1)}\Big(\frac{w^{m+1}+(h-z-w)^{m+1}}{(h-z)^{m-1}}-(h-z)^2\Big)\Bigg],
\end{aligned} \tag{25}
$$

which gives (15), and thus Theorem 4.1 is completed.
One can obtain (16) and (17) by following similar computational steps. □

Remark 4.2. *Let us consider a real valued function g defined on $\mathcal{R}_h$, then*

(i) $\Big(B^z_{n,\lambda}g\Big)(w,z) = g(w,z)$ *on* $\Gamma_1 \cup \Gamma_3$;

(ii) $\Big(B^z_{n,\lambda}e_{00}\Big)(w,z) = 1$,

(iii) $\Big(B^z_{n,\lambda}e_{01}\Big)(w,z) = z + \frac{\lambda}{n(n-1)}\Big(h-w-2z+\frac{z^{n+1}-(h-w-z)^{n+1}}{(h-w)^n}\Big)$,

(iv)
$$
\begin{aligned}
\Big(B^z_{n,\lambda}e_{02}\Big)(w,z) &= z^2 + \frac{z(h-w-z)}{n} + \lambda\Bigg[\frac{1}{n(n-1)}\Big(2z(h-w)-4z^2+\frac{2z^{n+1}}{(h-w)^{n-1}}\Big)\\
&+ \frac{1}{n^2(n-1)}\Big(\frac{z^{n+1}+(h-z-w)^{n+1}}{(h-w)^{n-1}}-(h-w)^2\Big)\Bigg],
\end{aligned}
$$

(v)
$$\Big(B^z_{n,\lambda}e_{03}\Big)(w,z) = z^3 + \frac{3z^2(h-z-w)}{n} + \frac{2z^3-3z^2(h-w)+z(h-w)^2}{n^2}$$
$$+\lambda\Bigg[\frac{1}{n^2}\Big(-6z^3+\frac{6z^{n+1}}{(h-w)^{n-2}}\Big)+\frac{1}{n(n-1)}\Big(3z^2(h-w)-\frac{3z^{n+1}}{(h-w)^{n-2}}\Big)$$
$$-\frac{1}{n^2(n-1)}\Big(-9z^2(h-w)+\frac{9z^{n+1}}{(h-w)^{n-2}}\Big)+\frac{1}{n^3(n-1)}\Big(-4z(h-w)^2$$
$$+\frac{4z^{n+1}}{(h-w)^{n-2}}\Big)+\frac{n+3}{n^3(n^2-1)}\Big((h-w)^3-\frac{z^{n+1}}{(h-w)^{n-2}}-\frac{(h-z-w)^{n+1}}{(h-w)^{n-2}}\Big)\Bigg],$$

(vi)
$$\Big(B^z_{n,\lambda}e_{04}\Big)(w,z) = z^4 + \frac{6z^3(h-z-w)}{n} + \frac{7z^2(h-w)^2-18z^3(h-w)+11z^4}{n^2}$$
$$+\frac{z(h-w)^3-7z^2(h-w)^2+12z^3(h-w)-6z^4}{n^3}$$
$$+\lambda\Bigg[\frac{1}{n^2}\Big(6z^2(h-w)^2-2z^3(h-w)-8z^4+\frac{4z^{n+1}}{(h-w)^{n-3}}\Big)$$
$$+\frac{1}{n^3}\Big(-z^2(h-w)^2-32z^3(h-w)-16z^4+\frac{17z^{n+1}}{(h-w)^{n-3}}\Big)$$
$$+\frac{1}{n^4}\Big(z(h-w)^3-\frac{z^{n+1}}{(h-w)^{n-3}}\Big)+\frac{1}{n^2(n-1)}\Big(7z^2(h-w)^2-\frac{7z^{n+1}}{(h-w)^{n-3}}\Big)$$
$$-\frac{1}{n^3(n-1)}\Big(z(h-w)^3-23z^2(h-w)^2+\frac{22z^{n+1}}{(h-w)^{n-3}}\Big)$$
$$+\frac{1}{n^4(n-1)}\Big(\frac{(h-z-w)^{n+1}}{(h-w)^{n-3}}-(h-w)^3(h-z-w)\Big)\Bigg].$$

Corollary 4.3. *If $\lambda \in [-1,1]$, $z \in [0,h]$, and fixed $w \in [0,h-z]$, then by Theorem 4.1, we get*

$$\Big(B^w_{m,\lambda}(s-w)\Big)(w,z) = \frac{1}{m(m-1)}\Big(h-z-2w+\frac{w^{m+1}}{(h-z)^m}-\frac{(h-z-w)^{m+1}}{(h-z)^m}\Big)\lambda$$
$$\leq \frac{1}{m(m-1)}\Big(h-z+2w+\frac{w^{m+1}}{(h-z)^m}+\frac{(h-z-w)^{m+1}}{(h-z)^m}\Big) := \phi_m(w,z); \tag{26}$$

and

$$\Big(B^w_{m,\lambda}(s-w)^2\Big)(w,z) = \frac{w(h-w-z)}{m}+\Bigg[\frac{1}{m(m-1)}\Big(\frac{2w(h-z-w)^{m+1}}{(h-z)^m}-\frac{2w^{m+1}}{(h-z)^{m-1}}-\frac{2w^{m+2}}{(h-z)^m}\Big)$$
$$+\frac{1}{m^2(m-1)}\Big(\frac{w^{m+1}+(h-z-w)^{m+1}}{(h-z)^{m-1}}-(h-z)^2\Big)\Bigg]\lambda$$
$$\leq \frac{w(h-w-z)}{m}+\frac{1}{m(m-1)}\Big(\frac{2w(h-z-w)^{m+1}}{(h-z)^m}+\frac{2w^{m+1}}{(h-z)^{m-1}}+\frac{2w^{m+2}}{(h-z)^m}\Big)$$
$$+\frac{1}{m^2(m-1)}\Big(\frac{w^{m+1}+(h-z-w)^{m+1}}{(h-z)^{m-1}}+(h-z)^2\Big) := \psi_m(w,z); \tag{27}$$

$$\lim_{m\to\infty} m\Big(B^w_{m,\lambda}(s-w)\Big)(w,z) = 0; \tag{28}$$

$$\lim_{m\to\infty} m\Big(B^w_{m,\lambda}(s-w)^2\Big)(w,z) = w(h-w-z), \quad w \in (0,h-z); \tag{29}$$

$$\lim_{m\to\infty} m^2\Big(B^w_{m,\lambda}(s-w)^4\Big)(w,z) = 3w^2(h-z)^2 - 6w^3(h-z) + 3w^4 + 6(w^2(h-z)^2 - w^3(h-z))\lambda, \quad w \in (0, h-z). \tag{30}$$

Remark 4.4. *If $\lambda \in [-1,1]$, $w \in [0,h]$, and fixed $z \in [0, h-w]$, then by remark 4.2, one can obtain*

$$\Big(B^z_{n,\lambda}(t-z)\Big)(w,z) \le \frac{1}{n(n-1)}\Big(h - w + 2z + \frac{z^{n+1}}{(h-w)^n} + \frac{(h-z-w)^{n+1}}{(h-w)^n}\Big) := \phi_n(w,z); \tag{31}$$

and

$$\begin{aligned}\Big(B^z_{n,\lambda}(t-z)^2\Big)(w,z) \le{}& \frac{z(h-w-z)}{n} + \frac{1}{n(n-1)}\Big(\frac{2z(h-w-z)^{n+1}}{(h-w)^n} + \frac{2z^{n+1}}{(h-w)^{n-1}} + \frac{2z^{n+2}}{(h-w)^n}\Big)\\ &+ \frac{1}{n^2(n-1)}\Big(\frac{z^{n+1} + (h-w-z)^{n+1}}{(h-w)^{n-1}} + (h-w)^2\Big) := \psi_n(w,z);\end{aligned} \tag{32}$$

$$\lim_{n\to\infty} n\Big(B^z_{n,\lambda}(t-z)\Big)(w,z) = 0;$$
$$\lim_{n\to\infty} n\Big(B^z_{n,\lambda}(t-z)^2\Big)(w,z) = z(h-w-z), \quad z \in (0, h-w);$$
$$\lim_{n\to\infty} n^2\Big(B^z_{n,\lambda}(t-z)^4\Big)(w,z) = 3z^2(h-w)^2 - 6z^3(h-w) + 3z^4 + 6(z^2(h-w)^2 - z^3(h-w))\lambda, \quad z \in (0, h-w).$$

5. Convergence properties

The approximation formula for the operator $B^w_{m,\lambda}$ is $g = B^w_{m,\lambda}g + R^w_{m,\lambda}g$.

Theorem 5.1. *For $g(\cdot, z) \in C[0, h-z]$, then*

$$\Big|\Big(R^w_{m,\lambda}g\Big)(w,z)\Big| \le \Big(1 + \frac{1}{\delta}\sqrt{\psi_m(w,z)}\Big)\omega\Big(g(\cdot,z);\delta\Big), \quad z \in [0,h], \tag{33}$$

where $\psi_m(w,z)$ is defined by (27) and $\omega\big(g(\cdot,z);\delta\big)$ denotes modulus of continuity with respect to the variable 'w' for the function g.

Proof. We have

$$\Big|\Big(R^w_{m,\lambda}g\Big)(w,z)\Big| \le \sum_{i=0}^{m} \tilde{p}_{m,i}(\lambda;w,z)\Big|g\Big(\frac{i}{m}(h-z), z\Big) - g(w,z)\Big|.$$

As

$$\Big|g\Big(\frac{i}{m}(h-z), z\Big) - g(w,z)\Big| \le \Big(1 + \frac{1}{\delta}\Big|w - \frac{i}{m}(h-z)\Big|\Big)\omega\Big(g(\cdot,z);\delta\Big),$$

one obtains

$$\begin{aligned}\Big|\Big(R^w_{m,\lambda}g\Big)(w,z)\Big| &\le \sum_{i=0}^{m} \tilde{p}_{m,i}(\lambda;w,z)\Big(1 + \frac{1}{\delta}\Big|w - \frac{i}{m}(h-z)\Big|\Big)\omega\Big(g(\cdot,z);\delta\Big)\\ &\le \Bigg[1 + \frac{1}{\delta}\Big(\sum_{i=0}^{m} \tilde{p}_{m,i}(\lambda;w,z)\Big(w - \frac{i(h-z)}{m}\Big)^2\Big)^{1/2}\Bigg]\omega\Big(g(\cdot,z);\delta\Big)\\ &\le \Bigg[1 + \frac{1}{\delta}\Big(\Big(B^w_{m,\lambda}(s-w)^2\Big)(w,z)\Big)^{1/2}\Bigg]\omega\Big(g(\cdot,z);\delta\Big).\end{aligned}$$

Since $\left(B^{w}_{m,\lambda}(s-w)^2\right)(w,z) \le \psi_m(w,z)$, we obtain

$$\left|\left(R^{w}_{m,\lambda}g\right)(w,z)\right| \le \left(1+\frac{1}{\delta}\sqrt{\psi_m(w,z)}\right)\omega\left(g(\cdot,z);\delta\right).$$

□

The approximation formula for the operator $B^{z}_{n,\lambda}$ is $g = B^{z}_{n,\lambda}g + R^{z}_{n,\lambda}g$.

Remark 5.2. *If $g(w,\cdot) \in C[0,h-w]$, then*

$$\left|\left(R^{z}_{n,\lambda}g\right)(w,z)\right| \le \left(1+\frac{1}{\delta}\sqrt{\psi_n(x,y)}\right)\omega\left(g(w,\cdot);\delta\right), \tag{34}$$

where $\psi_n(w,z)$ is defined by (32) and $\omega\left(g(w,\cdot);\delta\right)$ denotes modulus of continuity with respect to the variable 'z' for the function g.

Now, we study the rate of convergence of the operators $\left(B^{w}_{m,\lambda}g\right)(w,z)$ with the help of functions of Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \le 1$. A function $g(\cdot,z)$ belongs to $Lip_M(\alpha)$, if

$$\left|g(w_1,z) - g(w_2,z)\right| \le M|w_1 - w_2|, \quad (w_1, w_2, z \in \mathbb{R}). \tag{35}$$

Now, we present the following theorem.

Theorem 5.3. *Let $g(\cdot,z) \in Lip_M(\alpha)$, and $\lambda \in [-1,1]$, then we have*

$$\left|\left(B^{w}_{m,\lambda}g\right)(w,z) - g(w,z)\right| \le M\left[\psi_m(w,z)\right]^{\alpha/2},$$

for all $w \in [0,h-z]$ and $z \in [0,h]$, where $\psi_m(w,z)$ is defined by (27).

Proof. As $g(\cdot,z) \in Lip_M(\alpha)$ and $\left(B^{w}_{m,\lambda}g\right)(w,z)$ are linear positive operators, then

$$\begin{aligned}
\left|\left(B^{w}_{m,\lambda}g\right)(w,z) - g(w,z)\right| &\le \left(B^{w}_{m,\lambda}|g(s,z)-g(w,z)|\right)(w,z)\\
&= \sum_{i=0}^{m}\tilde{p}_{m,i}(\lambda;w,z)\left|g\left(\frac{i}{m}(h-z),z\right) - g(w,z)\right|\\
&\le M\sum_{i=0}^{m}\tilde{p}_{m,i}(\lambda;w,z)\left|\frac{i}{m}(h-z) - w\right|^{\alpha}\\
&\le M\sum_{i=0}^{m}\left[\tilde{p}_{m,i}(\lambda;w,z)\left|\frac{i}{m}(h-z) - w\right|^{2}\right]^{\frac{\alpha}{2}}\left[\tilde{p}_{m,i}(\lambda;w,z)\right]^{\frac{2-\alpha}{2}}.
\end{aligned}$$

By Hölder's inequality for sums, we have

$$\begin{aligned}
\left|\left(B^{w}_{m,\lambda}g\right)(w,z) - g(w,z)\right| &\le M\left[\sum_{i=0}^{m}\tilde{p}_{m,i}(\lambda;w,z)\left|\frac{i}{m}(h-z) - w\right|^{2}\right]^{\frac{\alpha}{2}}\left[\sum_{i=0}^{m}\tilde{p}_{m,i}(\lambda;w,z)\right]^{\frac{2-\alpha}{2}}\\
&= M\left[\left(B^{w}_{m,\lambda}(s-w)^2\right)(w,z)\right]^{\frac{\alpha}{2}}.
\end{aligned}$$

By equation (27), we get theorem 5.3. □

Remark 5.4. *For $\lambda \in [-1,1]$, and $g(w,\cdot) \in Lip_M(\alpha)$, we have*

$$\left|\left(B^z_{n,\lambda} g\right)(w,z) - g(w,z)\right| \le M\Big[\psi_n(w,z)\Big]^{\alpha/2},$$

for all $z \in [0, h-w]$ and $w \in [0,h]$, where $\psi_n(w,z)$ is defined by (32).

Next, we present a Voronovskaja asymptotic formula for $\left(B^w_{m,\lambda} g\right)(w,z)$.

Theorem 5.5. *Let us consider $g(w,z)$ is bounded on $\mathcal{R}_h$. Then, for any $\lambda \in [-1,1]$, $w \in (0, h-z)$, and $z \in [0,h]$ at which second order partial derivative of $g(w,z)$ with respect to w exists, we have*

$$\lim_{m\to\infty} m\Big[\left(B^w_{m,\lambda} g\right)(w,z) - g(w,z)\Big] = \frac{g^{(2,0)}(w,z)}{2}\Big[w(h-w-z)\Big]. \tag{36}$$

Proof. Let $w \in [0, h-z]$ be fixed. By the Taylor's formula, we may write

$$g(s,z) = g(w,z) + g^{(1,0)}(w,z)(s-w) + \frac{1}{2} g^{(2,0)}(w,z)(s-w)^2 + r(s;w,z)(s-w)^2, \tag{37}$$

where $r(s;w,z)$ denotes Peano's form of the remainder and $r(s;w,z) \in C[0, h-z]$, on applying L'Hôpital's rule, we have

$$\begin{aligned}\lim_{s\to w} r(s;w,z) &= \lim_{s\to w} \frac{g(s,z) - g(w,z) - g^{(1,0)}(w,z)(s-w) - \frac{1}{2} g^{(2,0)}(w,z)(s-w)^2}{(s-w)^2} \\ &= \lim_{s\to w} \frac{g^{(1,0)}(s,z) - g^{(1,0)}(w,z) - g^{(2,0)}(w,z)(s-w)}{2(s-w)} \\ &= \lim_{s\to w} \frac{g^{(2,0)}(s,z) - g^{(2,0)}(w,z)}{2} = 0.\end{aligned} \tag{38}$$

Applying $\left(B^w_{m,\lambda} g\right)(w,z)$ to (37), we obtain

$$\begin{aligned}\lim_{m\to\infty} m\Big[\left(B^w_{m,\lambda} g\right)(w,z) - g(w,z)\Big] &= g^{(1,0)}(w,z) \lim_{m\to\infty} m\Big(B^w_{m,\lambda}(s-w)\Big)(w,z) + \frac{g^{(2,0)}(w,z)}{2} \lim_{m\to\infty} m\Big(B^w_{m,\lambda}(s-w)\Big)(w,z) \\ &\quad + \lim_{m\to\infty} m\Big(B^w_{m,\lambda} r(s;w,z)(s-w)^2\Big)(w,z).\end{aligned} \tag{39}$$

By the Cauchy-Schwarz inequality, we have

$$\Big(B^w_{m,\lambda} r(s;w,z)(s-w)^2\Big)(w,z) \le \sqrt{\Big(B^w_{m,\lambda} r^2(s;w,z)\Big)(w,z)}\,\sqrt{\Big(B^w_{m,\lambda}(s-w)^4\Big)(w,z)}, \tag{40}$$

since $r^2(w;w,z) = 0$, therefore we can obtain

$$\lim_{m\to\infty} m\Big(B^w_{m,\lambda} r(s;w,z)(s-w)^2\Big)(w,z) = 0 \tag{41}$$

by (40) and (30). Finally, using (28), (29), (41), and (39), we get

$$\lim_{m\to\infty} m\Big[\left(B^w_{m,\lambda} g\right)(w,z) - g(w,z)\Big] = \frac{g^{(2,0)}(w,z)}{2}\Big[w(h-w-z)\Big],$$

and hence the proof is completed. □

Remark 5.6. *Let us consider $g(w,z)$ is bounded on $\mathcal{R}_h$. Then, for any $\lambda \in [-1,1]$, $w \in [0,h]$, and $z \in (0, h-w)$ at which the second order partial derivative of $g(w,z)$ with respect to z exists, we have*

$$\lim_{n\to\infty} n\Big[\left(B^z_{n,\lambda} g\right)(w,z) - g(w,z)\Big] = \frac{g^{(0,2)}(w,z)}{2}\Big[z(h-w-z)\Big].$$

6. Product operators

Let $P_{mn,\lambda} = B^w_{m,\lambda} B^z_{n,\lambda}$ and $Q_{nm,\lambda} = B^z_{n,\lambda} B^w_{m,\lambda}$ be the products of operators $B^w_{m,\lambda}$ and $B^z_{n,\lambda}$.
We have

$$\left(P_{mn,\lambda} g\right)(w,z) = \begin{cases} \sum_{i=0}^m \sum_{j=0}^n \tilde{p}_{m,i}(\lambda; w,z)\tilde{q}_{n,j}\left(\lambda; i\frac{(h-z)}{m}, z\right) g\left(i\frac{(h-z)}{m}, j\frac{(m-i)h+iz}{mn}\right), & (w,z) \in \mathcal{R}_h \setminus \{(0,h),(h,0)\}, \\ g(0,h), & (0,h) \in \mathcal{R}_h, \\ g(h,0), & (h,0) \in \mathcal{R}_h. \end{cases}$$

Remark 6.1. *The nodes of the operator $P_{mn,\lambda}$ are same as the nodes given in [10, Figure 2].*

Theorem 6.2. *The product operator $P_{mn,\lambda}$ satisfies the following relations:*

(i) $\left(P_{mn,\lambda} g\right)(w,0) = \left(B^w_{m,\lambda} g\right)(w,0)$,
(ii) $\left(P_{mn,\lambda} g\right)(0,z) = \left(B^z_{n,\lambda} g\right)(0,z)$,
(iii) $\left(P_{mn,\lambda} g\right)(w,h-w) = g(w,h-w), \quad w,z \in [0,h]$.

Above proofs follow from some simple computations. The property (i) or (ii) imply that $\left(P_{mn,\lambda} g\right)(0,0) = g(0,0)$.

Remark 6.3. *The product operator $P_{mn,\lambda}$ interpolates the function g at the vertex $(0,0)$ and on the hypotenuse $w+z=h$ of the triangle $\mathcal{R}_h$.*

The product operator

$$\left(Q_{nm,\lambda} g\right)(w,z) = \begin{cases} \sum_{i=0}^m \sum_{j=0}^n \tilde{p}_{m,i}\left(\lambda; w, j\frac{(h-w)}{n}\right)\tilde{q}_{n,j}(\lambda; w,z) g\left(i\frac{(n-j)h+jw}{mn}, j\frac{(h-w)}{n}\right), & (w,z) \in \mathcal{R}_h \setminus \{(0,h),(h,0)\}, \\ g(0,h), & (0,h) \in \mathcal{R}_h, \\ g(h,0), & (h,0) \in \mathcal{R}_h \end{cases}$$

has the nodes which are given in [10, Figure 3] and satisfies the following relations:

(i) $\left(Q_{nm,\lambda} g\right)(w,0) = \left(B^w_{m,\lambda} g\right)(w,0)$,
(ii) $\left(Q_{nm,\lambda} g\right)(0,z) = \left(B^z_{n,\lambda} g\right)(0,z)$,
(iii) $\left(Q_{nm,\lambda} g\right)(h-z,z) = g(h-z,z), \quad w,z \in [0,h]$.

The approximation formula for the operator $P_{mn,\lambda}$ is $g = P_{mn,\lambda} g + R^P_{mn,\lambda} g$.

Theorem 6.4. *If $g \in C(\mathcal{R}_h)$, then*

$$\left|\left(R^P_{mn,\lambda} g\right)(w,z)\right| \le \left(\frac{1}{\delta_1}\sqrt{\psi_m(w,z)} + \frac{1}{\delta_2}\sqrt{\psi_n(w,z)} + 1\right)\omega\left(g; \delta_1, \delta_2\right). \tag{42}$$

Proof. We have

$$\left|\left(R^P_{mn,\lambda} g\right)(w,z)\right| \le \left[\frac{1}{\delta_1}\sum_{i=0}^m \sum_{j=0}^n \tilde{p}_{m,i}(\lambda; w,z)\tilde{q}_{n,j}\left(\lambda; i\frac{(h-z)}{m}, z\right)\left|w - i\frac{(h-z)}{m}\right|\right.$$

$$+\frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\lambda;w,z)\tilde{q}_{n,j}\Big(\lambda;i\frac{(h-z)}{m},z\Big)\Big|z-j\frac{(m-i)h+iz}{mn}\Big|$$
$$+\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\lambda;w,z)\tilde{q}_{n,j}\Big(\lambda;i\frac{(h-z)}{m},z\Big)\Big]\omega\Big(g;\delta_1,\delta_2\Big).$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\lambda;w,z)\tilde{q}_{n,j}\Big(\lambda;i\frac{(h-z)}{m},z\Big)\Big|w-i\frac{(h-z)}{m}\Big| \leq \sqrt{\Big(B^{w}_{m,\lambda}(s-w)^2\Big)(w,z)},$$
$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\lambda;w,z)\tilde{q}_{n,j}\Big(\lambda;i\frac{(h-z)}{m},z\Big)\Big|z-j\frac{(m-i)h+iz}{mn}\Big| \leq \sqrt{\Big(B^{z}_{n,\lambda}(t-z)^2\Big)(w,z)},$$

while

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(\lambda;w,z)\tilde{q}_{n,j}\Big(\lambda;i\frac{(h-z)}{m},z\Big)=1.$$

It follows

$$\Big|\Big(R^{P}_{mn,\lambda}g\Big)(w,z)\Big| \leq \left(\frac{1}{\delta_1}\sqrt{\Big(B^{w}_{m,\lambda}(s-w)^2\Big)(w,z)}+\frac{1}{\delta_2}\sqrt{\Big(B^{z}_{n,\lambda}(t-z)^2\Big)(w,z)}\ +\ 1\right)\omega\Big(g;\delta_1,\delta_2\Big).$$

Since

$$\Big(B^{w}_{m,\lambda}(s-w)^2\Big)(w,z) \leq \psi_m(w,z), \quad \Big(B^{z}_{n,\lambda}(t-z)^2\Big)(w,z) \leq \psi_n(w,z),$$

we have

$$\Big|\Big(R^{P}_{mn,\lambda}g\Big)(w,z)\Big| \leq \left(\frac{1}{\delta_1}\sqrt{\psi_m(w,z)}+\frac{1}{\delta_2}\sqrt{\psi_n(w,z)}+1\right)\omega\Big(g;\delta_1,\delta_2\Big).$$

□

7. Boolean sum operators

Let

$$S_{mn,\lambda} := B^{w}_{m,\lambda}\oplus B^{z}_{n,\lambda} = B^{w}_{m,\lambda}+B^{z}_{n,\lambda}-B^{w}_{m,\lambda}B^{z}_{n,\lambda},$$
$$T_{nm,\lambda} := B^{z}_{n,\lambda}\oplus B^{w}_{m,\lambda} = B^{z}_{n,\lambda}+B^{w}_{m,\lambda}-B^{z}_{n,\lambda}B^{w}_{m,\lambda},$$

be the Boolean sums of the λ-Bernstein operators $B^{w}_{m,\lambda}$ and $B^{z}_{n,\lambda}$.

Theorem 7.1. *Let us consider a real valued function g defined on $\mathcal{R}_h$, then*

$$S_{mn,\lambda}g\Big|_{\partial\mathcal{R}_h} = g\Big|_{\partial\mathcal{R}_h}.$$

Proof. We have

$$S_{mn,\lambda}g=\Big(B^{w}_{m,\lambda}+B^{z}_{n,\lambda}-B^{w}_{m,\lambda}B^{z}_{n,\lambda}\Big)g.$$

The interpolation properties of $B^{w}_{m,\lambda}$, $B^{z}_{n,\lambda}$ together with the properties $(i)-(iii)$ of the operator $P_{mn,\lambda}$ imply that

$$\Big(S_{mn,\lambda}g\Big)(w,0)=\Big(B^{w}_{m,\lambda}g\Big)(w,0)+g(w,0)-\Big(B^{w}_{m,\lambda}g\Big)(w,0)=g(w,0),$$
$$\Big(S_{mn,\lambda}g\Big)(0,z)=g(0,z)-\Big(B^{z}_{n,\lambda}g\Big)(0,z)+\Big(B^{z}_{n,\lambda}g\Big)(0,z)=g(0,z),$$
$$\Big(S_{mn,\lambda}g\Big)(w,h-w)=g(w,h-w)+g(w,h-w)-g(w,h-w)=g(w,h-w), \quad \text{for all } w,z\in[0,h].$$

□

The approximation formula for the operator $S_{mn,\lambda}$ is given by $g = S_{mn,\lambda}g + R^S_{mn,\lambda}g$.

Theorem 7.2. *If $g \in C(\mathcal{R}_h)$, then*

$$\left|\left(R^S_{mn,\lambda}g\right)(w,z)\right| \leq \left(1 + \frac{1}{\delta_1}\sqrt{\psi_m(w,z)}\right)\omega\left(g(\cdot,z);\delta_1\right) + \left(1 + \frac{1}{\delta_2}\sqrt{\psi_n(w,z)}\right)\omega\left(g(w,\cdot);\delta_2\right)$$
$$+\left(\frac{1}{\delta_1}\sqrt{\psi_m(w,z)} + \frac{1}{\delta_2}\sqrt{\psi_n(w,z)} + 1\right)\omega\left(g;\delta_1,\delta_2\right), \tag{43}$$

for all $(w,z) \in \mathcal{R}_h$.

Proof. From the equality

$$g - S_{mn,\lambda}g = g - B^w_{m,\lambda}g + g - B^z_{n,\lambda}g - \left(g - P_{mn,\lambda}g\right),$$

we get

$$\left|\left(R^S_{mn,\lambda}g\right)(w,z)\right| \leq \left|\left(R^w_{m,\lambda}g\right)(w,z)\right| + \left|\left(R^z_{n,\lambda}g\right)(w,z)\right| + \left|\left(R^P_{mn,\lambda}g\right)(w,z)\right|.$$

Now, from (33, 34, 42), we follow the proof (43). □

Remark 7.3. *One can obtain the analogous relations for the remainders of the product approximation formula*

$$g = Q_{nm,\lambda}g + R^Q_{nm,\lambda}g = B^z_{n,\lambda}B^w_{m,\lambda}g + R^Q_{nm,\lambda}g,$$

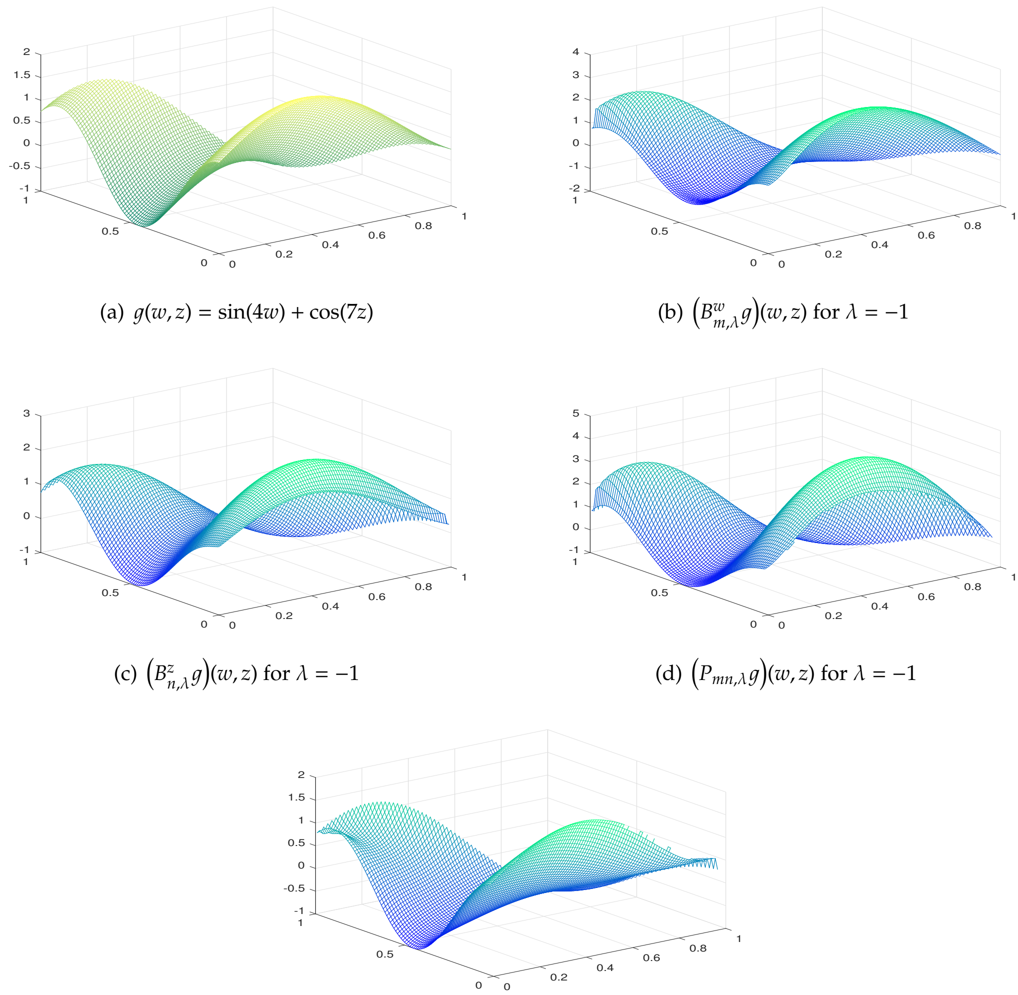
and for the Boolean sum formula

$$g = T_{nm,\lambda}g + R^T_{nm,\lambda}g = \left(B^z_{n,\lambda} \oplus B^w_{m,\lambda}\right)g + R^T_{nm,\lambda}g.$$

8. Graphical Analysis

Let us consider a function $g(w,z) = \sin(4w) + \cos(7z)$ for graphical analysis. In Figure 1(a), we have presented the graph of function g on the triangular domain. The graphs of λ-Bernstein operator $\left(B^w_{m,\lambda}g\right)(w,z)$ are shown in Figures 1(b), 2(a), 3(a), and the graphs of other operators $\left(B^z_{n,\lambda}g\right)(w,z)$, $\left(P_{mn,\lambda}g\right)(w,z)$, and $\left(S_{mn,\lambda}g\right)(w,z)$ that are approximating the function are shown in Figures 1(c) − 1(e), 2(b) − 2(d) and 3(b) − 3(d) for the values of $\lambda = -1$, $\lambda = -0.5$, and $\lambda = 0.5$, respectively. In all the figures, we have taken $m = 10$, $n = 10$ and $h = 1$. One can observe from the Figures 2(a), 2(b), 2(c), and 2(d) that the operators are approximating function better for some values of $\lambda \leq 0$ for fixed m and n.

Also from the remainder formulas, one can observe that the operators are approximating the function better with increasing values of m and n and by fixing λ on the triangular domain. Therefore, it can be concluded that the extra parameter λ is providing modeling flexibility for approximation on this triangular domain.



(a) $g(w,z) = \sin(4w) + \cos(7z)$

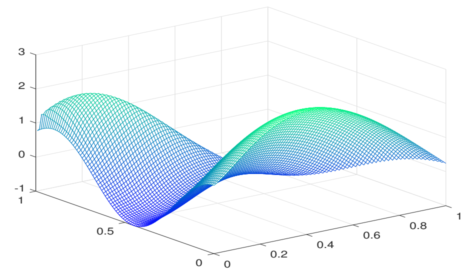
(b) $\left(B^{w}_{m,\lambda}g\right)(w,z)$ for $\lambda = -1$

(c) $\left(B^{z}_{n,\lambda}g\right)(w,z)$ for $\lambda = -1$

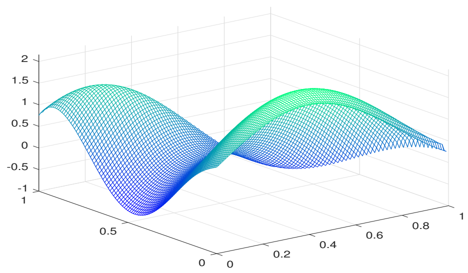
(d) $\left(P_{mn,\lambda}g\right)(w,z)$ for $\lambda = -1$

(e) $\left(S_{mn,\lambda}g\right)(w,z)$ for $\lambda = -1$

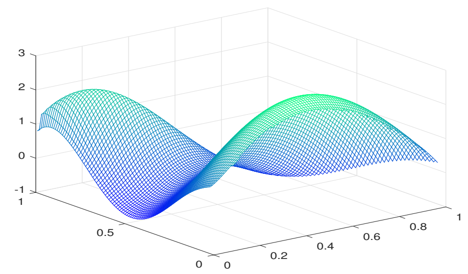
Figure 1: The graphs of the operators are approximating the graph of function for $\lambda = -1, h = 1, m = 10$, and $n = 10$.



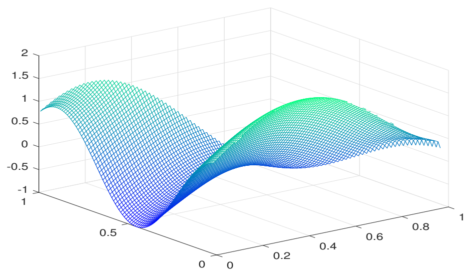
(a) $\left(B^{w}_{m,\lambda}g\right)(w,z)$ for $\lambda = -0.5$



(b) $\left(B^{z}_{n,\lambda}g\right)(w,z)$ for $\lambda = -0.5$

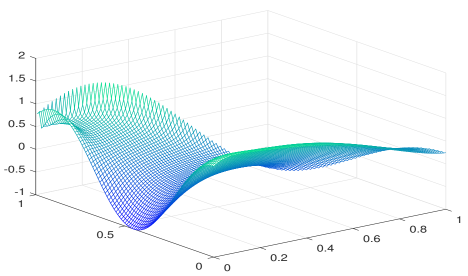


(c) $\left(P_{mn,\lambda}g\right)(w,z)$ for $\lambda = -0.5$

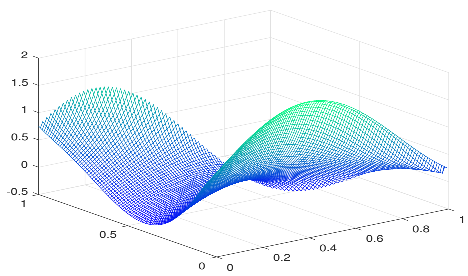


(d) $\left(S_{mn,\lambda}g\right)(w,z)$ for $\lambda = -0.5$

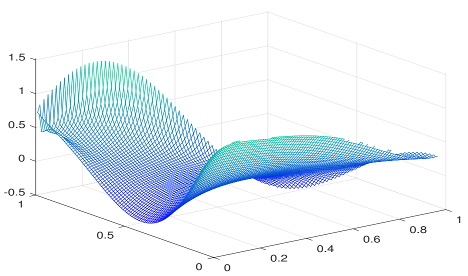
Figure 2: The graphs of the operators are approximating the graph of function for $\lambda = -0.5, h = 1, m = 10$, and $n = 10$.



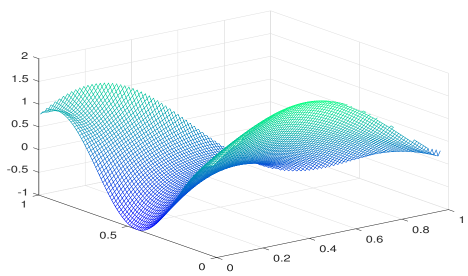
(a) $\left(B^{w}_{m,\lambda}g\right)(w,z)$ for $\lambda = 0.5$



(b) $\left(B^{z}_{n,\lambda}g\right)(w,z)$ for $\lambda = 0.5$



(c) $\left(P_{mn,\lambda}g\right)(w,z)$ for $\lambda = 0.5$



(d) $\left(S_{mn,\lambda}g\right)(w,z)$ for $\lambda = 0.5$

Figure 3: The graphs of the operators are approximating the graph of function for $\lambda = 0.5, h = 1, m = 10$, and $n = 10$.

Declarations:

Acknowledgments: None
Funding: None
Availability of data and materials: Not applicable.
Competing interests: The authors declare that they have no competing interests.
Author's contributions: All the authors contributed equally and they read and approved the final manuscript for publication.

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