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An existence study for a multiple system with $p-$Laplacian involving $\varphi-$Caputo derivatives

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Abstract. In this paper, we study the existence and uniqueness of solutions for a multiple system of fractional differential equations with nonlocal integro multi point boundary conditions by using the $p-$Laplacian operator and the $\varphi-$Caputo derivatives. The presented results are obtained by the two fixed point theorems of Banach and Krasnoselskii. An illustrative example is presented at the end to show the applicability of the obtained results. To the best of our knowledge, this is the first time where such problem is considered.

1. Introduction

The fractional calculus has many significant roles in various scientific fields of research, see for instance [14, 25, 27, 28, 31]. As applied results, the fractional order differential equations have attired attention of several scientists in different fields of research [9, 24]. However, most of the published works have been achieved by using the fractional derivatives of type Riemann-Liouville, Hadamard, Katugampola, Atangana-Baleanu, Grunwald Letnikov and Caputo. The fractional derivatives of functions with respect to some other functions [19] are different from the others since their kernels appear in terms of other functions (called φ). Recently, some fractional differential results have been considered in [3, 4, 13, 15].
In most of the present articles, Schauder, Krasnoselskii, Darbo, or Monch theories have been used to prove existence of solutions of nonlinear fractional differential equations with some restrictive conditions [2, 7, 8, 23, 26]. Some authors have worked on the solutions for fractional problems with $p-$Laplacian operators. We cite, for example [5, 6, 11, 17, 18, 22, 30] where it has been studied nonlinear fractional equation with $p-$Laplacian operator for the solutions.
Here, we will mention some other research works for the reader. We begin by A. Devi, A. Kumar, D. Baleanu and A. Khan [11] where they worked on the stability results, for the following nonlinear FDEs involving Caputo derivatives of distinct orders and ψ_p Laplacian operator:

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$$\begin{cases} {}^{c}\mathcal{D}^{r_1}\psi_p\left[{}^{c}\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^{m}v_i(t)\right)\right]=-w(t,u(t)),t\in(0,1] \\ \psi_p\left[{}^{c}\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^{m}v_i(t)\right)\right]\Big|_{t=0}=0, \\ u(0)=\sum_{i=1}^{m}v_i(0), \\ u'(1)=\sum_{i=1}^{m}v_i'(1), \\ u^{j}(0)=\sum_{i=1}^{m}v_i^{j}(0),\ \text{for}\quad j=2,3,...,n-1 \end{cases}$$

where $0<r_1\leq 1, n-1<r_2\leq n, n\geq 4$, and v_i, w are continuous functions. ${}^{c}\mathcal{D}^{r_1}$ and ${}^{c}\mathcal{D}^{r_2}$ denotes the derivative of fractional order r_1 and r_2 in Caputo's sense, respectively, and $\psi_p(z)=|z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1,\left(\psi_p\right)^{-1}=\psi_q$.

We can cite also the paper of A. Mahdjouba et al. [21] where they have investigated the study the existence and multiplicity of positive solutions of the following problem:

$$\begin{cases} \left(\psi_p\left[\mathcal{D}_{0^+}^{r}(u(t))\right]\right)'+a_1(t)f\left(u(\theta_1(t)),v(\theta_2(t)\right)=0,0<t<1, \\ \left(\psi_{\overline{p}}\left[\mathcal{D}_{0^+}^{r}(v(t))\right]\right)'+a_2(t)f\left(u(\theta_1(t)),v(\theta_2(t)\right)=0,0<t<1, \\ \mathcal{D}_{0^+}^{r}u(0)=u(0)=u'(0),\quad \mathcal{D}_{0^+}^{m}u(1)=\gamma\,\mathcal{D}_{0^+}^{m}u(\eta), \\ \mathcal{D}_{0^+}^{r}v(0)=v(0)=v'(0),\quad \mathcal{D}_{0^+}^{m}v(1)=\gamma\,\mathcal{D}_{0^+}^{m}v(\eta), \end{cases}$$

where $\eta\in(0,1),\gamma\in\left(0,\frac{1}{\eta^{r-m-1}}\right)$ $\mathcal{D}_{0^+}^{r},\mathcal{D}_{0^+}^{m}$, are the standard Riemann–Liouville fractional derivatives with $r\in(2,3), m\in(1,2)$ such that $r\geq m+1$, $p-$Laplacian operator is defined as $\psi_p(z)=z\,|z|^{p-2}, p>1$, and the functions $f,g\in C\left(\mathbb{R}^2,\mathbb{R}\right)$.

Then, S. Etemad with his co authors [12] have been concerned with the existence study for the following tripled impulsive fractional problem

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\kappa_m}x_m(t)=f_m(t,x(t)), m=1,2,3,\ \text{and}\ t\in J' \\ x_m(a)=\Phi_m x, x_m'(a)=\Theta_m x, \\ \Delta x_m|_{t=t_k}=I_{m,k}(x(t_k)),\ \Delta x_m'\big|_{t=t_k}=\overline{I}_{m,k}(x(t_k)), \end{cases}$$

where $J=[a,b], J'=J-\{t_1,t_2,...,t_p\}, a=t_0<t_1<\ldots<t_p<t_{p+1}=b$, ${}^{c}\mathcal{D}_{0^+}^{\kappa_m}, m=1,2,3$, are the Caputo fractional derivatives such that $\kappa_m\in(1,2]$, $f_m: J\times\mathbb{R}^3\to\mathbb{R}$,

$x(t)=(x_1(t),x_2(t),x_3(t)), I_{m,k},\overline{I}_{m,k}:\mathbb{R}^3\to\mathbb{R}, k=1,2,...,p$, are given functions, Φ_m,Θ_m are given operators, $\Delta x_m|_{t=t_k}=x(t_k^+)-x(t_k^-)$, $\Delta x_m'\big|_{t=t_k}=x'(t_k^+)-x'(t_k^-)$, and

$$x(t_k^+)=\lim_{h\to 0^+}x_m(t_k+h), x(t_k^-)=\lim_{h\to 0^-}x_m(t_k+h).$$

In the present research work, we study the existence and uniqueness of solutions for the following problem:

$$\begin{cases} \mathcal{D}_{0^+}^{r_{1m};\varphi}\psi_{p_m}\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t)-\mathcal{I}_{0^+}^{\sigma;\varphi}G_m(t,u_1(t),\ldots,u_n(t))\right)\right]=H_m(t,u_1(t),\ldots,u_n(t)), \\ m=\overline{1,n},\ \text{and}\ t\in J=(0,1] \\ \psi_{p_m}\left[\mathcal{D}_{0^+}^{r_2;\varphi}\left(u_m(t)-\mathcal{I}_{0^+}^{\sigma;\varphi}G_m(t,u_1(t),\ldots,u_n(t)\right)\right]\Big|_{t=0}=0, \\ u_m(0)=0,\ u_m(1)=\sum_{i=1}^{n}\lambda_{im}u_i\left(\zeta_{im}\right),\ \zeta_{im}\in(0,1] \\ \varphi(1)-\varphi(0)=K>0. \end{cases} \tag{1}$$

Here, we take $\mathcal{D}_{0^+}^{r_{im};\varphi}, i,m=\overline{1,n}$ as the $\varphi-$Caputo fractional derivatives of orders r_{im}, $0<r_{1m}<1<r_{2m}<2$, and $\mathcal{I}_{0^+}^{\sigma;\varphi}, 0<\sigma$, the fractional integral of order σ, $\lambda_{im}\in\mathbb{R}_+^*$, and $\varphi: J\to\mathbb{R}$ is an increasing function such that $\varphi'(t)\neq 0$, and $\psi_{p_m}(z)=|z|^{p_m-2}z$ denotes the p_m-Laplacian operator and satisfies $\frac{1}{p_m}+\frac{1}{q_m}=1,\left(\psi_{p_m}\right)^{-1}=\psi_{q_m}(q_m\geq 2)$. For all $t\in J$, $G_m, H_m: J\times\mathbb{R}^n\to\mathbb{R}$ is a given functions satisfying some assumptions that will be specified later.

2. $\varphi-$Caputo Derivatives

In this section, we introduce some notations and definitions of $\varphi-$Caputo approach, for details, see [4, 19, 24, 29].

Let $\varphi : J \to \mathbb{R}$ be an increasing function with $\varphi'(t) \neq 0$, for all $t \in J$.

And throughout the paper, let $C = C(J, \mathbb{R})$ denotes the Banach space of all continuous mappings from $[0,1]$ to $\mathbb{R}$ endowed with the norm $\|u\| = \sup_{t\in[0,1]} u(t)$. It is clear that the space C^n endowed with the norm $\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|$ is a Banach space.

We pose for all $r > 0$, and $t \in [0,1]$, $(t > s)$

$$\varphi_r(t,s) = \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{r-1}}{\Gamma(r)}.$$

Where the Gamma function $\Gamma(z)$(for $z \in \mathbb{R}$, such that $\Re(z) > 0$) is defined by the following integral:

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt.$$

Definition 2.1. *For $\alpha > 0$, the left-sided $\varphi-$Riemann Liouville fractional integral of order α for an integrable function $u : J \to \mathbb{R}$ with respect to another function $\varphi : J \to \mathbb{R}$ that is an increasing differentiable function such that $\varphi'(t) \neq 0$, for all $t \in J$ is defined as follows*

$$\mathcal{I}_{a^+}^{\alpha;\varphi} u(t) = \int_a^t \varphi_\alpha(t,s)\, u(s)ds, \tag{2}$$

Note that equation (2) is reduced to the Riemann Liouville and Hadamard fractional integrals when $\varphi(t) = t$ and $\varphi(t) = \ln t$, respectively.

Definition 2.2. *Let $n - 1 < \alpha \leq n$ and let $u \in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0$, for all $t \in J$. The left-sided $\varphi-$Riemann Liouville fractional derivative of a function u of order α is defined by*

$$\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\varphi} u(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \int_a^t \varphi_{n-\alpha}(t,s)\, u(s)ds,$$

where $n = [\alpha] + 1$.and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. *Let $n - 1 < \alpha \leq n$ and let $u \in C^{n-1}(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0$, for all $t \in J$. The left-sided $\varphi-$Caputo fractional derivative of a function u of order α is defined by*

$${}^c\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = \mathcal{D}_{a^+}^{\alpha;\varphi}\left[u(t) - \sum_{k=0}^{n-1} \frac{u_\varphi^{[k]}(a)}{k!}\left[\varphi(t) - \varphi(a)\right]^k\right]$$

where $u_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n u(t)$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, and $n = \alpha$ for $\alpha \in \mathbb{N}$. Further, if $u \in C^n(J)$ and $\alpha \notin \mathbb{N}$, then

$$\begin{aligned} {}^c\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) &= \mathcal{I}_{a^+}^{n-\alpha;\varphi}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n u(t), \\ &= \int_a^t \varphi_{n-\alpha}(t,s) u_\varphi^{[n]}(s)ds \end{aligned} \tag{3}$$

Thus, if $\alpha = n \in \mathbb{N}$, one has

$${}^c\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = u_\varphi^{[n]}(t).$$

2.1. Auxiliary Lemma

Lemma 2.4. *Let $\alpha, \beta > 0$, and $u \in L^1(J)$. Then*

$$\mathcal{I}_{a^+}^{\alpha;\varphi} \mathcal{I}_{a^+}^{\beta;\varphi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\varphi} u(t), \quad a.e.\ t \in J.$$

In particular,
If $u \in C(J)$, then $\mathcal{I}_{a^+}^{\alpha;\varphi} \mathcal{I}_{a^+}^{\beta;\varphi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\varphi} u(t)$, $t \in J$.

Next, we recall the property describing the composition rules for fractional φ-integrals and φ-derivatives.

Lemma 2.5. *Let $\alpha > 0$ The following holds:*
If $u \in C([a,b])$, then

$${}^c\mathcal{D}_{a^+}^{\alpha;\varphi} \mathcal{I}_{a^+}^{\alpha;\varphi} u(t) = u(t), t \in [a,b].$$

If $u \in C^n(J)$, $n-1 < \alpha < n$, then

$$\mathcal{I}_{a^+}^{\alpha;\varphi c} \mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\varphi^{[k]}(a)}{k!} \left[\varphi(t) - \varphi(a)\right]^k,$$

for all $t \in [a,b]$. In particular, if $0 < \alpha < 1$, we have

$$\mathcal{I}_{a^+}^{\alpha;\varphi c} \mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = u(t) - u(a).$$

Lemma 2.6. *Let $t > a$, $\alpha \geq 0$; and $\beta > 0$. Then*

- $\mathcal{I}_{a^+}^{\alpha;\varphi} \left[\varphi(t) - \varphi(a)\right]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left[\varphi(t) - \varphi(a)\right]^{\beta+\alpha-1}$,
- ${}^c\mathcal{D}_{a^+}^{\alpha;\varphi} \left[\varphi(t) - \varphi(a)\right]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left[\varphi(t) - \varphi(a)\right]^{\beta-\alpha-1}$,
- ${}^c\mathcal{D}_{a^+}^{\alpha;\varphi} \left[\varphi(t) - \varphi(a)\right]^{k} = 0$, *for all $k \in \{0, ..., n-1\}, n \in \mathbb{N}$.*

Lemma 2.7. *Let $\alpha > 0, n \in \mathbb{N}$; such that $n-1 < q \leq n$. Then:*

- ${}^c\mathcal{D}_{a^+}^{q;\varphi} \mathcal{I}_{a^+}^{\alpha;\varphi} u(t) ={}^c \mathcal{D}_{a^+}^{q-\alpha;\varphi} u(t)$; *if $q > \alpha$.*
- ${}^c\mathcal{D}_{a^+}^{q;\varphi} \mathcal{I}_{a^+}^{\alpha;\varphi} u(t) u(t) = \mathcal{I}_{a^+}^{\alpha-q;\varphi} u(t)$; *if $\alpha > q$.*

Lemma 2.8. *Given a function $u \in C^n[a,b]$ and $0 < q < 1$, we have*

$$\left|\mathcal{I}_{a^+}^{q;\varphi} u(t_2) - \mathcal{I}_{a^+}^{q;\varphi} u(t_1)\right| \leq \frac{2\|u\|}{\Gamma(q+1)} \left(\varphi(t_2) - \varphi(t_1)\right)^q.$$

Finally, we recall the fixed point theorems that will be used to prove the main results. (We have C a Banach space in each theorem).

Lemma 2.9. *(Banach fixed point theorem [10]) Let U be a closed set in C and $\mathcal{T} : U \to U$ satisfies*

$$|\mathcal{T}u - \mathcal{T}v| \leq \alpha |u - v|, \text{ for some } \alpha \in (0,1), \text{and for } u, v \in U.$$

Then $\mathcal{T}$ admits one fixed point in U.

Lemma 2.10. *(Krasnoselskii fixed point theorem [20]) Let M be a closed, bounded, convex and nonempty subset of a Banach space U. Let A, B be operators such that*
(i) $Ax + By \in M$ where $x, y \in M$,
(ii) A is compact and continuous and
(iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.11. *([16]) For the* $p-$*Laplacian operator* ψ_p*, the following conditions hold true:*
(1) If $|\delta_1|, |\delta_2| \geq \rho > 0, 1 < p \leq 2, \delta_1\delta_2 > 0,$ *then*

$$\left|\psi_p(\delta_1) - \psi_p(\delta_2)\right| \leq (p-1)\,\rho^{p-2}\,|\delta_1 - \delta_2|\,.$$

(2) If $p > 2, |\delta_1|, |\delta_2| \leq \rho_* > 0,$ *then*

$$\left|\psi_p(\delta_1) - \psi_p(\delta_2)\right| \leq (p-1)\,\rho_*^{p-2}\,|\delta_1 - \delta_2|\,.$$

Lemma 2.12. *[14] For nonnegative* $a_i, i = 1, ..., k,$

$$\left(\sum_{i=1}^{k} a_i\right)^q \leq k^{q-1}\left(\sum_{i=1}^{k} a_i^q\right), q \geq 1$$

Now, we pass to prove the following result.

Lemma 2.13. *For a given* $h_m, g_m \in L^1(J, , \mathbb{R}^3)(m = \overline{1,n})$*, the unique solution of the linear fractional initial value problem*

$$\begin{cases} \mathcal{D}_{0^+}^{r_{1m};\varphi}\psi_p\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t) - \mathcal{I}_{0^+}^{\sigma;\varphi} g_m(t)\right)\right] = h_m(t), \\ \qquad m = \overline{1,n}, \ \ and \ \ t \in J = (0,1] \\ \qquad \psi_{p_m}\left[\mathcal{D}_{0^+}^{r_2;\varphi}\left(u_m(t) - \mathcal{I}_{0^+}^{\sigma;\varphi} g_m(t)\right)\right]\Big|_{t=0} = 0, \\ u_m(0) = 0, \ \ u_m(1) = \sum\limits_{i=1}^{n}\lambda_{im}u_i(\zeta_{im}), \ \zeta_{im} \in (0,1] \\ \qquad \varphi(1) - \varphi(0) = K > 0. \end{cases}$$

is given by

$$\begin{aligned} & u_m(t) \\ = \ & \int_0^t \varphi_{r_{2m}}(t,s)\,\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)\,h_m(e)de\right]ds + \int_0^t \varphi_\sigma(t,s)\,g_m(s)ds \\ & -(\varphi(t) - \varphi(0))\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s \varphi_{r_{1m}}(s,e)\,h_m(e)de\right]ds \\ & +(\varphi(t) - \varphi(0))\left(\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_i(\zeta_{im}) - \frac{g_m(0)}{K}\right). \end{aligned} \tag{4}$$

Proof. For $0 < r_{1m} < 1 < r_{2m} < 2,$ Lemma 2.5 yields

$$\psi_{q_m}\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t) - \mathcal{I}_{0^+}^{\sigma;\varphi} g_m(t)\right)\right] = \mathcal{I}_{0^+}^{r_{1m};\varphi}h_m(t) + c_{1m}$$

by conditions $\psi_{q_m}\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}(u_m(t) - g_m(t)\right]\Big|_{t=0} = 0$, we gat $c_{1m} = 0$. Then

$$\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t) - \mathcal{I}_{0^+}^{\sigma;\varphi} g_m(t)\right)\right] = \psi_q\left[\mathcal{I}_{0^+}^{r_{1m};\varphi}h_m(t)\right]$$

so

$$u_m(t) = \mathcal{I}_{0^+}^{r_{2m};\varphi}\left[\psi_{q_m}\left[\mathcal{I}_{0^+}^{r_{1m};\varphi}h_m(t)\right]\right] + \mathcal{I}_{0^+}^{\sigma;\varphi} g_m(t) + c_{2m}(\varphi(t) - \varphi(0)),$$

by conditions $u_m(0) = 0$, and $u_m(1) = \sum\limits_{i=1}^{n}\lambda_{im}u_i(\zeta_{im})$, we gat

$$c_{2m} = \sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_i(\zeta_{im}) - \frac{g_m(0)}{K} - \mathcal{I}_{0^+}^{r_{2m};\varphi}\left[\psi_{q_m}\left[\mathcal{I}_{0^+}^{r_{1m};\varphi}h_m(t)\right]\right]\Big|_{t=1}. \tag{5}$$

□

3. Main Results

Taking into account Lemma 2.13, we define an operator $\mathcal{T}: C^n \to C^n$

$$\mathcal{T}(u_1,\ldots,u_n)(t) = (\mathcal{T}_1(u_1,\ldots,u_n)(t),\ldots,\mathcal{T}_n(u_1,\ldots,u_n)(t)), \tag{6}$$

where

$$\begin{aligned} & \mathcal{T}_m(u_1,\ldots,u_n)(t) \\ = & \int_0^t \varphi_{r_{2m}}(t,s)\,\psi_{q_m}\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\ldots,u_n(e))de\right]ds \\ & + \int_0^t \varphi_\sigma(t,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds \\ & -(\varphi(t)-\varphi(0))\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\ldots,u_n(e))de\right]ds \\ & +(\varphi(t)-\varphi(0))\left(\sum_{i=1}^n \frac{\lambda_{im}}{K}u_i(\zeta_{im}) - \frac{G_m(0,\ldots,0)}{K}\right), \quad m=\overline{1,n}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \varphi_{r_{2m}}(t,s) &= \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_{2m}-1}}{\Gamma(r_{2m})}, \varphi_{r_{1m}}(t,s) = \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r_{1m}-1}}{\Gamma(r_{1m})}, \\ \text{and} \quad \varphi_\sigma(t,s) &= \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\sigma-1}}{\Gamma(\sigma)}. \end{aligned}$$

For the sake of convenience, we use the following notations (for $m=\overline{1,n}$) :

$$\begin{aligned} \mathcal{K}_{1m} &= \frac{2^{q_m-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{k_m K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q_m-1}, \\ \mathcal{K}_{2m} &= \left(\frac{K^\sigma l_m}{\Gamma(1+\sigma)} + \sum_{i=1}^n |\lambda_{im}|\right), \\ \mathcal{K}_{3m} &= \frac{2^{q_m-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{\mathcal{N}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q_m-1} + \frac{K^\sigma\mathcal{M}}{\Gamma(1+\sigma)} + \mathcal{M}, \\ \mathcal{K}_{4m} &= \left(\frac{K^{r_{1m}}(\rho\mathcal{A}_m+\mathcal{N})}{\Gamma(1+r_{1m})}\right)^{q_m-2}, \\ \mathcal{K}_{5m} &= \frac{(q-1)(1+K)K^{r_{2m}+r_{1m}}\mathcal{A}_m\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})} + \frac{K^\sigma\mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^n |\lambda_{im}|. \end{aligned}$$

3.1. An Existence and Uniqueness Result

Here, by using the Banach contraction mapping principle, we prove an existence and uniqueness result.

Theorem 3.1. *Let $H_m, G_m : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ two continuous functions which satisfy the condition*
($\mathbb{A}_1$) there exist positive real constants $\mathcal{A}_m, \mathcal{B}_m$ such that, for all $t\in[0,1]$ and $u_i, v_i \in \mathbb{R}, i, m = \overline{1,n}$, we have

$$|H_m(t,u_1,\ldots,u_n) - H_m(t,v_1,\ldots,v_n)| \le \mathcal{A}_m\left(\sum_{i=1}^n |u_i - v_i|\right),$$

$$|G_m(t,u_1,\dots,u_n)-G_m(t,v_1,\dots,v_n)|\leq \mathcal{B}_m\left(\sum_{i=1}^{n}|u_i-v_i|\right).$$

Then, system (1) admits a unique solution on $[0,1]$ *provided that*

$$\sum_{m=1}^{n}\mathcal{K}_{2m}<\frac{1}{n+2},\ \textit{and}\ \sum_{m=1}^{n}\mathcal{K}_{5m}<1 \tag{8}$$

is valid.

Proof. We transform system (1) into a fixed point problem, $(u_1,\dots,u_n)(z)=\mathcal{T}(u_1,\dots,u_n)(z)$, where the operator $\mathcal{T}$ is defined as in (6). Applying the Banach contraction mapping principle (Lemma 2.9), we show that the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of system (1).

Let $\sup\limits_{t\in[0,1]}H_m(t,0,\dots,0)=\mathcal{N}<\infty$, and $\sup\limits_{t\in[0,1]}G_m(t,0,\dots,0)=\mathcal{M}<\infty$. Next, we set $\mathbb{U}\rho=\{(u_1,\dots,u_n)\in C^n,\|(u_1,\dots,u_n)\|\leq\rho\}$,

in which

$$\rho\geq\max\left\{\left((n+2)\sum_{m=1}^{n}\mathcal{K}_{1m}\right)^{\frac{1}{2-q_m}},(n+2)\sum_{m=1}^{n}\mathcal{K}_{3m}\right\}.$$

Observe that $\mathbb{U}\rho$ is a bounded, closed, and convex subset of C. First, we show that $\mathcal{T}\mathbb{U}\rho\subset\mathbb{U}\rho$.

For any $(u_1,\dots,u_n)\in\mathbb{U}\rho$, $t\in[0,1]$, using the condition $(\mathbb{A}_1)$, we have

$$\begin{aligned}|H_m(t,u_1,\dots,u_n)| &\leq |H_m(t,u_1,\dots,u_n)-H_m(e,0,\dots,0)|+|H_m(t,0,\dots,0)|\\ &\leq k_m\left(\sum_{i=1}^{n}|u_i|\right)+N\leq\rho\mathcal{A}_m+\mathcal{N},\end{aligned}$$

and

$$|G_m(t,u_1,\dots,u_n)|\leq\rho\mathcal{B}_m+\mathcal{M}.$$

Then, we obtain

$$\begin{aligned}&|\mathcal{T}_m(u_1,\dots,u_n)(t)|\\ \leq\ &\left|\int_0^t\varphi_{r_{2m}}(t,s)\,\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\right|\\ &+\left|\int_0^t\varphi_{\sigma}(t,s)\,G_m(s,u_1(s),\dots,u_n(s))ds\right|\\ &+\left|(\varphi(t)-\varphi(0))\right|\left|\int_0^1\varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\right|\\ &+\left|(\varphi(t)-\varphi(0))\right|\left(\sum_{i=1}^{n}\frac{|\lambda_{im}|}{K}|u_i(\zeta_{im})|+\frac{|G_m(0,\dots,0,0)|}{K}\right),\end{aligned}$$

by Lemma 2.8 we get

$$\begin{aligned}&|\mathcal{T}_m(u_1,\dots,u_n)(t)|\\ \leq\ &\frac{(K+1)\,K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]\right|\\ &+\frac{K^{\sigma}(\rho\mathcal{B}_m+\mathcal{M})}{\Gamma(1+\sigma)}+\left(\sum_{i=1}^{n}\rho|\lambda_{im}|+\mathcal{M}\right),\end{aligned}$$

and by $\psi_q(z) = |z|^{q-2} z$, we have

$$
\begin{aligned}
& |\mathcal{T}_m(u_1, \ldots, u_n)(t)| \\
\leq\; & \left(\frac{(\rho \mathcal{A}_m + \mathcal{N}) K^{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{q_m - 1} \frac{(K+1) K^{r_{2m}}}{\Gamma(1 + r_{2m})} + \frac{K^{\sigma}(\rho \mathcal{B}_m + \mathcal{M})}{\Gamma(1+\sigma)} + \left(\sum_{i=1}^{n} \rho |\lambda_{im}| + \mathcal{M} \right) \\
\leq\; & \frac{(K+1) K^{r_{2m}}}{\Gamma(1 + r_{2m})} \left(\frac{K^{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{q_m - 1} (\rho \mathcal{A}_m + \mathcal{N})^{q_m - 1} \\
& + \left(\frac{K^{\sigma} \mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^{n} |\lambda_{im}| \right) \rho + \left(\frac{K^{\sigma} \mathcal{M}}{\Gamma(1+\sigma)} + \mathcal{M} \right).
\end{aligned}
$$

Thanks to Lemma 2.12 , for all $m = \overline{1, n}$ we get

$$
\begin{aligned}
& |\mathcal{T}_m(u_1, \ldots, u_n)(t)| \\
\leq\; & \frac{2^{q_m - 2}(K+1) K^{r_{2m}}}{\Gamma(1 + r_{2m})} \left(\frac{K^{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{q_m - 1} \left(\mathcal{A}_m^{q_m - 1} \rho^{q_m - 1} + \mathcal{N}^{q_m - 1} \right) \\
& + \left(\frac{K^{\sigma} \mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^{n} |\lambda_{im}| \right) \rho + \frac{K^{\sigma} \mathcal{M}}{\Gamma(1+\sigma)} + \mathcal{M} \\
\leq\; & \frac{2^{q_m - 2}(K+1) K^{r_{2m}}}{\Gamma(1 + r_{2m})} \left(\frac{\mathcal{A}_m K^{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{q_m - 1} \rho^{q_m - 1} + \left(\frac{K^{\sigma} \mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^{n} |\lambda_{im}| \right) \rho \\
& + \frac{2^{q_m - 2}(K+1) K^{r_{2m}}}{\Gamma(1 + r_{2m})} \left(\frac{\mathcal{N} K^{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{q_m - 1} + \frac{K^{\sigma} \mathcal{M}}{\Gamma(1+\sigma)} + \mathcal{M} \\
\leq\; & \mathcal{K}_{1m} \rho^{q_m - 1} + \mathcal{K}_{2m} \rho + \mathcal{K}_{3m}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \|\mathcal{T}(u_1, \ldots, u_n)\| \\
\leq\; & \sum_{m=1}^{n} \left(\mathcal{K}_{1m} \rho^{q_m - 1} + \mathcal{K}_{2m} \rho + \mathcal{K}_{3m} \right) \leq \rho,
\end{aligned}
\tag{9}
$$

which gives us $\mathcal{T} \mathbb{U}\sigma \subset \mathbb{U}\sigma$.

Next, we show that $\mathcal{T} : C^n \to C^n$ is a contraction.

Using condition $(\mathbb{A}_1)$, for any $(u_1, \ldots, u_n), (v_1, \ldots, v_n) \in C^n$ and for each $t \in [0, 1]$, we have

$$\begin{aligned}
& |\mathcal{T}_m(u_1, \ldots, u_n) - \mathcal{T}_m(v_1, \ldots, v_n)| \\
\leq\ & \left| \int_0^t \varphi_{r_{2m}}(t,s)\, \psi_q \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, u_1(e), \ldots, u_n(e)) de \right] ds - \right. \\
& \left. \int_0^t \varphi_{r_{2m}}(t,s)\, \psi_q \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, v_1(e), \ldots, v_n(e)) de \right] ds \right| \\
& + \left| \int_0^t \varphi_\sigma(t,s) \left(G_m(s, u_1(s), \ldots, u_n(s)) - G_m(s, v_1(s), \ldots, v_n(s)) \right) ds \right| \\
& + K \left| \int_0^1 \varphi_{r_{2m}}(1,s)\, \psi_q \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, u_1(e), \ldots, u_n(e)) de \right] ds \right. \\
& \left. - \int_0^1 \varphi_{r_{2m}}(1,s)\, \psi_q \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, v_1(e), \ldots, v_n(e)) de \right] ds \right| \\
& + \sum_{i=1}^n |\lambda_{im}| \left| u_i(\zeta_{im}) - v_i(\zeta_{im}) \right|,
\end{aligned}$$

by Lemma 2.8 and Lemma 2.11, we get

$$\begin{aligned}
& |\mathcal{T}_m(u_1, \ldots, u_n) - \mathcal{T}_m(v_1, \ldots, v_n)| \\
\leq\ & \frac{(1+K)\, K^{r_{2m}}}{\Gamma(1+r_{2m})} \left| \psi_{q_m} \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, u_1(e), \ldots, u_n(e)) de \right] - \right. \\
& \left. \psi_{q_m} \left[\int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, v_1(e), \ldots, v_n(e)) de \right] \right| \\
& + \left(\frac{K^\sigma \mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^n |\lambda_{im}| \right) \sum_{i=1}^n |u_i - v_i| \\
\leq\ & \frac{(1+K)\, K^{r_{2m}} (q_m - 1) \mathcal{K}_{4m}}{\Gamma(1+r_{2m})} \left| \int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, u_1(e), \ldots, u_n(e)) de - \right. \\
& \left. \int_0^s \varphi_{r_{1m}}(s,e)\, H_m(e, v_1(e), \ldots, v_n(e)) de \right| + \frac{K^\sigma \mathcal{B}_m}{\Gamma(1+\sigma)} \sum_{i=1}^n |u_i - v_i| \\
\leq\ & \left[\frac{(q_m - 1) \mathcal{A}_m (1+K) K^{r_{2m}+r_{1m}} \mathcal{K}_{4m}}{\Gamma(1+r_{2m})\, \Gamma(1+r_{1m})} + \frac{K^\sigma \mathcal{B}_m}{\Gamma(1+\sigma)} + \sum_{i=1}^n |\lambda_{im}| \right] \sum_{i=1}^n |u_i - v_i| \\
\leq\ & \mathcal{K}_{5m} \sum_{i=1}^n |u_i - v_i|.
\end{aligned}$$

Hence,

$$\begin{aligned}
& |\mathcal{T}(u_1, \ldots, u_n) - \mathcal{T}(v_1, \ldots, v_n)| \\
\leq\ & \left(\sum_{m=1}^n \mathcal{K}_{5m} \right) \sum_{i=1}^n |u_i - v_i|.
\end{aligned}$$

Since $\left(\sum_{m=1}^n \mathcal{K}_{5m} \right) < 1$, by (8), the operator $\mathcal{T}$ is a contraction. Therefore, using the Banach contraction mapping principle (Lemma 2.9), the operator $\mathcal{T}$ has a unique fixed point. Hence, system (1) has a unique solution on $[0, 1]$. The proof is completed. □

3.2. *An Existence Result*

Now we apply Krasnoselskii fixed point theorem (Lemma 2.10) to prove our second existence result. So, consider the following operator

$$\begin{aligned}\mathcal{T}(u_1,\dots,u_n))(t) &= (\mathcal{T}_1(u_1,\dots,u_n)(t),\dots,\mathcal{T}_2(u_1,\dots,u_n)(t))\\ &= \mathcal{P}_1(u_1,\dots,u_n)(t)+\mathcal{P}_2(u_1,\dots,u_n)(t),\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}_1(u_1,\dots,u_n)(t) &= (\mathcal{P}_{11}(u_1,\dots,u_n)(t),\dots,\mathcal{P}_{1n}(u_1,\dots,u_n)(t))\\ \mathcal{P}_2(u_1,\dots,u_n)(t) &= (\mathcal{P}_{21}(u_1,\dots,u_n)(t),\dots,\mathcal{P}_{2n}(u_1,\dots,u_n)(t))\end{aligned}$$

and

$$\begin{aligned}&\mathcal{P}_{1m}(u_1,\dots,u_n)(t)\\ =\;& \int_0^t \varphi_{r_{2m}}(t,s)\,\psi_{q_m}\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\\ &-(\varphi(t)-\varphi(0))\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\end{aligned}$$

and

$$\begin{aligned}&\mathcal{P}_{2m}(u_1,\dots,u_n)(t)\\ =\;& \int_0^t \varphi_{\sigma}(t,s)\,G_m(s,u_1(s),\dots,u_n(s))ds\\ &+(\varphi(t)-\varphi(0))\left(\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_i(\zeta_{im})-\frac{G_m(0,u_1(0),\dots,u_n(0))}{K}\right).\end{aligned}$$

Theorem 3.2. *Let $H_m, G_m : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ be continuous functions which satisfy condition $(\mathbb{A}_1)$ in Theorem 3.1. In addition, we assume that there exist two positive constants $\Upsilon_{1m}, \Upsilon_{2m}$ such that, for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$, $i, m = \overline{1,n}$, we have.*

$$\begin{aligned}|H_m(t,u_1,\dots,u_n)| &\le \Upsilon_{1m},\\ |G_m(t,u_1,\dots,u_n)| &\le \Upsilon_{2m}.\end{aligned}$$

Moreover, assume that

$$\sum_{i=1}^{n}\sum_{m=1}^{n}|\lambda_{im}|\le 1,\ \textit{and}\ \left(\sum_{m=1}^{n}\frac{(q_m-1)\,\mathcal{A}_m\,(1+K)\,K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\,\Gamma(1+r_{1m})}\right)<1.$$

Then, problem (1) admits at least one solution on $[0,1]$

Proof. The proof will be given in several steps. Let $\mathbb{U}_\delta = \{(u_1,\dots,u_n)\in C^n, \|(u_1,\dots,u_n)\|\le\delta\}$, in which

$$\delta \ge \left[\frac{\sum\limits_{m=1}^{n}\left(\left(\frac{\Upsilon_{1m}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q_m-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}+\frac{K^{\sigma}\Upsilon_{2m}}{\Gamma(1+\sigma)}+\mathcal{M}\right)}{1-\sum\limits_{i=1}^{n}\sum\limits_{m=1}^{n}|\lambda_{im}|}\right].$$

First step: We prove that

$$|(\mathcal{T}(u_1,\dots,u_n))(t)|\le\delta.$$

Let $(u_1,\dots,u_n)\in\mathbb{U}_\delta$. As in the proof of Theorem 3.1, we have

$$\begin{aligned}&|\mathcal{P}_{1m}(u_1,\dots,u_n)(t)+\mathcal{P}_{2m}(u_1,\dots,u_n)(t)|\\ \leq\;&\left(\frac{\Upsilon_{1m}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q_m-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}+\frac{K^{\sigma}\Upsilon_{2m}}{\Gamma(1+\sigma)}+\mathcal{M}+\left(\sum_{i=1}^{n}|\lambda_{im}|\right)\delta.\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{T}(u_1,\dots,u_n)(t)\| &= \|(\mathcal{T}_1(u_1,\dots,u_n)(t),\dots,\mathcal{T}_3(u_1,\dots,u_n)(t))\|\\ &\leq \sum_{i=1}^{n}\|\mathcal{T}_i(u_1,\dots,u_n)(t)\|\\ &\leq \sum_{m=1}^{n}\sup_{t\in[0,1]}|\mathcal{P}_{1m}(u_1,\dots,u_n)(t)+\mathcal{P}_{2m}(u_1,\dots,u_n)(t)|\\ &\leq \sum_{m=1}^{n}\left(\left(\frac{\Upsilon_{1m}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q_m-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}+\frac{K^{\sigma}\Upsilon_{2m}}{\Gamma(1+\sigma)}+\mathcal{M}\right)\\ &\quad+\left(\sum_{m=1}^{n}\sum_{i=1}^{n}|\lambda_{im}|\right)\delta\\ &\leq \delta.\end{aligned}$$

Accordingly, $\mathcal{T}\mathbb{U}_\delta\subset\mathbb{U}_\delta$ and the condition (i) of Lemma 2.10 is satisfied.

Second Step : $\mathcal{P}_1$ is a contraction.

Let $(u_1,\dots,u_n),(v_1,\dots,v_n)\in\mathbb{U}_\delta$, we have the following estimate

$$\begin{aligned}&|\mathcal{P}_{1m}(u_1,\dots,u_n)(t)-\mathcal{P}_{1m}(v_1,\dots,v_n)(t)| \qquad (10)\\ \leq\;&\left|\int_0^t\varphi_{r_{2m}}(t,s)\,\psi_q\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\right.\\ &\left.-\int_0^t\varphi_{r_{2m}}(t,s)\,\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),\dots,v_n(e))de\right]ds\right|\\ &+K\left|\int_0^1\varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]ds\right.\\ &\left.-\int_0^1\varphi_{r_{2m}}(1,s)\,\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),\dots,v_n(e))de\right]ds\right|\\ \leq\;&\frac{(1+K)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\psi_{q_m}\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),\dots,u_n(e))de\right]-\right.\\ &\left.\psi_q\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),\dots,v_n(e))de\right]\right|\\ \leq\;&\frac{(q_m-1)\,\mathcal{A}_m(1+K)K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}\sum_{i=1}^{n}|u_i-v_i|.\end{aligned}$$

So

$$\begin{aligned}&|\mathcal{P}_1(u_1,\dots,u_n)(t)-\mathcal{P}_1(v_1,\dots,v_n)(t)|\\ \leq\;&\left(\sum_{m=1}^{n}\frac{(q_m-1)\,\mathcal{A}_m(1+K)K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}\right)\sum_{i=1}^{n}|u_i-v_i|.\end{aligned}$$

Since $\left(\sum_{m=1}^{n}\frac{(q_m-1)\mathcal{A}_m(1+K)K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}\right)<1$, the operator $\mathcal{P}_1$ is a contraction.

Third Step : $\mathcal{P}_2$ is compact and continuous.

Since H_m, G_m are a continuous functions, this implies that the operator $\mathcal{P}_2$ is continuous on $\mathbb{U}_\delta$. Moreover, $\mathcal{P}_2(u_1,\ldots,u_n)$ is uniformly bounded by (9). Next, we show equicontinuity. Let $(u_1,\ldots,u_n)\in\mathbb{U}_\delta$, we have

$$\begin{aligned}
&|\mathcal{P}_{2m}(u_1,\ldots,u_n)(t)|\\
\leq\ & \left|\int_0^t \varphi_\sigma(t,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds\right|\\
&+K\left(\sum_{i=1}^{n}\frac{|\lambda_{im}|}{K}|u_i(\zeta_{im})|+\frac{|G_m(0,u_1(0),\ldots,u_n(0))|}{K}\right)\\
\leq\ & \left(\frac{\Upsilon_{2m}K^\sigma}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|+\mathcal{M}\right)\delta.
\end{aligned}$$

So

$$|\mathcal{P}_2(u_1,\ldots,u_n)(t)|\leq\sum_{m=1}^{n}\left(\frac{\Upsilon_{2m}K^\sigma}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|+\mathcal{M}\right)\delta. \tag{11}$$

Moreover, $\mathcal{P}_2(u_1,\ldots,u_n)$ is uniformly bounded by (11). Next, we show equicontinuity.

and $t_1,t_2\in[0,1]$ such that , $t_1<t_2$ we have

$$\begin{aligned}
&|\mathcal{P}_{2m}(u_1,\ldots,u_n)(t_2)-\mathcal{P}_{2m}(u_1,\ldots,u_n)(t_1)|\\
\leq\ & \left|\int_0^{t_2}\varphi_\sigma(t_2,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds-\int_0^{t_1}\varphi_\sigma(t_1,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds\right|\\
\leq\ & \frac{\Upsilon_{2m}}{\Gamma(1+\sigma)}(\varphi(t_2)-\varphi(t_1))^\sigma.
\end{aligned}$$

So

$$\begin{aligned}
&|\mathcal{P}_2(u_1,\ldots,u_n)(t_2)-(\mathcal{P}_2(u_1,\ldots,u_n)(t_1)|\\
\leq\ & \left|\int_0^{t_2}\varphi_\sigma(t_2,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds-\int_0^{t_1}\varphi_\sigma(t_1,s)\,G_m(s,u_1(s),\ldots,u_n(s))ds\right|\\
\leq\ & \sum_{m=1}^{n}\left(\frac{\Upsilon_{2m}}{\Gamma(1+\sigma)}\right)(\varphi(t_2)-\varphi(t_1))^\sigma.
\end{aligned}$$

Consequently,

$$|\mathcal{P}_2(u_1,\ldots,u_n)(t_2)-\mathcal{P}_2(u_1,\ldots,u_n)(t_1)|\to 0,\ \text{ as } t_1\to t_2.$$

This shows that $\mathcal{P}_2\mathbb{U}_\delta$ is equicontinuous. Hence, by Arzelià-Ascoli theorem $\mathcal{P}_2$ is completely continuous on $\mathbb{U}_\delta$. As a consequence of Krasnoselskii's fixed point theorem, we conclude that has a fixed point which is a solution of (1). The proof of Theorem 3.2 is thus completely achieved. □

3.3. An Illustrative example

Example 3.3. *Consider the following nonlinear equation for all* $t \in (0,1]$, $n = 3, p_m = 2$

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};t^2}\psi_2\left[{}^{c}\mathcal{D}_{0^+}^{\frac{1}{4};t^2}\left(u(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{e^t+1}{1+(u(t))^2}\right)\right)\right] = \frac{1}{1+t^2}\left(\frac{u(t)}{1+(u(t))^2}\right), \\ {}^{c}\mathcal{D}_{0^+}^{\frac{1}{4};t^2}\psi_2\left[{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};t^2}\left(v(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{t^2+1}{1+(v(t))^2}\right)\right)\right] = \frac{t}{1+e^t}\left(\frac{v(t)}{1+(v(t))^2}\right), \\ {}^{c}\mathcal{D}_{0^+}^{\frac{3}{4};t^2}\psi_2\left[{}^{c}\mathcal{D}_{0^+}^{\frac{3}{4};t^2}\left(w(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{e^t}{1+(w(t))^2}\right)\right)\right] = \frac{e^t}{1+t^2}\left(\frac{w(t)}{1+(w(t))^2}\right), \\ \psi_2\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(u(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{e^t+1}{1+(u(t))^2}\right)\right)\right]\Big|_{t=0} = 0, \\ \psi_2\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(v(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{t^2+1}{1+(v(t))^2}\right)\right)\right]\Big|_{t=0} = 0, \\ \psi_2\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(w(t) - \mathcal{I}_{0^+}^{\frac{3}{2};t^2}\left(\frac{e^t}{1+(w(t))^2}\right)\right)\right]\Big|_{t=0} = 0, \\ u(0) = v(0) = w(0) = 0, \\ u(1) = \sum_{i=1}^{n}\frac{1}{7(i!)}u(\zeta_i), v(1) = \sum_{i=1}^{n}\frac{1}{9(i!)}v(\zeta_i), w(1) = \sum_{i=1}^{n}\frac{1}{11(i!)}w(\zeta_i),\ \zeta_i \in (0,1] \end{cases} \tag{12}$$

and

$$\begin{aligned} K &= 1, \\ \Upsilon_{11} &= \mathcal{A}_1 = \frac{1}{2}, \Upsilon_{12} = \mathcal{A}_2 = \frac{1}{4}, \Upsilon_{13} = \mathcal{A}_3 = \frac{e}{2}, \\ \Upsilon_{21} &= \mathcal{B}_1 = \frac{1+e}{2}, \Upsilon_{22} = \mathcal{B}_2 = 1, \Upsilon_{23} = \mathcal{B}_3 = \frac{e}{2}. \end{aligned}$$

Thus, the assumptions ($\mathbb{A}_1$) *are satisfied and Theorem 3.1-3.2 implies that (12) has a unique solution on* $[0,1]$.

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