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A family of chaotic dynamical systems on the Cantor dust $C \times C$

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Abstract. In this paper, we give a family of dynamical systems on the Cantor dust ($C \times C$) by using elements of 4th Dihedral group with shift map and then express them via code representations of the points of $C \times C$. Moreover, we prove that these systems are chaotic in the sense of Devaney and also we investigate the topological equivalence of these dynamical systems.

1. Introduction

The theory of dynamical system, which has several applications in different branches of engineering and many human sciences such as economics besides fundamental sciences such as physics, chemistry and biology, is one of the important research areas in mathematics ([1, 8, 10, 11, 14, 15, 18]). Predicting how the orbit of a point will behave is one of the main problems of in this theory. Although this problem can be solved for many dynamical systems, it has been observed that the solution of the problem is quite difficult in some examples such as the dynamics model of the celestial bodies of Poincare, May's population model and Lorenz's atmosphere dynamics model. This is due to the fact that very small differences in the initial conditions in these systems lead to large differences in the behavior of the orbits. This property is called the "sensitive dependence to initial conditions". Dynamical systems satisfying this property besides "topological transitivity" and "density of periodic points" are called chaotic dynamical systems and they are widely encountered ([9, 10, 14]).

Fractal geometry, also known as the geometry of nature, is originally composed of extraordinary geometric objects, can represent many phenomena better when compared with solid and constrained shapes such as line, circle, triangle in Euclidean geometry ([7, 12, 13]). Thanks to this feature, fractals have increasing number of application areas [11, 17, 19]. Fractals are often occurred as the attractor of an iterated function system (IFS), Julia sets of some complex functions, etc ([2, 7, 12, 13]). Moreover, when a complex function is restricted to the related Julia set, a discrete time chaotic dynamical system is obtained. Thus, fractals are often mentioned together with chaotic dynamical systems (see [1, 10, 14] for more details).

In the literature, one of the well-known examples of the chaotic dynamical system is the shift map defined on a code space. This dynamical system guides in defining many different chaotic systems. Another important examples are Tent map and Doubling map. Also, the restricted Tent map and the Smale's horseshoe map are fundamental examples of chaotic dynamical systems on C and $C \times C$ respectively. Both

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the Tent map and the Smale's horseshoe map can be expressed with the help of code representations of the points of these fractals and they can be expressed by using special cases of a shift map ([6]). On the other hand, there have been different studies on chaotic dynamical systems obtained by using folding maps on fractals such as Sierpinski triangle, Sierpinski tetrahedron and Box Fractal (see [3–5, 20]).

Although there exist many classical fractal models in the literature, dynamical system examples defined on self-similar sets are very few and no detailed investigations has been made. In order to overcome this deficiency, Cantor and Cantor like sets, which seem possible to generalize, can firstly be considered.

We first give a small brief for the code representation of the points of C and $C \times C$ and we remind the elements of Dihedral group D_4. In Section 3, as the main purpose of the present paper, we define a new family of dynamical system on $C \times C$ by using special shift maps and the elements of D_4. In the last section, we compare these dynamical systems whether they are topologically conjugate or not.

From now on we will use the notation C for Cantor set and C^2 for Cantor dust.

2. The code representations of Cantor dust C^2 and its symmetries

2.1. The code representations of the points of C^2

We use code representations of the points of the Cantor set to describe a family of dynamical systems in the next section. Although there are different ways for coding the points of the Cantor set (such as ternary form of the unit interval etc.), we prefer the following one:

Consider the Cantor set as an attractor of an iterated function system $\{\mathbb{R}; f_0, f_2\}$ where

$$f_0(x) = \frac{x}{3} \text{ and } f_2(x) = \frac{x}{3} + \frac{2}{3}.$$

Let $C_0 := f_0(C)$ and $C_2 := f_2(C)$. Note that $C_0 \cup C_2 = C$ and $C_0 \cap C_2 = \emptyset$. For a word $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \in \{0, 2\}^k$ with length k, let $C_\alpha := f_\alpha(C)$ where $f_\alpha = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_k}$. We call C_α as a sub-Cantor of level k. We set $\alpha = \emptyset$ if $k = 0$, and $C_\alpha = C$. For an infinite word $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \ldots$, it is obvious that

$$C_{\alpha_1} \supset C_{\alpha_1\alpha_2} \supset C_{\alpha_1\alpha_2\alpha_3} \supset \cdots \supset C_{\alpha_1\alpha_2\ldots\alpha_k} \supset \cdots$$

and by the Cantor Intersection Theorem, the infinite intersection $\bigcap_{k=1}^{\infty} C_{\alpha_1\alpha_2\ldots\alpha_k}$ is a singleton, say $\{a\}$ where $a \in C$. We call the sequence (or the infinite word) $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \ldots$ as the code representation of the point a. One can easily show that every point of C has a unique code representation in the above sense.

Using the code representations of the points of the Cantor set constructed above, we can directly represent the points of the Cantor dust C^2 as follows: Let $(a, b) \in C^2$, $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \ldots$ and $\beta = \beta_1\beta_2 \ldots \beta_k \ldots$ be the code representations of a and b respectively. Then we take (α, β) as the (unique) code representation of the point $(a, b) \in C^2$.

Throughout the paper, although $(a, b) \in C^2$ and (α, β) is an element of the code space (we are not dealing with the notion of the code space in this article, we refer [16] for detailed information on the code space theory), we write $a = \alpha \in C$ and $(a, b) = (\alpha, \beta) \in C^2$ for simplicity.

For two finite words $\alpha = \alpha_1\alpha_2 \ldots \alpha_k, \beta = \beta_1\beta_2 \ldots \beta_k \in \{0, 2\}^k$ with lengths k, let $C_{\alpha,\beta} := f_\alpha(C) \times f_\beta(C)$. We call $C_{\alpha,\beta}$ as a sub-Cantor dust of level k (See Figure 1 for some examples).

For a point $a = \alpha_1\alpha_2\alpha_3 \ldots \in C$, we define the point $\tilde{a} := \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 \ldots$ where $\tilde{x} = 2 - x$ for $x \in \{0, 2\}$, i.e. $\tilde{x} = 2$ if $x = 0$ and $\tilde{x} = 0$ if $x = 2$. By the definition, it is obvious that the point $\tilde{a}$ is the reflection of the point a about the point $x = 1/2$ (see Figure 2(a)). Similarly, let $(a, b) \in C^2$. Then the points $(\tilde{a}, b)$ and $(a, \tilde{b})$ are the reflections of the point (a, b) about the lines $x = 1/2$ and $y = 1/2$ respectively and $(\tilde{a}, \tilde{b})$ is the point obtained by the rotating the point (a, b) about the origin O of the unit square (see Figure 2(b)).

2.2. Symmetries on C^2

To construct a family of chaotic dynamical systems on the Cantor dust C^2, we use 4th Dihedral group D_4 which is the symmetry group of the unit square (the convex hull of C^2). In Figure 3, we give the pictorial description of the eight maps $\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1$ and δ_2 on the unit square.

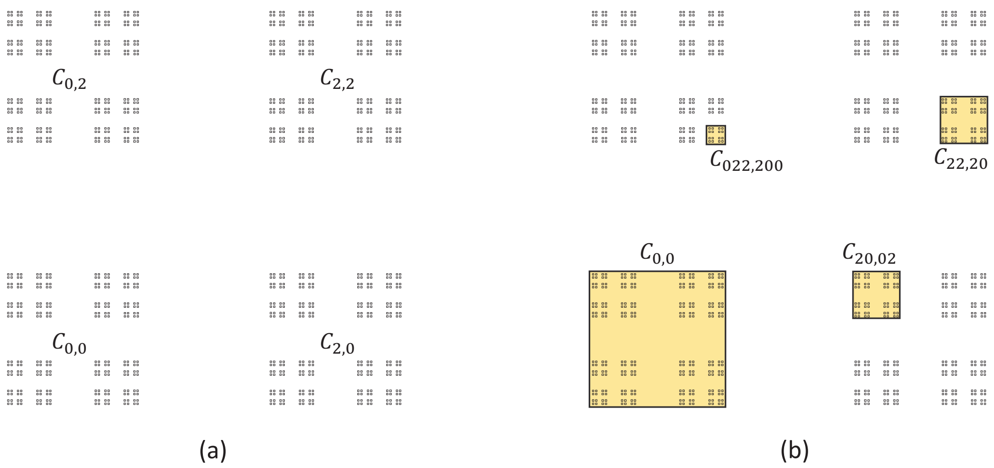

Figure 1: (a) 1-level sub-Cantor sets, (b) sub-Cantor sets of different levels.

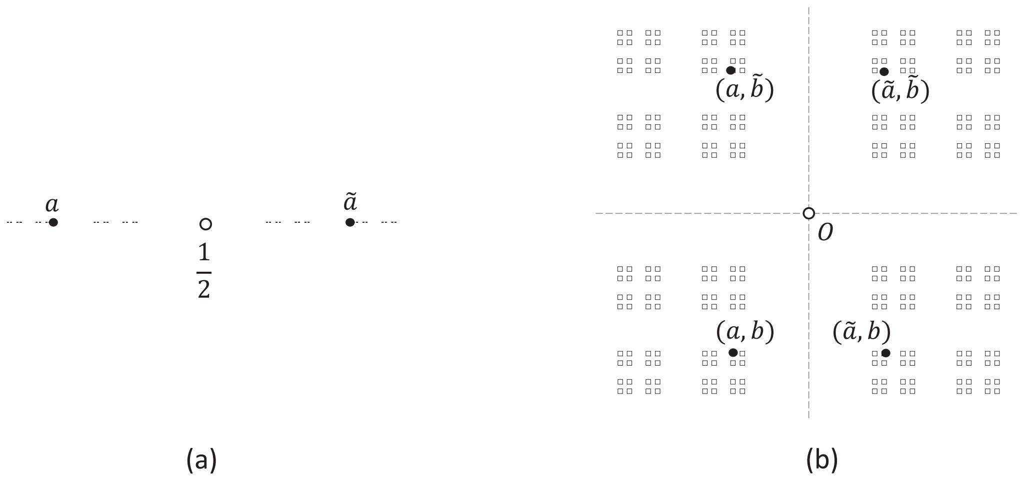

Figure 2: (a) The points a and $\tilde{a}$ on C, (b) the points (a, b), $(\tilde{a}, b)$, $(a, \tilde{b})$ and $(\tilde{a}, \tilde{b})$ on C^2.

In this work, we use these elements to define a dynamical system on C^2, thus we give the analytical expressions of these eight maps on C^2 using the code representations of the points (notice that these maps are well-defined on C^2). Using code representation of the point (a, b), one can obtain the images of this point as in Figure 4 (please follow Figure 2 to verify Figure 4).

Let $(a, b) = (\alpha_1\alpha_2\alpha_3 \ldots, \beta_1\beta_2\beta_3 \ldots) \in C^2$.

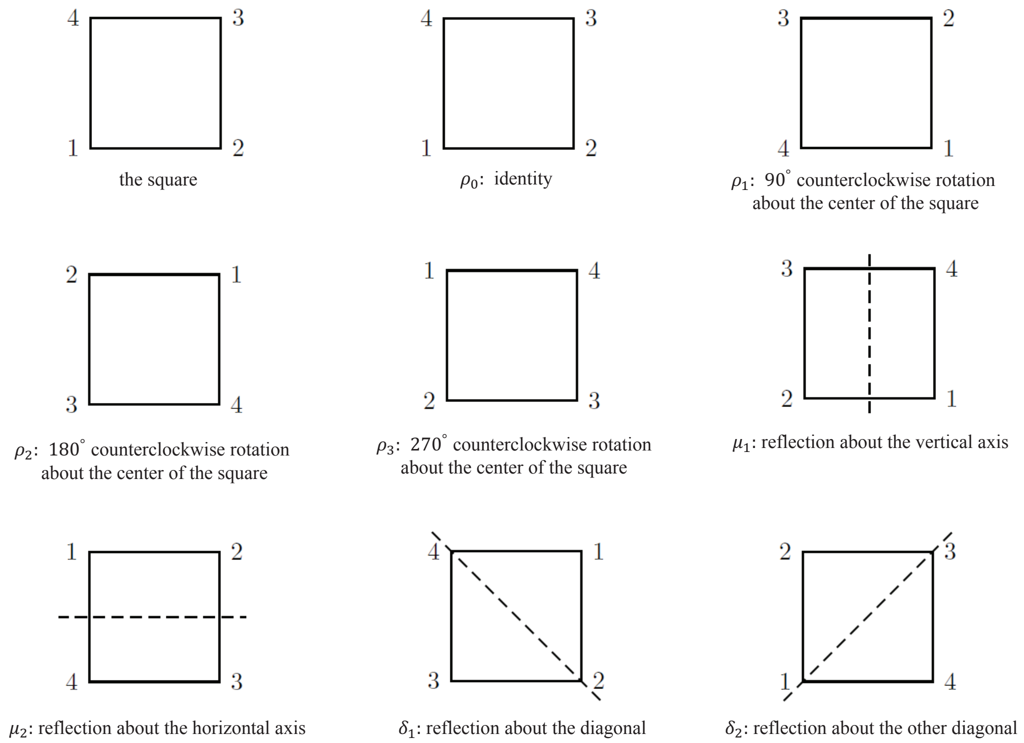

Figure 3: Pictorial description of the elements of the Dihedral group D_4.

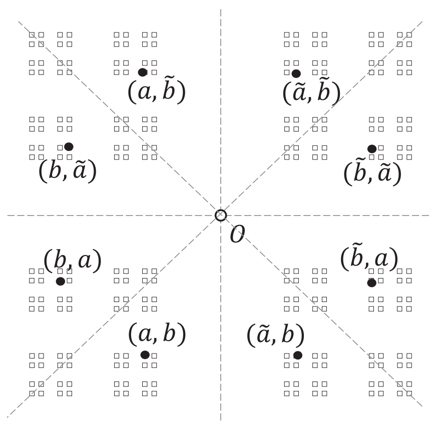

Figure 4: Images of the point $(a, b) \in C_{0220,0022} \subset C^2$ under the maps of D_4.

More clearly, the eight maps (on C^2) mentioned above can be expressed explicitly as follows :

$$\begin{aligned}
\rho_0(a,b) &= (\alpha_1\alpha_2\alpha_3\ldots,\beta_1\beta_2\beta_3\ldots) &&= (a,b)\\
\rho_1(a,b) &= (\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\ldots,\alpha_1\alpha_2\alpha_3\ldots) &&= (\tilde{b},a)\\
\rho_2(a,b) &= (\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3\ldots,\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\ldots) &&= (\tilde{a},\tilde{b})\\
\rho_3(a,b) &= (\beta_1\beta_2\beta_3\ldots,\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3\ldots) &&= (b,\tilde{a})\\
\mu_1(a,b) &= (\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3\ldots,\beta_1\beta_2\beta_3\ldots) &&= (\tilde{a},b)\\
\mu_2(a,b) &= (\alpha_1\alpha_2\alpha_3\ldots,\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\ldots) &&= (a,\tilde{b})\\
\delta_1(a,b) &= (\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\ldots,\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3\ldots) &&= (\tilde{b},\tilde{a})\\
\delta_2(a,b) &= (\beta_1\beta_2\beta_3\ldots,\alpha_1\alpha_2\alpha_3\ldots) &&= (b,a)
\end{aligned}$$

These equalities can be easily verified by the curious readers using the code representation of the points.

3. The construction of a family of chaotic dynamical systems on C^2

Let $(a,b) = (\alpha_1\alpha_2\alpha_3\ldots,\beta_1\beta_2\beta_3\ldots) \in C^2$ and let $\sigma : C^2 \longrightarrow C^2$,

$$\sigma(\alpha_1\alpha_2\alpha_3\ldots,\beta_1\beta_2\beta_3\ldots) = (\alpha_2\alpha_3\ldots,\beta_2\beta_3\ldots)$$

be the shift map on C^2.

In this section, we give a general proof to show that special shift maps defined by using the code representations of points of C^2 are chaotic in the sense of Devaney. For this aim, we take the compositions of shift map and symmetries of the square. To be more precise, let us consider the dynamical system $\{C^2; F\}$ such that $F : C^2 \longrightarrow C^2$,

$$F(a,b) = (\sigma \circ f)(a,b) \tag{1}$$

where $f : C^2 \longrightarrow C^2$

$$f(a,b) = \begin{cases} \eta_1(a,b) & , \quad (a,b) \in C_{0,0} \\ \eta_2(a,b) & , \quad (a,b) \in C_{2,0} \\ \eta_3(a,b) & , \quad (a,b) \in C_{0,2} \\ \eta_4(a,b) & , \quad (a,b) \in C_{2,2} \end{cases}$$

and $\eta_i \in D_4 = \{\rho_0,\rho_1,\rho_2,\rho_3,\mu_1,\mu_2,\delta_1,\delta_2\}$ for $i = 1,2,3,4$. If we apply the elements of D_4 to $C_{0,0}, C_{0,2}, C_{2,0}$ and $C_{2,2}$ respectively, then we get 2^{12} dynamical systems. In one of these, the image of $C_{2,2}$ under F for $\eta_4 = \sigma \circ \rho_1$ is depicted in Figure 5.

Remark 3.1. *The case $\eta_i = \rho_0$ for $i = 1,2,3,4$ is the most well-known and simple one. In this case, the function F is the shift map where*

$$\sigma(\alpha_1\alpha_2\alpha_3\ldots,\beta_1\beta_2\beta_3\ldots) = (\alpha_2\alpha_3\ldots,\beta_2\beta_3\ldots).$$

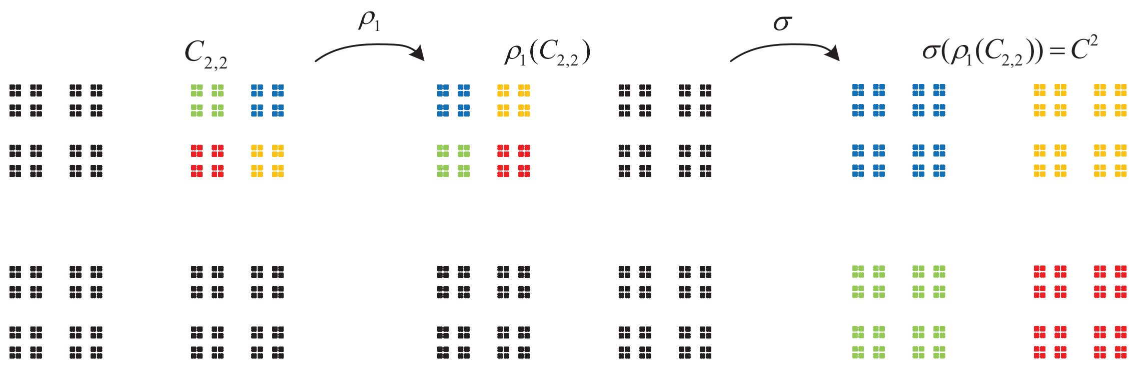

Figure 5: $F(C_{2,2})$ where $\eta_4 = \sigma \circ \rho_1$

Actually, this dynamical system can be considered by taking the inverse of the similarities in the iterated function system $\{C^2; w_1, w_2, w_3, w_4\}$ *where* $w_1(x, y) = (x/3, y/3)$, $w_2(x, y) = (x/3 + 2/3, y/3)$, $w_3(x, y) = (x/3, y/3 + 2/3)$ *and* $w_4(x, y) = (x/3 + 2/3, y/3 + 2/3)$. *Obviously, this system is expressed as*

$$F(x,y) = \begin{cases} (3x, 3y) & , \quad (x,y) \in C_{0,0} \\ (3x-2, 3y) & , \quad (x,y) \in C_{2,0} \\ (3x, 3y-2) & , \quad (x,y) \in C_{0,2} \\ (3x-2, 3y-2) & , \quad (x,y) \in C_{2,2} \end{cases}$$

on C^2.

Remark 3.2. *Note that there are eight different cases for* $F^n(a, b)$ *since* D_4 *is a group. That is, there is an element* ξ *of* D_4 *such that*

$$F^n(a, b) = (\sigma^n \circ \xi)(a, b).$$

Lemma 3.3. *Let*

$$(a, b) = (\alpha_1\alpha_2\alpha_3 \ldots \alpha_n\alpha_{n+1} \ldots, \beta_1\beta_2\beta_3 \ldots \beta_n\beta_{n+1} \ldots) \in C^2.$$

Some periodic points which close enough to (a, b) *are in the following forms:*
Case 1: *If* $F^n(a, b) = (\sigma^n \circ \rho_0)(a, b)$, *then one of* $n-$*periodic points is*

$$(\overline{\alpha_1\alpha_2\alpha_3 \ldots \alpha_n}, \overline{\beta_1\beta_2\beta_3 \ldots \beta_n}).$$

Case 2: *If* $F^n(a, b) = (\sigma^n \circ \rho_1)(a, b)$, *then one of* $n-$*periodic points is*

$$(\overline{\alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_n\tilde{\alpha}_1\tilde{\alpha}_2 \ldots \tilde{\alpha}_n\tilde{\beta}_1\tilde{\beta}_2 \ldots \tilde{\beta}_n}, \overline{\beta_1\beta_2 \ldots \beta_n\tilde{\alpha}_1\tilde{\alpha}_2 \ldots \tilde{\alpha}_n\tilde{\beta}_1\tilde{\beta}_2 \ldots \tilde{\beta}_n\alpha_1\alpha_2 \ldots \alpha_n}).$$

Case 3: *If* $F^n(a, b) = (\sigma^n \circ \rho_2)(a, b)$, *then one of* $n-$*periodic points is*

$$(\overline{\alpha_1\alpha_2 \ldots \alpha_n\tilde{\alpha}_1\tilde{\alpha}_2 \ldots \tilde{\alpha}_n}, \overline{\beta_1\beta_2 \ldots \beta_n\tilde{\beta}_1\tilde{\beta}_2 \ldots \tilde{\beta}_n}).$$

Case 4: *If* $F^n(a, b) = (\sigma^n \circ \rho_3)(a, b)$, *then one of* $n-$*periodic points is*

$$(\overline{\alpha_1\alpha_2 \ldots \alpha_n\tilde{\beta}_1\tilde{\beta}_2 \ldots \tilde{\beta}_n\tilde{\alpha}_1\tilde{\alpha}_2 \ldots \tilde{\alpha}_n\beta_1\beta_2 \ldots \beta_n}, \overline{\beta_1\beta_2 \ldots \beta_n\alpha_1\alpha_2 \ldots \alpha_n\tilde{\beta}_1\tilde{\beta}_2 \ldots \tilde{\beta}_n\tilde{\alpha}_1\tilde{\alpha}_2 \ldots \tilde{\alpha}_n}).$$

Case 5: *If $F^n(a,b) = (\sigma^n \circ \mu_1)(a,b)$, then one of $n-$periodic points is*

$$(\overline{\alpha_1\alpha_2\dots\alpha_n\tilde{\alpha_1}\tilde{\alpha_2}\dots\tilde{\alpha_n}}, \overline{\beta_1\beta_2\dots\beta_n}).$$

Case 6: *If $F^n(a,b) = (\sigma^n \circ \mu_2)(a,b)$, then one of $n-$periodic points is*

$$(\overline{\alpha_1\alpha_2\dots\alpha_n}, \overline{\beta_1\beta_2\dots\beta_n\tilde{\beta_1}\tilde{\beta_2}\dots\tilde{\beta_n}}).$$

Case 7: *If $F^n(a,b) = (\sigma^n \circ \delta_1)(a,b)$, then one of $n-$periodic points is*

$$(\overline{\alpha_1\alpha_2\dots\alpha_n\tilde{\beta_1}\tilde{\beta_2}\dots\tilde{\beta_n}}, \overline{\beta_1\beta_2\dots\beta_n\tilde{\alpha_1}\tilde{\alpha_2}\dots\tilde{\alpha_n}}).$$

Case 8: *If $F^n(a,b) = (\sigma^n \circ \delta_2)(a,b)$, then one of $n-$periodic points is*

$$(\overline{\alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_n}, \overline{\beta_1\beta_2\dots\beta_n\alpha_1\alpha_2\dots\alpha_n}).$$

Note that, $\overline{\alpha_1\alpha_2\alpha_3\dots\alpha_n}$ stands for $\alpha_1\alpha_2\alpha_3\dots\alpha_n\alpha_1\alpha_2\alpha_3\dots\alpha_n\dots$.

Proof. We only give the proof of Case 2 which is a little more complicated than the other cases. The remaining cases can be proved in a similar way. Suppose that

$$F^n(\alpha_1\alpha_2\alpha_3\dots,\beta_1\beta_2\beta_3\dots) = (\sigma^n \circ \rho_1)(\alpha_1\alpha_2\dots\alpha_n\dots,\beta_1\beta_2\dots\beta_n\dots).$$

Since $\rho_1(a,b) = (\tilde{b},a)$ we compute that

$$\begin{aligned} F^n(\alpha_1\alpha_2\dots\alpha_n\dots,\beta_1\beta_2\dots\beta_n\dots) &= (\sigma^n \circ \rho_1)(\alpha_1\alpha_2\dots\alpha_n\dots,\beta_1\beta_2\dots\beta_n\dots) \\ &= (\tilde{\beta}_{n+1}\tilde{\beta}_{n+2}\dots\tilde{\beta}_{2n}\dots,\alpha_{n+1}\alpha_{n+2}\dots\alpha_{2n}\dots). \end{aligned}$$

And then we get $\alpha_i = \tilde{\beta}_{n+i}$ and $\beta_i = \alpha_{n+i}$ for $i = 1,2,3,\dots n$. It follows that $\alpha_{n+i} = \beta_i$ and $\beta_{n+i} = \tilde{\alpha}_i$ for $i = 1,2,3,\dots n$.

As $\rho_1^2(a,b) = (\tilde{a},\tilde{b})$, we have

$$F^{2n}(\alpha_1\alpha_2\dots\alpha_n\dots,\beta_1\beta_2\dots\beta_n\dots) = (\tilde{\alpha}_{2n+1}\tilde{\alpha}_{2n+2}\dots\tilde{\alpha}_{3n}\dots,\tilde{\beta}_{2n+1}\tilde{\beta}_{2n+2}\dots\tilde{\beta}_{3n}\dots).$$

Thus, we obtain $\alpha_i = \tilde{\alpha}_{2n+i}$ and $\beta_i = \tilde{\beta}_{2n+i}$ for $i = 1,2,3,\dots n$. This shows that $\alpha_{2n+i} = \tilde{\alpha}_i$ and $\beta_{2n+i} = \tilde{\beta}_i$ for $i = 1,2,3,\dots n$.

For $\rho_1^3(a,b) = (b,\tilde{a})$, we obtain

$$F^{3n}(\alpha_1\alpha_2\dots\alpha_n\dots,\beta_1\beta_2\dots\beta_n\dots) = (\beta_{3n+1}\beta_{3n+2}\dots\beta_{4n}\dots,\tilde{\alpha}_{3n+1}\tilde{\alpha}_{3n+2}\dots\tilde{\alpha}_{4n}\dots).$$

Therefore, we compute $\alpha_i = \beta_{3n+i}$ and $\beta_i = \tilde{\alpha}_{3n+i}$ for $i = 1,2,3,\dots n$. That is, $\alpha_{3n+i} = \tilde{\beta}_i$ and $\beta_{3n+i} = \alpha_i$ for $i = 1,2,3,\dots n$.

There will be recurrent after the first $4n$ terms of the code representation of the point (a,b) owing to the fact that $\rho_1^4(a,b) = (a,b)$.

Consequently, it is obtained that

$$F^n(\overline{\alpha_1\dots\alpha_n\beta_1\dots\beta_n\tilde{\alpha}_1\dots\tilde{\alpha}_n\tilde{\beta}_1\dots\tilde{\beta}_n}, \overline{\beta_1\dots\beta_n\tilde{\alpha}_1\dots\tilde{\alpha}_n\tilde{\beta}_1\dots\tilde{\beta}_n\alpha_1\dots\alpha_n})$$
$$= (\overline{\alpha_1\dots\alpha_n\beta_1\dots\beta_n\tilde{\alpha}_1\dots\tilde{\alpha}_n\tilde{\beta}_1\dots\tilde{\beta}_n}, \overline{\beta_1\dots\beta_n\tilde{\alpha}_1\dots\tilde{\alpha}_n\tilde{\beta}_1\dots\tilde{\beta}_n\alpha_1\dots\alpha_n}).$$

and thus it is a periodic point (with period n) which has same the first n terms with the code representation of the point (a,b). □

Theorem 3.4. *$\{C^2;F\}$ defined in (1) is a chaotic dynamical system in the sense of Devaney.*

Proof. **i) Density of periodic points:** We must show that there exists a periodic point close enough to an arbitrary point of the Cantor dust. Let

$$(a,b) = (\alpha_1\alpha_2\alpha_3\ldots\alpha_n\alpha_{n+1}\ldots,\beta_1\beta_2\beta_3\ldots\beta_n\beta_{n+1}\ldots) \in C^2.$$

Due to Lemma 3.3, we find a periodic point close enough to (a,b).
ii) Topological transitivity: For a pair of non-empty open sets U and V in C^2, we must show that there is a natural number n such that $F^n(U) \cap V \neq \emptyset$. Given an open set V. Because of self-similarity of C^2, any open set U includes a subset $C_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n}$ which can be expressed as

$$\{(\alpha_1\ldots\alpha_n x_{n+1}x_{n+2}\ldots,\beta_1\ldots\beta_n y_{n+1}y_{n+2}\ldots) \mid x_{n+i}, y_{n+i} \text{ are arbitrary for } i = 1,2,3,\ldots\}.$$

Then, $F^n(C_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n})$ is the set

$$\{(x'_{n+1}x'_{n+2}x'_{n+3}\ldots,y'_{n+1}y'_{n+2}y'_{n+3}\ldots) \mid x'_{n+i}, y'_{n+i} \text{ are arbitrary for } i = 1,2,3,\ldots\}.$$

Therefore, we obtain $F^n(C_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n}) = C^2$. Since $C_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n} \subseteq U$, for any open set U there is a natural number n such that $F^n(U) = C^2$ and thus we get $F^n(U) \cap V \neq \emptyset$.
iii) Sensitivity dependence on initial conditions: Let (a,b) be arbitrary point of C^2 with the code representation $(\alpha_1\alpha_2\alpha_3\ldots\alpha_n\alpha_{n+1}\ldots,\beta_1\beta_2\beta_3\ldots\beta_n\beta_{n+1}\ldots)$. We choose the (a',b') of C^2 with the code representations

$$(\alpha_1\alpha_2\alpha_3\ldots\alpha_n\alpha'_{n+1}\alpha'_{n+2}\ldots,\beta_1\beta_2\beta_3\ldots\beta_n\beta'_{n+1}\beta'_{n+2}\ldots).$$

where $\alpha_{n+1} \neq \alpha'_{n+1}$ and $\beta_{n+1} \neq \beta'_{n+1}$. Obviously, we compute

$$d((a,b),(a',b')) < \frac{\sqrt{2}}{3^n}.$$

where d is the well-known code space metric on the code space. Moreover, $F^n(a,b)$ and $F^n(a',b')$ equal to

$$(\alpha''_{n+1}\alpha''_{n+2}\alpha''_{n+3}\ldots,\beta''_{n+1}\beta''_{n+2}\beta''_{n+3}\ldots)$$

and

$$(\alpha'''_{n+1}\alpha'''_{n+2}\alpha'''_{n+3}\ldots,\beta'''_{n+1}\beta'''_{n+2}\beta'''_{n+3}\ldots).$$

respectively. Note that $\alpha''_{n+1} \neq \alpha'''_{n+1}$ and $\beta''_{n+1} \neq \beta'''_{n+1}$ and this shows that

$$d(F(a,b),F(a',b')) \geq \frac{2\sqrt{2}}{3}.$$

As a result, the dynamical system $\{C^2;F\}$ is chaotic in the sense of Devaney. □

4. Some topologically conjugate dynamical systems on C^2

In this section, we investigate which of dynamical systems obtained by using the elements of D_4 on C^2 are topologically equivalent. To this end, we first determine the elements of D_4 on C^2 which are conjugate maps. Then using these maps, we discuss topological equivalence of dynamical systems on C^2. In the following, we first recall the definition of conjugate dynamical systems:

The dynamical systems $\{X_1;f_1\}$ and $\{X_2;f_2\}$ are said to be topologically conjugate (or topologically equivalent), if there is a homeomorphism $h : X_1 \to X_2$ such that $f_2 = h \circ f_1 \circ h^{-1}$ which means $\forall x \in X_1$, $h(f_1(x)) = f_2(h(x))$. Then f_1 and f_2 are called conjugate maps and h is called a conjugacy (see [7]).

Lemma 4.1. *The following elements of D_4 given on C^2 are conjugate maps by indicated conjugacy h:*

i) μ_1 *and* μ_2 *are conjugate maps via conjugacies* δ_1, δ_2 *or* ρ_1.

ii) ρ_1 *and* ρ_3 *are conjugate maps via conjugacies* μ_1 *or* μ_2.

iii) δ_1 *and* δ_2 *are conjugate maps via conjugacies* ρ_1, μ_1 *or* μ_2.

Proof. We only give the proof of the first case since (ii) and (iii) are similarly done.

i) To prove μ_1 and μ_2 are conjugate maps via conjugacy δ_2, for all $(a, b) \in C^2$ with the code representation $(\alpha_1\alpha_2\alpha_3 \dots, \beta_1\beta_2\beta_3 \dots)$ we must show that

$$(\delta_2 \circ \mu_1)(a, b) = (\mu_2 \circ \delta_2)(a, b),$$

that is, the diagram given in Figure 6 is commutative:

$$\begin{tikzcd} C^2 \arrow[r, "\mu_2"] \arrow[d, "\delta_2"'] & C^2 \arrow[d, "\delta_2"] \\ C^2 \arrow[r, "\mu_1"'] & C^2 \end{tikzcd}$$

Figure 6: The commutative diagram of the conjugate maps μ_1 and μ_2

We compute that

$$(\delta_2 \circ \mu_1)(a, b) = \delta_2(\tilde{\alpha_1}\tilde{\alpha_2}\tilde{\alpha_3} \dots, \beta_1\beta_2\beta_3 \dots) = (\beta_1\beta_2\beta_3\beta_4 \dots, \tilde{\alpha_1}\tilde{\alpha_2}\tilde{\alpha_3}\tilde{\alpha_4} \dots)$$

and

$$(\mu_2 \circ \delta_2)(a, b) = \mu_2(\beta_1\beta_2\beta_3 \dots, \alpha_1\alpha_2\alpha_3 \dots) = (\beta_1\beta_2\beta_3 \dots, \tilde{\alpha_1}\tilde{\alpha_2}\tilde{\alpha_3} \dots)$$

due to the fact that $\mu_2(a, b) = (a, \tilde{b})$, $\mu_1(a, b) = (\tilde{a}, b)$ and $\delta_2(a, b) = (b, a)$. This shows that

$$(\delta_2 \circ \mu_1)(a, b) = (\mu_2 \circ \delta_2)(a, b)$$

for all $(\alpha_1\alpha_2\alpha_3 \dots, \beta_1\beta_2\beta_3 \dots) \in C^2$.

If the maps δ_1 or ρ_1 can be taken instead of δ_2, then topological conjugacy of μ_1 and μ_2 can be shown in a similar way. □

Remark 4.2. *From now on, we denote the conjugate maps of* f *as* $f_\sim$. *Thus,* $\mu_{1_\sim} = \mu_2$, $(\mu_{2_\sim} = \mu_1)$ *and* $\rho_{1_\sim} = \rho_3$, $(\rho_{3_\sim} = \rho_1)$ *and* $\delta_{1_\sim} = \delta_2$, $(\delta_{2_\sim} = \delta_1)$.

Theorem 4.3. *Let* $f : C^2 \to C^2$

$$f(a, b) = \begin{cases} \eta_1(a, b), & (a, b) \in C_{0,0} \\ \eta_2(a, b), & (a, b) \in C_{2,0} \\ \eta_3(a, b), & (a, b) \in C_{0,2} \\ \eta_4(a, b), & (a, b) \in C_{2,2} \end{cases}$$

and $g : C^2 \to C^2$

$$g(a, b) = \begin{cases} \eta'_1(a, b), & (a, b) \in C_{0,0} \\ \eta'_2(a, b), & (a, b) \in C_{2,0} \\ \eta'_3(a, b), & (a, b) \in C_{0,2} \\ \eta'_4(a, b), & (a, b) \in C_{2,2} \end{cases}$$

be two different dynamical systems on C^2 *where* $\eta_i, \eta'_i \in D_4$ *for* $i = 1, 2, 3, 4$. $\{C^2; f\}$ *and* $\{C^2; g\}$ *are topologically conjugate if the following conditions satisfy:*

i) If $\eta_i \in \{\rho_0, \rho_2\}$, then $\eta_i' = \eta_i$,

ii) Otherwise, $\eta_i' = \eta_i$ or $\eta_i' = \eta_{i\sim}$

for $i \in \{1, 2, 3, 4\}$.

Proof. Let $\eta_i = \rho_0$ (or ρ_2). Then, $\eta_i' = \eta_i = \rho_0$ (or ρ_2) if and only if $\eta_i \circ h = h \circ \eta_i'$ for any $h \in D_4$. Because, the elements ρ_0 and ρ_2 satisfy the following equalities:

$$(\rho_0 \circ \eta)(a,b) = (\eta \circ \rho_0)(a,b)$$

and

$$(\rho_2 \circ \eta)(a,b) = (\eta \circ \rho_2)(a,b)$$

for $\forall \eta \in D_4$ and $\forall (a,b) \in C^2$.

If $\eta_i = \rho_1$ (or ρ_3), then either $\eta_i' = \eta_i$, since in this case we get $\rho_1 \circ \rho_3 = \rho_3 \circ \rho_1$ where $h = \rho_3$ (or ρ_1), or $\eta_i' = \eta_{i\sim}$, due to the fact that $\rho_1 \circ h = h \circ \rho_3$ where $h = \mu_1, \mu_2$ from Lemma 4.1 and the commutative property

$$(\rho_1 \circ \rho_3)(a,b) = (\rho_3 \circ \rho_1)(a,b), \ \forall (a,b) \in C^2.$$

Moreover the other cases ($\eta_i \in \{\mu_1, \mu_2, \delta_1, \delta_2\}$) can easily be shown by using Lemma 4.1 and the properties

$$(\delta_1 \circ \delta_2)(a,b) = (\delta_2 \circ \delta_1)(a,b)$$

and

$$(\mu_1 \circ \mu_2)(a,b) = (\mu_2 \circ \mu_1)(a,b)$$

for all $(a,b) \in C^2$. This concludes the proof. □

Corollary 4.4. *If f and g are topologically conjugate maps then $F = \sigma \circ f$ and $G = \sigma \circ g$ are also topologically conjugate dynamical systems via the same conjugacy map.*

Example 4.5. *Given $f : C^2 \to C^2$ and $g : C^2 \to C^2$ such that*

$$f(a,b) = \begin{cases} \rho_1(a,b), & (a,b) \in C_{0,0} \\ \delta_1(a,b), & (a,b) \in C_{2,0} \\ \mu_2(a,b), & (a,b) \in C_{0,2} \\ \rho_3(a,b), & (a,b) \in C_{2,2}. \end{cases}$$

and

$$g(a,b) = \begin{cases} \rho_3(a,b), & (a,b) \in C_{0,0} \\ \delta_2(a,b), & (a,b) \in C_{2,0} \\ \mu_2(a,b), & (a,b) \in C_{0,2} \\ \rho_1(a,b), & (a,b) \in C_{2,2} \end{cases}$$

respectively. Then, the dynamical systems $F = \sigma \circ f$ and $G = \sigma \circ g$ are topologically conjugate via the conjugacy maps $h = \mu_1$ or $h = \mu_2$ from Lemma 4.1 and Corollary 4.4.

In addition, one can easily verify that $(\overline{020220}, \overline{220020})$ is a 3−periodic point for F. Thus, we obtain that $\mu_1(\overline{020220}, \overline{220020}) = (\overline{202002}, \overline{002202})$ is also a 3−periodic point for G from the well-known proposition

" if x^* is a point of period n for F, then $h(x^*)$ is a point of period n for G."

5. Conclusion

In the present paper, we define a new family of chaotic dynamical systems by using the elements of D_4 on the first level of the Cantor dust $C \times C$ and determine topological equivalence classes of them. Many investigations can also be done on different levels of the Cantor dust $C \times C$ by using the elements of D_4 to define various chaotic dynamical systems in the light of this paper.

For future works, a new family of chaotic dynamical systems on the Cantor dust $C \times C \times C$ can be constructed similarly. In fact, these construction may be consider for the Cantor dust C^n with the help of the elements of D_4 on the various level of C^n. Moreover, dynamical systems that can be defined in a similar way can also be investigated for polygonal fractals such as the Pentagasket.

References

[1] K. T. Alligood, T. D. Sauer, J. A. Yorke, *Chaos an Introduction to Dynamical Systems*, New York, USA: Springer-Verlag, 1996.
[2] N. Aslan, M. Saltan, B. Demir, *A Different Construction of the Classical Fractals via the Escape Time Algorithm*, Journal of Abstract and Computational Mathematics **4** (2018), 1–15.
[3] N. Aslan, M. Saltan, B. Demir, *The intrinsic Metric Formula and a Chaotic Dynamical System on the Code Set of the Sierpinski Tetrahedron*, Chaos, Solitons Fractals **123** (2019), 422–428.
[4] N. Aslan, M. Saltan, B. Demir, *On Topological Conjugacy of Some Chaotic Dynamical Systems on the Sierpinski Gasket*, Filomat **35(7)** (2021), 2317–2391.
[5] N. Aslan, M. Saltan, *On the Construction of Chaotic Dynamical Systems on the Box Fractal*, Researches in Mathematics, **29(2)** (2021), 3–14.
[6] J. Bank, V. Dragan, *Smale's Horseshoe Map Via Ternary Numbers*, SIAM Review **36(2)** (1994), 265–271.
[7] M. Barnsley, *Fractals Everywhere*, New York, Dover Publications, 2012.
[8] R. Brown, *A Modern Introduction to Dynamical Systems*, Oxford University Press, 2018.
[9] B. Demir, Ş . Koçak, *A Note on Positive Lyapunov Exponent and Sensitive Dependence on Initial Conditions*, Chaos, Solitons Fractals **12** (2001), 2119–2121.
[10] R. L. Devaney, *A First Course in Chaotic Dynamical Systems*, CRC Press, Taylor and Francis Group, 2020.
[11] G. Drzewiecki, *Fundamentals of Chaos and Fractals for Cardiology*, Springer, 2021.
[12] G. Edgar, *Measure, Topology, and Fractal Geometry*, Springer:Berlin/Heidelberg, Germany, Springer Undergraduate Texts in Mathematics, 2008.
[13] K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, Hoboken NJ, USA, John Wiley and Sons, 2004.
[14] D. Gulick, *Encounters with Chaos and Fractals*, CRC Press, Taylor and Francis Group, 2012.
[15] W. M. Haddad, *A Dynamical Systems Theory of Thermodynamics*, Princeton University Press, 2019.
[16] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.
[17] Y. Özdemir, M. Saltan and B. Demir, *The Intrinsic Metric on the Box Fractal*, Bull. Iran. Math. Soc. **45** (2019), 1269–1281.
[18] H.O. Peitgen, H. Jürgens and D. Saupe, *Chaos and Fractals, New frontiers of science*, Springer-Verlag, 2004.
[19] M. Saltan, Y. Özdemir, B. Demir, *An Explicit Formula of the Intrinsic Metric on the Sierpinski Gasket via Code Representation*, Turk. J. Math. **42** (2018), 716–725.
[20] M. Saltan, N. Aslan, B. Demir, *A Discrete Chaotic Dynamical System on the Sierpinski Gasket*, Turk. J. Math. **43** (2019), 361-372.