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Some properties of extended eigenvalues for operators pair

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Abstract. In this paper, we determine some properties of extended eigenvalues for operators pair. Furthermore, the relationship between this kind of operators pair and the operators pencils in Hilbert space is established.

1. Introduction

The concepts of extended eigenvalues of operators and their extended eigenoperators have been investigated by such authors as H. Alkanjo [3] A. Biswas, A. Lambert and S. Petrovic [5] and M. T. Karaev [10]. This issue was particularly explored by S. Brown [7], R. L. Moore and C. M. Pearcy [11] who demonstrated that the extended eigenvalues of operators are related to the generalization of Lomonosov's theorem concerning the existence of a non trivial invariant subspace. If A is bounded linear operator, then a complex number λ is an extended eigenvalue of A, if there is nonzero operator X such that $AX = \lambda XA$, we denote $\sigma_{ext}(A)$ the set of extended eigenvalues of A. In [15], M. Rosenblum asserted that if $\sigma(A) \cap \sigma(B) = \emptyset$, then $X = 0$ stands for a unique solution of equation $AX - XB = 0$, which implies the following important inclusion

$$\sigma_{ext}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\},$$

where $\sigma(A)$ is the spectrum of A. Such inclusion will turn into an equality in the case of finite dimension (see [6]). In [5], A. Biswas, A. Lambert and S. Petrovic introduced this concept by depicting that the extended spectrum of operator A has a dense image corresponding to the point spectrum of another operator from A. In the same paper, the authors considered the Volterra operator defined on the Hilbert space with $L^2([0, 1])$ by

$$V(f(x)) = \int_0^t f(s)ds, \tag{1.1}$$

and they portrayed that $\sigma_{ext}(V) =]0, +\infty[$, by giving a non-trivial solution to the equation $AV = \lambda VA$. In [4], R. Bhatia and P. Rosenthal identified several different explicit forms of the solution of the operator equation $AX - XB = Y$, and they reported how these are useful in the perturbation theory. Furthermore, S. Shakrin in [16] highlighted that there exists a quasinilpotent compact operator in which its extended eigenvalue is

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equal to {1}, which makes it possible to classify this type of operators. In [2], A. Lambert demonstrated that the existence of extended eigenvalues leads to invariant or hyperinvariant subspaces. In [8], G. Cassier and H. Alkanjo introduced the notion of intertwining values of operators pair and they also gave the complete description of the intertwining set associated with the quasi-normal operator and the operator of form $A \otimes S$ such that S is the shift operator. In the same paper, G. Cassier and H. Alkanjo described the set of extended eigenvalues and extended eigenoperators of quasi-normal operators.

As far as our research work is concerned, we attempt to investigate the extended eigenvalues of operators pair in Hilbert space. Besides, having special interest in operators pencils, we will exhibit some links between this spectrum and other spectra in order to find a relationship between the operators pencils and the extended eigenvalues.

Our paper is organized as follows: In Section 2, we display the preliminary and the auxiliary properties that will be needed to prove the main results of the other sections. In Section 3, we establish the relationship between the extended eigenvalues and the operators pencils in case Hilbert space is in finite and infinite dimensions. Subsequently, we shall portray the structure of the set of extended eigenvalues of direct sum of pair operators defined in direct sum of Hilbert space. In addition, we enact a relation between extended eigenvalues of pair operators and their powers. In Section 4, we explore the simalirity application presented by R. Bhatia and P. Rosenthal within [4] in case we have the operators pencils.

2. Preliminary and auxiliary results

In this section, we will introduce some basic results which will be needed in the sequel. Let H be a Hilbert space, we denote by $\mathcal{L}(H)$ the set of all bounded linear operators on H. For $A \in \mathcal{L}(H)$, we use $N(A)$ to denote the null space, and A^* the adjoint of A such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for all } x, y \in H,$$

where $\langle ., .\rangle$ is inner product.

Definition 2.1. A set D in the complex plane which is called a Cauchy domain if the following conditions are satisfied:
(i) D is bounded and open.
(ii) D has a finite number of components, of which the closures of any two are disjoint.
(iii) The boundary of D is composed of a finite positive number of closed rectifiable Jordan curves of which any two are unable to intersect. ◊

Theorem 2.1. [15, Theorem 2.1] Let F be a closed and G be a bounded open subset of the complex plane with $F \subset G$, then there exists a Cauchy domain D such that $F \subset D$ and $\overline{D} \subset G$. ◊

Definition 2.2. Let $A \in \mathcal{L}(H)$. The point spectrum, the resolvent set and the spectrum of A are, respectively, defined as

$$\begin{aligned}
\sigma_p(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective}\},\\
\rho(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective and } (\lambda I - A)^{-1} \in \mathcal{L}(H)\},\\
\sigma(A) &= \mathbb{C}\backslash\rho(A).
\end{aligned}$$

Definition 2.3. Let $A, S \in \mathcal{L}(H)$ such that $S \neq A$ and $S \neq 0$, we define the S-point spectrum, the S-resolvent set and the S-spectrum of A, respectively, by

$$\begin{aligned}
\sigma_{p,S}(A) &= \{\lambda \in \mathbb{C} : \lambda S - A \text{ is not injective}\},\\
\rho_S(A) &= \{\lambda \in \mathbb{C} : \lambda S - A \text{ is injective and } (\lambda S - A)^{-1} \in \mathcal{L}(H)\},\\
\sigma_S(A) &= \mathbb{C}\backslash\rho_S(A).
\end{aligned}$$

Proposition 2.1. [9, Proposition 3.3.1]Let $A, S \in \mathcal{L}(H)$ such that $S \neq 0$ and $S \neq A$. Then, the S-resolvent set $\rho_S(A)$ is open. ◊

Definition 2.4. (*i*) Let $A \in \mathcal{L}(H)$. We denote by $F(A)$ the family of all functions, which are analytic on some neighborhood of $\sigma(A)$.

(*ii*) Let $f \in F(A)$ and $\mathcal{U}$ be an open subset of $\mathbb{C}$, whose boundary $\partial\mathcal{U}$ consists of a finite number of rectifiable Jordan curves. We assume that $\partial\mathcal{U}$ is oriented, such that

$$\int_{\partial\mathcal{U}} (\lambda - \mu)^{-1} d\lambda = \begin{cases} 2\pi i, & \text{if } \mu \in \mathcal{U}; \\ 0, & \text{if } \mu \notin (\mathcal{U} \cup \partial\mathcal{U}). \end{cases}$$

Suppose that $\sigma_S(A) \subseteq \mathcal{U}$. Then, for all $S \in \mathcal{L}(H)$ the bounded operator $f(A)$ is defined by

$$f(A) = \frac{1}{2\pi i} \int_{\partial\mathcal{U}} f(\lambda)(\lambda S - A) d\lambda. \qquad \diamond$$

Definition 2.5. Let $A, B \in \mathcal{L}(H)$. We say that operators A and B are similar if there exists an isomorphism P such that $A = P^{-1}BP$. ◊

Definition 2.6. Let $A \in \mathcal{L}(H)$. Then,

(*i*) An operator A is self-adjoint if $A^* = A$.

(*ii*) A is called normal operator if $A^*A = A^*A$.

(*iii*) A is called quasi-normal operator if $A(A^*A) = (A^*A)A$. ◊

Now, we recall the definition introduced by G. Cassier and H. Alkanjo in [8], and some properties of extended spectrum of operators pair in Hilbert space H.

Definition 2.7. (*i*) Let $A, B \in \mathcal{L}(H)$. We say that $\lambda \in \mathbb{C}$ is an extended eigenvalue of operators pair (A, B), if there exists a non-zero operator $C \in \mathcal{L}(H)$ such that

$$AC = \lambda CB.$$

(*ii*) The set $\sigma_{ext}(A, B) = \left\{\lambda \in \mathbb{C} : \exists C \neq 0, AC = \lambda CB\right\}$ is called the set of extended eigenvalues of operators pair (A, B).

(*iii*) The subspace generated by extended eigenopertors corresponding to λ will be denoted by $E_{ext}((A, B), \lambda)$. ◊

Remark 2.1. Let $A \in \mathcal{L}(H)$, then

(*i*) $\lambda \in \sigma_{ext}\left(A, \frac{A}{\lambda}\right)$ for all $\lambda \in \mathbb{C}^*$.

(*ii*) $\sigma_{ext}(A, A) = \sigma_{ext}(A)$. For example, if $H = L^2([0, 1])$ and V be the Volterra operator defined in Eq. (1.1). Then,

$$\sigma_{ext}(V, V) = \sigma_{ext}(V) = [0, +\infty[. \qquad \diamond$$

Proposition 2.2. Let $A, B, S \in \mathcal{L}(H)$. Then,

(*i*) If $0 \in \rho(A) \cap \rho(B)$, then for $\lambda \in \mathbb{C}^*$, we have

$$\lambda \in \sigma_{ext}(A, B) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, B^{-1}).$$

(*ii*) If $0 \in \rho(S)$, then $\sigma_{ext}(S^{-1}A, I) = \sigma_{p,S}(A)$.

(*iii*) If S commutes with A and $\lambda \in \sigma_{ext}(SA, SA)$, then

$$\lambda^n \in \sigma_{ext}((SA)^n, (SA)^n) \text{ for all } n \in \mathbb{N}^*. \qquad \diamond$$

Proof. (*i*) Let $\lambda \in \sigma_{ext}(A, B)$, then there exists $C \neq 0$ such that $AC = \lambda CB$, which implies that $CB^{-1} = \lambda A^{-1}C$. Therefore, $A^{-1}C = \frac{1}{\lambda}CB^{-1}$. We infer that

$$\frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, B^{-1}).$$

Conversely, if $\frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, B^{-1})$, then there exists $C \neq 0$ such that $A^{-1}C = \frac{1}{\lambda}CB^{-1}$. We infer that $CB^{-1} = \lambda A^{-1}C$, which implies that $ACB^{-1} = \lambda C$. Therefore,

$$AC = \lambda CB.$$

Hence, $\lambda \in \sigma_{ext}(A, B)$.

(*ii*) Let $\lambda \in \sigma_{ext}(S^{-1}A, I)$, then there exists $C \neq 0$ such that $S^{-1}AC = \lambda C$. This implies that $AC = \lambda SC$. Thus,

$$\lambda \in \sigma_{p,S}(A).$$

Conversely, let us assume that $\lambda \in \sigma_{p,S}(A)$, then there exists $C \neq 0$ such that $AC = \lambda SC$. As a matter of fact, $S^{-1}AC = \lambda C$. This implies that

$$\lambda \in \sigma_{ext}(S^{-1}A, I).$$

(*iii*) Proceeding by induction, the case $n = 1$ is obvious. For this reason, we focus on the inductive step. Therefore,

$$S^{n+1}A^nC = \lambda^n SCS^nA^n.$$

This implies that

$$S^{n+1}A^{n+1}C = \lambda^n SACS^nA^n.$$

We can deduce that

$$(SA)^{n+1}C = \lambda^{n+1}C(SA)^{n+1}.$$

Therefore, the property holds for $n + 1$, and we infer that

$$\lambda^n \in \sigma_{ext}((SA)^n, (SA)^n). \qquad \square$$

Remark 2.2. It follows immediately, from Proposition 2.2 that

(*i*) If $B = A^{-1}$, then for $\lambda \in \mathbb{C}^*$, we get

$$\lambda \in \sigma_{ext}(A, A^{-1}) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, A).$$

(*ii*) If $S = I$, then $\sigma_{ext}(A, I) = \sigma_p(A)$.

(*iii*) For $S = I$, we have

$$if \lambda \in \sigma_{ext}(A, A), \text{ then } \lambda^n \in \sigma_{ext}(A^n, A^n). \qquad \diamond$$

3. Main results

The major purpose of this section is to characterize the relationship between extended spectrum of operators pairs and operators pencils.

Theorem 3.1. Let $E, F, C,$ and $S \in \mathcal{L}(H)$ such that $SC = CS$. Then, $\sigma_S(SE) \cap \sigma_S(SF) = \emptyset$ if, and only if, $C = 0$ is the only solution of the operator equation $SEC - CSF = 0$. $\diamond$

Proof. We will consider the following operator equation

$$SEC - CSF = Q, \tag{3.1}$$

with $Q \in \mathcal{L}(H)$. According to Theorem 2.1, we infer that there exists a Cauchy domain D such that

$$\sigma_S(SF) \subset D \text{ and } \sigma_S(SE) \cap \overline{D} = \emptyset.$$

Now, we suppose that C is a solution of Eq. (3.1) and $z \in \partial D$. Then,

$$SEC - CSF = (SE - zS)C + C(zS - SF). \tag{3.2}$$

Hence, by using Eqs. (3.1) and (3.2), we deduce that

$$(SE - zS)C + C(zS - SF) = Q.$$

Since $z \in \rho_S(SE) \cap \rho_S(SF)$, then

$$(SE - zS)^{-1}C + C(zS - SF)^{-1} = (SE - zS)^{-1}Q(zS - SF)^{-1}. \tag{3.3}$$

Thus, it follows from the properties of Dunford's functional calculus [9] that

$$\frac{1}{2\pi i}\int_{\partial D}(SE - zS)^{-1}Cdz = 0,$$

and

$$\frac{1}{2\pi i}\int_{\partial D}C(zS - SF)^{-1}dz = C.$$

Finally, according to Eq.(3.3), we conclude that

$$C = \frac{1}{2\pi i}\int_{\partial D}(SE - zS)^{-1}Q(zS - SF)^{-1}dz.$$

Assuming that $Q = 0$, we get the result.
By replacing E and F by A and λB, respectively, we get the following proposition. □

Proposition 3.1. Let $A, B, S \in \mathcal{L}(H)$. Then,

$$\sigma_{ext}(SA, SB) \subset \Big\{\lambda \in \mathbb{C} : \sigma_S(SA) \cap \sigma_S(\lambda SB) \neq \emptyset\Big\}.$$ ◊

Remark 3.1. It follows immediately, from Proposition 3.1 that for $S = I$

$$\sigma_{ext}(A, B) \subset \Big\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \emptyset\Big\}.$$

In particular, $\sigma_{ext}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\}$ (see [15]). ◊

In the finite dimension, we get the following result.

Theorem 3.2. Let $A, B, S \in \mathcal{L}(H)$ such that S is self-adjoint operator. Then,

$$\sigma_{ext}(SA, SB) = \Big\{\lambda \in \mathbb{C} : \sigma_S(SA) \cap \sigma_S(\lambda SB) \neq \emptyset\Big\}.$$ ◊

Proof. We shall discuss these two cases:
<u>1^{st} case:</u> if SA and SB are not invertible, then $N(SA) \neq 0$ and $N((SB)^*) \neq 0$. Considering on the one hand the non-zero operator T defined by

$$\begin{array}{rcl} T : N((SB)^*) & \rightarrow & N(SA) \\ P & \rightarrow & TP, \end{array}$$

where P is the orthogonal projection on H onto $N((SB)^*)$. Defining on the other hand the operator $K = TP \neq 0$, implies that

$$SAK = \lambda KSB = 0 \text{ for all } \lambda \in \mathbb{C}.$$

We infer that $\sigma_{ext}(SA, SB) = \mathbb{C}$. Since SA and SB are not invertible, then $0 \in \sigma_S(SA) \cap \sigma_S(\lambda SB)$ for all $\lambda \in \mathbb{C}$. This implies that

$$\sigma_{ext}(SA, SB) = \Big\{\lambda \in \mathbb{C} : \sigma_S(SA) \cap \sigma_S(\lambda SB) \neq \emptyset\Big\} = \mathbb{C}.$$

<u>2^{nd} case:</u> if SA and SB are invertible, then $0 \notin \sigma_S(SA)$ and $0 \notin \sigma_S(\lambda SB)$. We shall demonstrate that

$$\Big\{\lambda \in \mathbb{C} : \sigma_S(SA) \cap \sigma_S(\lambda SB) \neq \emptyset\Big\} \subset \sigma_{ext}(SA, SB).$$

Let $\beta \in \sigma_S(SA)$, then there exists a vector x such that $SAx = \beta Sx$. On the other side, $\beta \in \sigma_S(\lambda SB)$, then $\frac{\beta}{\lambda} \in \sigma_S(SB)$ for all $\lambda \neq 0$, which provides $\overline{\frac{\beta}{\lambda}} \in \sigma_S((SB)^*)$. This implies that $(SB)^*y = \overline{\frac{\beta}{\lambda}} Sy$. We consider the operator $C = (x \otimes y)z = \langle z, y\rangle x$ for all $z \in H$, then we can easily verify that $SAC = \lambda CSB$. Indeed,

$$\begin{array}{rcl} SAC & = & \langle z, y\rangle SAx \\ & = & \langle z, y\rangle \beta Sx \\ & = & \beta S\langle z, y\rangle x. \end{array}$$

We can conclude that $SAC = \beta SC$, and

$$\begin{array}{rcl} CSB & = & \langle z, (SB)^* y\rangle x \\ & = & \frac{\beta}{\lambda}\langle Sz, y\rangle x. \end{array}$$

We infer that $CSB = \frac{\beta}{\lambda} CS$. Thus, $SAC = \lambda CSB$, which yields that

$$\lambda \in \sigma_{ext}(SA, SB).$$

□

Remark 3.2. It follows immediately, from Theorem 3.2 that for $S = I$, we have

$$\sigma_{ext}(A, B) = \Big\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \emptyset\Big\}.$$

In particular, $\sigma_{ext}(A) = \Big\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\Big\}$ (see [6]). ◊

Corollary 3.1. Let $n \geq 1$ and H_n be a Hilbert space, such that $H = \bigoplus_{n=1}^{\infty} H_n$. Let $A_n, B_n, S_n \in \mathcal{L}(H_n)$ such that $A = \bigoplus_{n=1}^{\infty} A_n$, $B = \bigoplus_{n=1}^{\infty} B_n$, $S = \bigoplus_{n=1}^{\infty} S_n$. If H is finite dimensional, then

$$\sigma_{ext}(SA, SB) = \bigcup_{n,m=1}^{\infty} \Big\{\lambda \in \mathbb{C} : \sigma(S_n A_n) \cap \sigma(\lambda S_m B_m) \neq \emptyset\Big\}.$$ ◊

Theorem 3.3. Let $n \geq 1$ and H_n be a Hilbert space, such that $H = \bigoplus_{n=1}^{\infty} H_n$. Let $A_n, B_n, S_n \in \mathcal{L}(H_n)$ such that $A = \bigoplus_{n=1}^{\infty} A_n$, $B = \bigoplus_{n=1}^{\infty} B_n$, $S = \bigoplus_{n=1}^{\infty} S_n$. Then,

$$\bigcup_{n=1}^{\infty} \sigma_{ext}(S_nA_n, S_nB_n) \subset \sigma_{ext}(SA, SB). \qquad \diamond$$

Proof. We assume that $\lambda \in \bigcup_{n=1}^{\infty} \sigma_{ext}(S_nA_n, S_nB_n)$, then there exists a natural number n_λ and a nonzero operator $C_{n_\lambda} \in \mathcal{L}(H)$ such that

$$S_{n_\lambda}A_{n_\lambda}C_{n_\lambda} = \lambda C_{n_\lambda}S_{n_\lambda}B_{n_\lambda}.$$

We consider the operator $C_\lambda : H \to H$ defined by

$$C_\lambda = \Big\{0, ..., 0, C_{n_\lambda}, 0, ...\Big\}, \tag{3.4}$$

where operator C_{n_λ} is placed in n_λth index. Therefore,

$$SAC_\lambda = \Big\{0, ..., 0, \lambda C_{n_\lambda}S_{n_\lambda}B_{n_\lambda}, 0, ...\Big\}.$$

We deduce that $SAC_\lambda = \lambda C_\lambda SB$, where $C_\lambda \in \mathcal{L}(H)$ and $C_\lambda \neq 0$. We infer that $\lambda \in \sigma_{ext}(SA, SB)$. Then,

$$\bigcup_{n=1}^{\infty} \sigma_{ext}(S_nA_n, S_nB_n) \subset \sigma_{ext}(SA, SB). \qquad \square$$

Corollary 3.2. It follows immediately, from Theorem 3.3 that for $S = I$, we have

$$\bigcup_{n=1}^{\infty} \sigma_{ext}(A_n, B_n) \subset \sigma_{ext}(A, B).$$

In particular, $\bigcup_{n=1}^{\infty} \sigma_{ext}(A_n) \subset \sigma_{ext}(A)$. $\diamond$

Remark 3.3. Let $n \geq 1$ and H_n be a Hilbert space, such that $H = \bigoplus_{n=1}^{\infty} H_n$. Let $A_n, B_n, S_n \in \mathcal{L}(H_n)$ such that $A = \bigoplus_{n=1}^{\infty} A_n$, $B = \bigoplus_{n=1}^{\infty} B_n$, $S = \bigoplus_{n=1}^{\infty} S_n$. Then,

$$\sigma_{ext}(SA, SB) \neq \bigcup_{n=1}^{\infty} \sigma_{ext}(S_nA_n, S_nB_n).$$

Indeed, let H_1 and H_2 be any Hilbert spaces such that $H = \mathbb{R}^2 \oplus i\mathbb{R}^2$. We define the following bounded linear matrices operators on $\mathbb{C}^4$ by :

$$A_1 = \begin{pmatrix} -7 & 6 \\ 5 & -6 \end{pmatrix}, B_1 = \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}, S_1 = \begin{pmatrix} -7 & -6 \\ -5 & -8 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} i & -2i \\ -i & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 2i & -4i \\ -2i & 0 \end{pmatrix}, S_2 = \begin{pmatrix} -2i & i \\ -i & 0 \end{pmatrix},$$

where $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ and $S = S_1 \oplus S_2$. After a simple computation, we obtain $\sigma(S_1A_1) = \{24, 13\}$, $\sigma(S_2B_2) = \{4, -2\}$, $\sigma(S_2A_2) = \{2, -1\}$, $\sigma(S_1B_1) = \{13, 2\}$. In this case, we get

$$\begin{aligned}
\{\lambda \in \mathbb{C} : \sigma(S_1A_1) \cap \sigma(\lambda S_1B_1) \neq \emptyset\} &= \{\frac{24}{13}, \frac{13}{2}, 1, 12\} \\
\{\lambda \in \mathbb{C} : \sigma(S_2A_2) \cap \sigma(\lambda S_2B_2) \neq \emptyset\} &= \{\frac{1}{2}, \frac{-1}{4}, -1\} \\
\{\lambda \in \mathbb{C} : \sigma(S_2A_2) \cap \sigma(\lambda S_1B_1) \neq \emptyset\} &= \{\frac{2}{13}, \frac{-1}{13}, \frac{-1}{2}, 1\} \\
\{\lambda \in \mathbb{C} : \sigma(S_1A_1) \cap \sigma(\lambda S_2B_2) \neq \emptyset\} &= \{\frac{13}{4}, \frac{-13}{2}, -12, 6\}.
\end{aligned}$$

Based on Corollary 3.1, we get
$\sigma_{ext}(SA, SB) = \{1, -1, -12, 6, 12, \frac{13}{2}, \frac{24}{13}, \frac{-1}{4}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{13} \frac{2}{13}, \frac{-13}{2}, \frac{13}{4}\}$.
On the other side, we have
$\sigma_{ext}(S_1A_1, S_1B_1) \bigcup \sigma_{ext}(S_2A_2, S_2B_2) = \{1, 12, -1, \frac{24}{13}, \frac{13}{2}, \frac{1}{2}, \frac{-1}{4}\}$.
This implies that

$$\sigma_{ext}(SA, SB) \neq \bigcup_{n=1}^{2} \sigma_{ext}(S_nA_n, S_nB_n). \quad \diamond$$

Proposition 3.2. Let $n \geq 1$ and H_n be a Hilbert space, such that $H = \bigoplus_{n=1}^{\infty} H_n$. Let $A_n, B_n \in \mathcal{L}(H_n)$ such that $A = \bigoplus_{n=1}^{\infty} A_n$, $B = \bigoplus_{n=1}^{\infty} B_n$. If $0 \in \rho(A) \cap \rho(B)$ and $\lambda \in \bigcup_{n=1}^{\infty} \sigma_{ext}(A_n, B_n)$, then

$$\frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, B^{-1}). \quad \diamond$$

Proof. If we assume that $\lambda \in \bigcup_{n=1}^{\infty} \sigma_{ext}(A_n, B_n)$, then there exists natural numbers n_λ and $C_{n_\lambda} \neq 0$ such that

$$A_{n_\lambda} C_{n_\lambda} = \lambda C_{n_\lambda} B_{n_\lambda}.$$

We consider the operator C_λ defined in Eq.(**??**). This implies that $AC_\lambda = \lambda C_\lambda B$. We infer that $C_\lambda B = \frac{1}{\lambda} AC_\lambda$. Hence,

$$A^{-1} C_\lambda B = \frac{1}{\lambda} C_\lambda.$$

We conclude that

$$A^{-1} C_\lambda = \frac{1}{\lambda} C_\lambda B^{-1}.$$

From this perspective, $\frac{1}{\lambda} \in \sigma_{ext}(A^{-1}, B^{-1})$. □

Theorem 3.4. Let A, B and S be invertible bounded linear operators. Then,

$$\sigma_{ext}(SA, SB) \subset \left\{ \lambda \in \mathbb{C} : \quad \frac{1}{\|A^{-1}S^{-1}\| \|SB\|} \leq |\lambda| \right\}. \quad \diamond$$

Proof. Since A and S are invertible, then SA is invertible. So, it exists a constant ($\gamma = \|A^{-1}S^{-1}\|^{-1} > 0$). Therefore,

$$\gamma \|x\| \leq \|SAx\| \text{ for all } x \in H.$$

Let $\lambda \in \sigma_{ext}(SA, SB)$. Then, there exists $C \neq 0$ such that $SAC = \lambda CSB$. We infer that

$$\begin{aligned} |\gamma|\|Cx\| &\leq \|SACx\| \\ &= |\lambda|\|CSBx\| \\ &\leq |\lambda|\|C\|\|SBx\|. \end{aligned}$$

This implies that

$$\begin{aligned} \gamma\|C\| &\leq |\lambda|\|C\|\|SB\| \\ \frac{\gamma}{\|SB\|} &\leq |\lambda| \\ \frac{1}{\|A^{-1}S^{-1}\|\|SB\|} &\leq |\lambda|. \end{aligned}$$

Now, we can conclude that $\dfrac{1}{\|A^{-1}S^{-1}\|\|SB\|} \leq |\lambda|$. □

Corollary 3.3. It follows immediately, from Theorem 3.4 that for $S = I$, we have

$$\sigma_{ext}(A, B) \subset \left\{ \lambda \in \mathbb{C} : \quad \frac{1}{\|A^{-1}\|\|B\|} \leq |\lambda| \right\}.$$

In particular, $\sigma_{ext}(A) \subset \left\{ \lambda \in \mathbb{C} : \quad \frac{1}{\|A^{-1}\|\|A\|} \leq |\lambda| \right\}$.

Theorem 3.5. Let A, B and $S \in \mathcal{L}(H)$. Then,

(*i*) If $0 \in \sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*)$, then $\sigma_{ext}(SA, SB) = \mathbb{C}$.

(*ii*) Let $0 \in \rho(S)$. If $\sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*) \neq \emptyset$, then for all $\lambda \in \mathbb{C}^*$

$$\left\{ \frac{\lambda}{\overline{\lambda}} : \lambda \in \sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*) \neq \emptyset \right\} \subset \sigma_{ext}\Big(S^{-1}A, BS^{-1}\Big). \qquad \diamond$$

Proof. (*i*) Let $0 \in \sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*)$, then there exists $x, y \in H\backslash\{0\}$ such that

$$Ax = B^*y = 0. \tag{3.5}$$

We define the operator $C = (x \otimes y)z = \langle z, y\rangle x$ for all $z \in H$. By using Eq.(3.5), we get

$$AC = \langle z, y\rangle Ax = 0 \text{ and } CB = \langle z, B^*y\rangle x = 0.$$

We infer that

$$SAC = \lambda CBS = 0, \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

(*ii*) Let $\sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*) \neq \emptyset$, then there exists $\lambda \in \mathbb{C}^*$ such that

$$\lambda \in \sigma_{p,S}(A) \cap \sigma_{p,S^*}(B^*).$$

This implies that

$$Ax = \lambda Sx \text{ and } B^*y = \lambda S^*y \text{ for all } \quad x, y \in \mathcal{L}(H)\backslash\{0\}.$$

We define the operator $C = x \otimes y$ by

$$(x \otimes y)z = \langle z, y\rangle x \text{ for all } z \in H.$$

Then,

$$AC = \lambda SC \text{ and } CB = \overline{\lambda}CS.$$

Indeed,

$$\begin{aligned} AC &= \langle z, y\rangle Ax \\ &= \langle z, y\rangle \lambda Sx \\ &= \lambda SC, \end{aligned}$$

and

$$\begin{aligned} CB &= \langle z, B^* y\rangle x \\ &= \overline{\lambda}\langle Sz, y\rangle x \\ &= \overline{\lambda} CS. \end{aligned}$$

We can deduce that $S^{-1}AC = \dfrac{\lambda}{\overline{\lambda}} CBS^{-1}$. □

Theorem 3.6. Let $A, B \in \mathcal{L}(H)$. If A^* is a quasi-normal operator and A is not normal such that $0 \notin \sigma_p(A^*) \cup \sigma_p(B)$, then

$$\left\{ \frac{\lambda_i}{\overline{\lambda_j}} \in \mathbb{C} : \lambda_i \in \sigma_p(A^*), \lambda_j \in \sigma_p(B) \right\} \cup \left\{0\right\} \subset \sigma_{ext}\Big(A^*, B^*\Big). \qquad \diamond$$

Proof. The proof of Theorem 3.6 is checked in a similar way to that in [13, Theorem 2.2] □

Proposition 3.3. Let $n \geq 1$ and H_n be a Hilbert space, such that $H = \bigoplus_{n=1}^{\infty} H_n$. Let $A_n, B_n \in \mathcal{L}(H_n)$ such that $A = \bigoplus_{n=1}^{\infty} A_n$, $B = \bigoplus_{n=1}^{\infty} B_n$. If $A_m = 0$ for some $m \in \mathbb{N}$ and $B_k = 0$ for some $k \in \mathbb{N}$. Then,

$$\sigma_{ext}(A, B) = \mathbb{C}. \qquad \diamond$$

Proof. The proof may be achieved in a similar way as [14, Theorem 2.4]. □

Example 3.1. Let H be a Hilbert space and let A and B be normal compact operators. If we assume that $H = l^2(\mathbb{N})$ and for any $x = (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$, then

$$A(x_n) = \Big\{0, u_2x_2, 0, u_4x_4, ..., 0, u_{2n}x_{2n}, ...\Big\},$$

$$B(x_n) = \Big\{0, v_2x_2, 0, v_4x_4, ..., 0, v_{2n}x_{2n}, ...\Big\},$$

where (u_n) and (v_n) are sequences of complex numbers such that

$$\lim_{n\to\infty} u_n = 0 \ \text{ and } \ \lim_{n\to\infty} v_n = 0.$$

As a matter of fact, A and B are compact operators in $l^2(\mathbb{N})$. Defining the following operator $C : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ by $C(x_n) = \Big\{x_1, 0, x_3, 0, ..., 0, x_{2n+1}, 0, ...\Big\}$, we get

$$AC = \lambda CB = 0 \ \text{ for all } \ \lambda \in \mathbb{C}.$$

Consequently,

$$\sigma_{ext}(A, B) = \mathbb{C}. \qquad \diamond$$

Theorem 3.7. Let A_1, B_1, A_2 and $B_2 \in \mathcal{L}(H)$, then

$$\sigma_{ext}(A_1, B_1)\sigma_{ext}(A_2, B_2) \subset \sigma_{ext}(A_1 \otimes A_2, B_1 \otimes B_2),$$

where $A_1 \otimes A_2$, $B_1 \otimes B_2$ are the tensor products of A_1 and A_2, B_1 and B_2, respectively. ◊

Proof. Let $\lambda_i \in \sigma_{ext}(A_i, B_i)$ and $X_i \in E_{ext}(A_i, B_i, \lambda_i) \setminus \{0\}$ with $i = 1, 2$. If $\lambda_1 \in \sigma_{ext}(A_1, B_1)$, then there exists $X_1 \neq 0$ such that

$$A_1X_1 = \lambda_1 X_1 B_1,$$

and if $\lambda_2 \in \sigma_{ext}(A_2, B_2)$, then there exists $X_2 \neq 0$ such that

$$A_2X_2 = \lambda_2 X_2 B_2.$$

We consider $X = X_1 \otimes X_2$, then X is a non zero operator in $E_{ext}(A_1 \otimes A_2, B_1 \otimes B_2, \lambda_1\lambda_2)$ which gives

$$(A_1 \otimes A_2)X = \lambda_1\lambda_2 X(B_1 \otimes B_2).$$

This implies

$$\lambda_1\lambda_2 \in \sigma_{ext}(A_1 \otimes A_2, B_1 \otimes B_2).$$ □

Remark 3.4. Let A_1, B_1, A_2 and $B_2 \in \mathcal{L}(H)$, then

$$\sigma_{ext}(A_1 \otimes A_2, B_1 \otimes B_2) \neq \sigma_{ext}(A_1, B_1)\sigma_{ext}(A_2, B_2).$$

In fact, if we define the following matrices operators

$$A_1 = \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -4 & 3 \\ 2 & -5 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}.$$

After a simple computation, we get

$$\sigma(A_1) = \{-2, -5\}, \sigma(B_1) = \{3, -1\}, \sigma(A_2) = \{-2, -7\}, \sigma(B_2) = \{-5, 2\}.$$

In this case, we obtain

$$\begin{aligned} \{\lambda \in \mathbb{C} : \sigma(A_1) \cap \sigma(\lambda B_1) \neq \emptyset\} &= \{2, 5, \frac{-2}{3}, \frac{-5}{3}\}, \\ \{\lambda \in \mathbb{C} : \sigma(A_2) \cap \sigma(\lambda B_2) \neq \emptyset\} &= \{-1, \frac{-7}{2}, \frac{7}{5}, \frac{2}{5}\}, \end{aligned}$$

which gives
$\sigma_{ext}(A_1, B_1)\sigma_{ext}(A_2, B_2) = \{-5, -2, -7, 7, 2, \frac{2}{3}, -\frac{5}{3}, \frac{-35}{2}, \frac{14}{6}, \frac{35}{6}, \frac{14}{5}, \frac{-14}{15}, \frac{-35}{15}, \frac{4}{5}, \frac{-4}{15}, \frac{-2}{3}\}$, and
$\sigma_{ext}(A_1 \otimes A_2, B_1 \otimes B_2) = \{-7, 7, 2, -5, -2, 6, \frac{-7}{3}, \frac{2}{3}, \frac{4}{5}, \frac{-4}{15}, \frac{5}{3}, \frac{-2}{3}, \frac{-35}{2}, \frac{-14}{15}, \frac{7}{3}, \frac{14}{5}\}$. Then,

$$\sigma_{ext}(A_1, B_1)\sigma_{ext}(A_2, B_2) \neq \sigma_{ext}(A_1 \otimes A_2, B_1 \otimes B_2).$$ ◊

Theorem 3.8. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then, λ is an extended eigenvalue of $((SA)^n, (SB)^n)$ if, and only if, there exists a complex number $\mu \in \sigma_{ext}(SA, SB)$ with the property $\mu^n = \lambda$, for $n \in \mathbb{N}$. ◊

To prove Theorem 3.8, we need the following lemma.

Lemma 3.1. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and A, B and S be bounded linear operators on Hilbert space H. We assume that μ is any nth root of λ and ξ be the primitive nth root of 1 (we use the term primitive nth root of 1 to denote $e^{2\pi i/n}$). Then,

$$(SA)^n C - \lambda C(SB)^n = F_{\mu\xi^{n-1}} F_{\mu\xi^{n-2}} ... F_{\mu\xi} F_\mu(C),$$

where F_λ is defined by

$$\begin{array}{rcl} F_\lambda : \mathcal{L}(H) & \rightarrow & \mathcal{L}(H) \\ C & \rightarrow & SAC - \lambda CSB. \end{array}$$

Proof. For any non negative integer k, there exist complex numbers defined by $\beta_{k_0}, \beta_{k_1}, ..., \beta_k, \beta_{k+1}$ such that

$$F_{\mu\xi^k} F_{\mu\xi^{k-1}} ... F_{\mu\xi} F_\mu(C) = \sum_{i=0}^{k+1} \beta_{ki} (SA)^{k+1-i} C(SB)^i.$$

Proceeding by induction, the case $k = 0$ is obvious, then we move on to verify the next step. Hence,

$$\begin{array}{rcl} F_{\mu\xi^{k+1}} F_{\mu\xi^k} ... F_{\mu\xi} F_\mu(C) & = & SAF_{\mu\xi^k} F_{\mu\xi^{k-1}} ... F_{\mu\xi} F_\mu(C) \\ & - & \mu\xi^{k+1} F_{\mu\xi^k} F_{\mu\xi^{k-1}} ... F_{\mu\xi} F_\mu(C) SB \\ & = & \displaystyle\sum_{i=0}^{k+1} \beta_{ki} (SA)^{k+2-i} C(SB)^i \\ & - & \mu\xi^{k+1} \displaystyle\sum_{i=0}^{k+1} \beta_{ki} (SA)^{k+1-i} C(SB)^{i+1}. \end{array}$$

By using a simple change of the index of summation, we draw the result. □

Proof of Theorem 3.8 It is obvious that if $\mu^n = \lambda$ and $SAC = \mu CSB$, then

$$(SA)^n C = \lambda C(SB)^n.$$

Thus, we shall focus upon the converse. If $\lambda \in \sigma_{ext}((SA)^n, (SB)^n)$, then there exists $C \neq 0$ such that

$$(SA)^n C = \lambda C(SB)^n$$

and let μ satisfy $\mu^n = \lambda$, $\xi = e^{2\pi i/n}$. By using Lemma 3.1, we get

$$(SA)^n C - \lambda C(SB)^n = 0.$$

This implies

$$F_{\mu\xi^{n-1}} F_{\mu\xi^{n-2}} ... F_{\mu\xi} F_\mu(C) = 0.$$

We have two cases:
<u>1^{st} case:</u> if $F_\mu(C) = 0$ then $\mu \in \sigma_{ext}(SA, SB)$.
<u>2^{nd} case:</u> if $F_\mu(C) \neq 0$, let m be the smallest integer such that

$$F_{\mu\xi^m} F_{\mu\xi^{m-1}} F_{\mu\xi^{m-2}} ... F_{\mu\xi} F_\mu(C) = 0,$$

therefore

$$W := F_{\mu\xi^{m-1}} F_{\mu\xi^{n-2}} ... F_{\mu\xi} F_\mu(C) \neq 0 \text{ and } F_{\mu\xi^m}(W) = 0.$$

Since $(\mu\xi^m)^n = \lambda$, we infer that $\mu\xi^m \in \sigma_{ext}(SA, SB)$.

Corollary 3.4. It follows immediately from Theorem 3.8 that for $S = I$, we have
$\lambda \in \sigma_{ext}(A^n, B^n)$ if, and only if, there exists a complex number $\mu \in \sigma_{ext}(A, B)$ with $\mu^n = \lambda$ for $n \in \mathbb{N}$.
In particular, $\lambda \in \sigma_{ext}(A^n)$ if, and only if, there exists a complex number $\mu \in \sigma_{ext}(A)$ with $\mu^n = \lambda$ for $n \in \mathbb{N}$ (see[6]) ◊

Theorem 3.9. Let A, B, S and $C \in \mathcal{L}(H)$ such that $SC = CS$. If $\sigma_S(A) \cap \sigma_S(B) = \emptyset$. Then,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \text{ is similar to } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

◊

Proof. Since $\sigma_S(A) \cap \sigma_S(B) = \emptyset$ and $SC = CS$, then resting upon Theorem 3.1, the equation $AX - XB = C$ has a unique solution X for any operator C. On the other side,

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AX - XB \\ 0 & B \end{pmatrix}. \tag{3.6}$$

Since $AX - XB = C$ has solution X, then Eq.(3.6) can be equivalently written as

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Let $P = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ be invertible, and $P^{-1} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$. Hence,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \text{ is similar to } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

□

Remark 3.5. The Theorem 3.9 generalizes the result given by R. Bhatia and P. Rosenthal in [4].

Example 3.2. Let V be the Volterra integral operator and A the invertible bounded linear operator such that $\sigma(V) \cap \sigma(A) = \emptyset$. Then, for all $C \in \mathcal{L}(H)$ such that $SC = CS$ we have

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} V & VX - XA \\ 0 & A \end{pmatrix}.$$

Since $VX - XA = C$ has a unique solution X, then

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} V & C \\ 0 & A \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} V & C \\ 0 & A \end{pmatrix} \text{ is similar to } \begin{pmatrix} V & 0 \\ 0 & A \end{pmatrix}.$$

◊

Theorem 3.10. Let A, B, C and $S \in \mathcal{L}(H)$ such that $SC = CS$. If $\sigma_S(A) \cap \sigma_S(B) = \emptyset$, then if $(A + B)$ and (AB) commutes with C, this implies that C commutes with A and B. ◊

Proof. First, we have

$$(A+B)C = C(A+B), \tag{3.7}$$

and

$$(AB)C = C(AB). \tag{3.8}$$

By multiplying Eq. (3.7) by A and making use of Eq. (3.8), we get

$$AAC + CAB = ACA + ACB.$$

This implies,

$$A(AC - CA) = (AC - CA)B.$$

Relying upon Theorem 3.1, we must have

$$AC - CA = 0. \tag{3.9}$$

Thus, C commutes with A and by using Eqs. (3.9) and (3.7), then C commutes also with B. □

Example 3.3. Let V be the Volterra integral operator and I be the identity operator such that $\sigma(V) \cap \sigma(I) = \emptyset$, then we have, for every operator C which commutes with $(V+I)$ and (VI), the following results

$$(V+I)C = C(V+I)$$

and

$$(VI)C = C(VI).$$

By using Theorem 3.1, we must get $VC - CV = 0$ and $IC = CI$. Hence, C commutes with V and I. ◊

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