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Subordination results for some subclasses of analytic functions using generalized q-Dziok-Srivastava-Catas operator

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Abstract. We introduce two classes of analytic functions related to conic domains, using a new generalized q-Dziok-Srivastava-Catas operator $\mathfrak{D}_{q,\tau,\ell}^{m,s,r}(m \in \mathbb{N}_0 = \{0, 1, ..\}, r \leq s+1; r, s \in \mathbb{N}_0, 0 < q < 1, \tau \geq 0, \ell \geq 0)$. Basic properties of these classes are studied, such as coefficients estimate. For these new function classes, we establish subordination theorems and also, point out some new and known consequences of our main results.

1. Introduction and Preliminaries

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

For functions $f(z) \in A$, given by (1.1), and $g(z) \in A$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \qquad (z \in \mathbb{U}).$$

Definition 1.1. [10, Chapter 6, p. 190] (Subordination Principle). *For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write*

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

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if there exists a Schwarz function $\varphi(z)$, analytic in $\mathbb{U}$ with

$$\varphi(0) = 0 \qquad \textit{and} \qquad |\varphi(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\varphi(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \textit{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Given $\eta\,(0 \leq \eta < 1)$, a function $f \in A$ is said to be in the class of starlike functions of order η in $\mathbb{U}$, denoted by $\mathcal{ST}(\eta)$, (see [32]) if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \eta, \quad (z \in \mathbb{U}, 0 \leq \eta < 1).$$

On the other hand, a function $f \in A$ is said to be in the class $\mathcal{CV}(\eta)$ of convex functions of order η in $\mathbb{U}$ if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \eta, \quad (z \in \mathbb{U}, 0 \leq \eta < 1).$$

In particular, the classes $\mathcal{CV} \equiv \mathcal{CV}(0)$ and $\mathcal{ST} \equiv \mathcal{ST}(0)$ are, respectively, the familiar classes of convex and starlike functions in $\mathbb{U}$.

A function $f \in A$ is said to be in the class of uniformly convex functions of order η and type δ, denoted by $\mathcal{UCV}(\delta, \eta)$ (see [7]) if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \eta\right\} > \delta\left|\frac{zf''(z)}{f'(z)}\right|,$$

where $\delta \geq 0$, $\eta \in [-1, 1)$ and $\delta + \eta \geq 0$ and is said to be in a corresponding class denoted by $\mathcal{SP}(\delta, \eta)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \eta\right\} > \delta\left|\frac{zf'(z)}{f(z)} - 1\right|,$$

where $\delta \geq 0, \eta \in [-1, 1)$and $\delta + \eta \geq 0$.

It is obvious that $f(z) \in \mathcal{UCV}(\delta, \eta)$ if and only if $zf'(z) \in \mathcal{SP}(\delta, \eta)$. These classes generalize various other classes. For $\delta = 0$, we get, respectively, the classes $\mathcal{CV}(\eta)$ and $\mathcal{ST}(\eta)$. The class $\mathcal{UCV}(1, 0) \equiv \mathcal{UCV}$ is called uniformly convex functions introduced by Goodman with geometric interpretation in [16]. The class $\mathcal{SP}(1, 0) \equiv \mathcal{SP}$ is defined by Ronning in [33]. The classes $\mathcal{UCV}(1, \eta) \equiv \mathcal{UCV}(\eta)$ and $\mathcal{SP}(1, \eta) \equiv \mathcal{SP}(\eta)$ are investigated by Ronning in [34]. For $\eta = 0$, the classes $\mathcal{UCV}(\delta, 0) \equiv \delta - \mathcal{UCV}$ and $\mathcal{SP}(\delta, 0) \equiv \delta - \mathcal{SP}$, respectively, are defined by Kanas and Wisniowska in [23] and [24](see also Kanas and Srivastava [22]).

Geometric interpretation [2] (see also [42]). $f \in \mathcal{UCV}(\delta, \eta)$ and $f \in \mathcal{SP}(\delta, \eta)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ and $\frac{zf'(z)}{f(z)}$, respectively, take all the values in the conic domain $R_{\delta,\eta}$ which is included in the right half plane such that

$$R_{\delta,\eta} = \left\{u + iv : u > \delta\sqrt{(u-1)^2 + v^2} + \eta\right\}. \tag{2}$$

Denote by $\rho(P_{\delta,\eta})(\delta \geq 0, -1 \leq \eta < 1)$, the family of functions p, such that $p \in \rho$ and $p \prec P_{\delta,\eta}$ in $\mathbb{U}$, where ρ denotes the well-known class of Caratheodory functions and the function $P_{\delta,\eta}$ maps the unit disc conformally onto the domain $R_{\delta,\eta}$ such that $1 \in R_{\delta,\eta}$ and $\partial R_{\delta,\eta}$ is a curve defined by the equality

$$\partial R_{\delta,\eta} = \left\{ u + iv : u^2 = \left(\delta \sqrt{(u-1)^2 + v^2} + \eta \right)^2 \right\}.$$

From elementary computations we see that $\partial R_{\delta,\eta}$ represents the conic sections symmetric about the real axis. Thus $R_{\delta,\eta}$ is an elliptic domain for $\delta > 1$, a parabolic domain for $\delta = 1$, a hyperbolic domain for $0 < \delta < 1$ and a right half plane $u > \eta$ for $\delta = 0$.

The functions, which play the role of extremal functions of the class $\rho(P_{\delta,\eta})$, were obtained in [2] as follows:

$$P_{\delta,\eta}(z) = \begin{cases} \frac{1+(1-2\eta)z}{1-z} & \delta = 0, \\ 1 + \frac{2(1-\eta)}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & \delta = 1, \\ \frac{1-\eta}{1-\delta^2} \cos \left\{ \left(\frac{2}{\pi} \cos^{-1} \delta \right) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{\delta^2-\eta}{1-\delta^2} & 0 < \delta < 1, \\ \frac{(1-\eta)}{\delta^2-1} \sin \left(\frac{\pi}{2K(t)} \right) \int\limits_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}} dx + \frac{\delta^2-\eta}{\delta^2-1} & \delta > 1, \end{cases} \tag{3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in \mathbb{U}$ and t is chosen such that $\delta = \cosh \frac{\pi K'(t)}{4K(t)}$, $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$.

For $\delta = 0$ obviously $P_{0,\eta}(z) = 1 + 2(1-\eta)z + 2(1-\eta)z^2 + ...$, for $\delta = 1$ (compare [28] and [34]) $P_{1,\eta}(z) = 1 + \frac{8}{\pi^2}(1-\eta)z + \frac{16}{3\pi^2}(1-\eta)z^2 + ...$, by comparing Taylor series expansion in [25], we get for $0 < \delta < 1$

$$P_{\delta,\eta}(z) = 1 + \frac{(1-\eta)}{1-\delta^2} \sum_{k=1}^{\infty} \left[\sum_{l=1}^{2k} 2^l \binom{B}{l} \binom{2k-1}{2k-l} \right] z^k,$$

where $B = \frac{2}{\pi} \cos^{-1} \delta$ and for $\delta > 1$,

$$P_{\delta,\eta}(z) = 1 + \frac{\pi^2(1-\eta)}{4\sqrt{t}(\delta^2-1)K^2(t)(1+t)} \times \left\{ z + \frac{4K^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}K^2(t)(1+t)} z^2 + ... \right\}.$$

In the recent years, practical applications of $q-$calculus (quantum calculus) in the fields of $q-$difference equation, optimal control, $q-$transform analysis and number theory are an efficient area of research. Jackson [19, 20] was the successful first to develop q-integral and $q-$derivative in a systematic way and later geometrical interpretation of the $q-$analysis has been recognized through studies of quantum groups.

Fractional calculus appears more and more frequently for the modelling of relevant systems in several fields of applied sciences. Fractional $q-$calculus is the $q-$extension of ordinary fractional calculus. Researchers have claimed it to construct and investigated several classes of analytic and bi-univalent functions and their interesting results are extremely numerous to discuss.

Definition 1.2. *Jackson [19] defined the $q-$derivative of a function $f(z)$ of the form (1) as follows*

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \qquad (z \neq 0), \tag{4}$$

where

$$[k]_q = \begin{cases} \sum_{l=0}^{k-1} q^l = 1 + q + q^2 + \cdots + q^{k-1} & (k \in \mathbb{N} = \{1, 2, ...\}) \\ 0 & (k = 0) \end{cases} \tag{5}$$

and

$$\lim_{q\to 1^-} D_q f(z) = f'(z).$$

A $q-$analog of the class of starlike functions was first introduced by Ismail et al. [18] by means of the $q-$difference operator $D_q f(z)$, $f(z) \in \mathrm{A}$ and $0 < q < 1$. Also, several authors studied many applications of q-calculus and it's generalization associated with various families of analytic and univalent (or multivalent) functions (for example see [26, 31, 39, 43, 45, 46]).

In 2004, Gasper and Rahman [15, Page 4] defined a $q-$hypergeometric series which is given by

$${}_r\varphi_s(\xi_1, ..., \xi_r, \zeta_1, ..., \zeta_s; q, z) = \sum_{k=0}^{\infty} \frac{(\xi_1; q)_k(\xi_2; q)_k...(\xi_r; q)_k}{(q; q)_k(\zeta_1; q)_k(\zeta_2; q)_k...(\zeta_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k,$$

where $\binom{k}{2} = \frac{k(k-1)}{2}$, $r, s \in \mathbb{N}_0$, $r \leq s+1$, $q \neq 0$, $\xi_j\,(j = 1, 2, ..., r)$ and $\zeta_j\,(j = 1, 2, ..., s)$ are complex numbers, $\zeta_j \neq q^{-n}(j = 1, 2, ..., s,\ n \in \mathbb{N}_0)$ are such that the denominator factors in the series are never zero.

Definition 1.3. *[37] For ν, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, q-shifted factorial is defined by*

$$(\nu; q)_0 = 1, \qquad (\nu; q)_k = \prod_{l=0}^{k-1}(1 - \nu q^l), \tag{6}$$

and in terms of the basic (or $q-$) gamma function

$$(q^\nu; q)_k = \frac{(1-q)^k \Gamma_q(\nu + k)}{\Gamma_q(\nu)} \qquad k \in \mathbb{N}_0,$$

where the $q-$gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{1-x}(q; q)_\infty}{(q^x; q)_\infty} \qquad (|q| < 1,\ x \in \mathbb{N}_0)$$

where

$$(\nu; q)_\infty = \prod_{l=0}^{\infty}(1 - \nu q^l) \qquad |q| < 1.$$

For the q-gamma function $\Gamma_q(x)$, it is known that (see [15])

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x)$$

where $[x]_q$ denotes by (5). It is also known that

$$\lim_{q\to 1^-} \frac{(q^\nu; q)_n}{(1-q)^n} = (\nu)_n = \nu(\nu+1)(\nu+2)...(\nu+n-1).$$

Note that the series ${}_r\varphi_s(\xi_1, ..., \xi_r, \zeta_1, ..., \zeta_s; q, z)$ converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. Further, note that

$$\lim_{q\to 1^-} {}_r\varphi_s(q^{\xi_1}, ..., q^{\xi_r}, q^{\zeta_1}, ..., q^{\zeta_s}; q, (q-1)^{1+s-r}, z) = {}_r\mathfrak{F}_s(\xi_1, ..., \xi_r, \zeta_1, ..., \zeta_s; z)$$

which is a well known generalized hypergeometric functions [15].

Corresponding to the function ${}_r\varphi_s(\xi_1,...,\xi_r,\zeta_1,...,\zeta_s;q,z)$ defined by

$$\begin{aligned}{}_rH_s(\xi_1,...,\xi_r,\zeta_1,...,\zeta_s;q,z) &= z\,{}_r\varphi_s(\xi_1,...,\xi_r,\zeta_1,...,\zeta_s;q,z)\\ &= z+\sum_{k=2}^{\infty}\frac{(\xi_1;q)_{k-1}(\xi_2;q)_{k-1}...(\xi_r;q)_{k-1}}{(q;q)_{k-1}(\zeta_1;q)_{k-1}(\zeta_2;q)_{k-1}...(\zeta_s;q)_{k-1}}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k.\end{aligned}$$

Bhardwaj and Sharma [8] are defined a linear operator $\mathfrak{H}_s^r(\xi_1;q)=\mathfrak{H}_s^r(\xi_i;\zeta_j;q,z):A\to A$ by

$$\mathfrak{H}_s^r(\xi_1;q)f(z)={}_rH_s(\xi_1,...,\xi_r,\zeta_1,...,\zeta_s;q,z)*f(z). \tag{7}$$

For a function $f(z)$ of the form (1), the series expansion of $\mathfrak{H}_s^r(\xi_1;q)f(z)$ is given by

$$\mathfrak{H}_s^r(\xi_1;q)f(z)=z+\sum_{k=2}^{\infty}\frac{(\xi_1;q)_{k-1}(\xi_2;q)_{k-1}...(\xi_r;q)_{k-1}}{(q;q)_{k-1}(\zeta_1;q)_{k-1}(\zeta_2;q)_{k-1}...(\zeta_s;q)_{k-1}}\left[(-1)^{k-1} q^{\binom{k-1}{2}}\right]^{1+s-r} a_k z^k \tag{8}$$

which converges absolutely in $\mathbb{U}$ if $r\le s+1$. The operator $\mathfrak{H}_s^r(\xi_1;q)$ is called a $q-$analogue of Dziok-Srivastava operator.

Let

$$\Gamma(\xi_1;q,k)=\frac{(\xi_1;q)_{k-1}(\xi_2;q)_{k-1}...(\xi_r;q)_{k-1}}{(q;q)_{k-1}(\zeta_1;q)_{k-1}(\zeta_2;q)_{k-1}...(\zeta_s;q)_{k-1}}\left[(-1)^{k-1} q^{\binom{k-1}{2}}\right]^{1+s-r},$$

then (8) reduces to

$$\mathfrak{H}_s^r(\xi_1;q)f(z)=z+\sum_{k=2}^{\infty}\Gamma(\xi_1;q,k)a_k z^k. \tag{9}$$

We introduce the linear extended $q-$analogue of Dziok-Srivastava-Catas operator $\mathfrak{D}_{q,\tau,\ell}^{m,s,r}$ as following:

$$\begin{aligned}\mathfrak{D}_{q,\tau,\ell}^{0,s,r}f(z) &= f(z)\\ \mathfrak{D}_{q,\tau,\ell}^{1,s,r}f(z) &= (1-\tau)\mathfrak{H}_s^r(\xi_1;q)f(z)+\frac{\tau}{[1+\ell]_q z^{\ell-1}}D_q(z^\ell\mathfrak{H}_s^r(\xi_1;q)f(z))=\mathfrak{D}_{q,\tau,\ell}^{s,r}f(z)\\ &= z+\sum_{k=2}^{\infty}\frac{[1+\ell]_q+\tau([k+\ell]_q-[1+\ell]_q)}{[1+\ell]_q}\Gamma(\xi_1;q,k)a_k z^k\\ &\ \ \vdots\\ \mathfrak{D}_{q,\tau,\ell}^{m,s,r}f(z) &= \mathfrak{D}_{q,\tau,\ell}^{s,r}(\mathfrak{D}_{q,\tau,\ell}^{m-1,s,r}f(z))\end{aligned} \tag{10}$$

where $m\in\mathbb{N}_0$, $\tau\ge0$ and $\ell\ge0$. It follows from (1) and (10) that

$$\mathfrak{D}_{q,\tau,\ell}^{m,s,r}f(z)=z+\sum_{k=2}^{\infty}\Theta_{q,k}^{m}(\tau,\ell,\xi_1)a_k z^k \tag{11}$$

where

$$\Theta_{q,k}^{m}(\tau,\ell,\xi_1)=\left[\frac{[1+\ell]_q+\tau([k+\ell]_q-[1+\ell]_q)}{[1+\ell]_q}\Gamma(\xi_1;q,k)\right]^m. \tag{12}$$

By virtue of (7) and (11), $\mathfrak{D}_{q,\tau,\ell}^{m,s,r}f(z)$ can be written in terms of convolution as follows:

$$\mathfrak{D}_{q,\tau,\ell}^{m,s,r}f(z)=\underbrace{\left[\left(\mathfrak{H}_s^r(\xi_1;q)\star\mathfrak{G}_{\ell,\tau}^{q}(z)\right)\star\cdots\star\left(\mathfrak{H}_s^r(\xi_1;q)\star\mathfrak{G}_{\ell,\tau}^{q}(z)\right)\right]}_{m-times}\star f(z)$$

where

$$\mathfrak{G}^{q}_{\ell,\tau}(z)=\frac{z-\left(1-\frac{q^{\ell}}{[1+\ell]_q}\tau\right)qz^2}{(1-z)(1-qz)}.$$

Remark 1.4. *Note that the operator $\mathfrak{D}^{m,s,r}_{q,\tau,\ell}$ generalizes several previously studied familiar operators, and we will mention some of the interesting particular cases as follows:*

(i) *For $r=2$, $s=1$, $\xi_1=q^2$, $\xi_2=q$, $\zeta_1=q^{2-\varrho}$, and $\ell=0$ we obtain the operator $\mathfrak{D}^{\varrho,m}_{q,\tau}$studied by Abelman et al. [1] (see also [27, with $\ell=0$]);*

(ii) *For $r=2$, $s=1$, $\xi_1=q^2$, $\xi_2=q$, $\zeta_1=q^{2-\varrho}$, $\tau=\ell=0$ and $m=1$ we obtain the operator $\mathfrak{D}^{\varrho}_{q,z}$studied by Purohit and Raina [29];*

(iii) *For $r=2$, $s=1$, $\xi_1=q^{1+\lambda}\,(\lambda>-1)$, $\xi_2=q$, $\zeta_1=q$, $\tau=\ell=0$ and $m=1$ we obtain the operator $\mathfrak{D}_{q,\lambda+1}$studied by Kanas and Raducanu [21];*

(iv) *For $r=2$, $s=1$, $\xi_1=\xi_2=q$, $\zeta_1=q^{n+1}\,(n>-1)$, $\ell=\tau=0$ and $m=1$ we obtain the operator $\mathfrak{F}^{n+1}_{q}$studied by Arif et al. [5];*

(v) *For $r=2$, $s=1$, $\xi_1=\xi_2=\zeta_1=q$ we obtain the operator $\mathfrak{I}^{m}_{q}(\tau,\ell)$studied by Aouf and Madian [4];*

(vi) *For $r=2$, $s=1$, $\xi_1=\xi_2=\zeta_1=q$, $\ell=0$ and $\tau=1$ we obtain the operator $\mathfrak{I}^{m}_{q}$studied by Govindaraj and Sivasubaramanian [17];*

(vii) *For $q\to 1^-$ we obtain the operator $\mathfrak{D}^{m,s,r}_{\tau,\ell}$ studied by El-Ashwah et al. [13];*

(viii) *For $q\to 1^-$, $r=2$, $s=1$, $\xi_1=2$, $\xi_2=1$, $\zeta_1=2-\alpha\,(\alpha\neq 2,3,4,\ldots)$ and $\ell=0$, we obtain the operator $\mathfrak{D}^{m,1,2}_{\tau,0}f(z)=\mathfrak{D}^{m,\alpha}_{\tau}f(z)$ studied by Al-Oboudi and Al-Amoudi [3];*

(ix) *For $q\to 1^-$, $r=2$, $s=1$, $\xi_1=a\,(a>0)$, $\xi_2=1$, $\zeta_1=c\,(c>0)$ and $\ell=0$, we obtain the operator $\mathfrak{D}^{m,1,2}_{\tau,0}f(z)=I^{m}_{\tau,a,c}f(z)$ studied by Prajapat and Raina [30];*

(x) *For $q\to 1^-$, $r=2$, $s=1$ and $\xi_1=\xi_2=\zeta_1=1$, we obtain the operator $\mathfrak{D}^{m,1,2}_{\tau,\ell}f(z)=I^{m}(\tau,\ell)f(z)$ studied by Catas [9];*

(xi) *For $q\to 1^-$, $m=1$ and $\tau=\ell=0$, we obtain the operator $\mathfrak{D}^{1,s,r}_{0,0}f(z)=H_{s,r}(\xi_1)$ studied by Dziok and Srivastava [11, 12].*

Making use of the linear extended $q-$analogue of Dziok-Srivastava-Catas operator given by (11), we introduce the subclass $\mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ of q-uniformly starlike functions of order η and type δ in $\mathbb{U}$ and the subclass $\mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ of q-uniformly convex functions of order η and type δ in $\mathbb{U}$ as follows:

$$\operatorname{Re}\left(\frac{zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)\right)}{\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)}-\eta\right)>\delta\left|\frac{zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)\right)}{\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)}-1\right| \tag{13}$$
$$(0<q<1,\ \delta\geq 0,\ \eta\in[-1,1),\ \eta+\delta\geq 0),$$

and

$$\operatorname{Re}\left(\frac{D_q\left(zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)\right)\right)}{D_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)\right)}-\eta\right)>\delta\left|\frac{D_q\left(zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z)\right)\right)}{D_q(\mathfrak{D}^{m,s,r}_{q,\tau,\ell}f(z))}-1\right| \tag{14}$$
$$(0<q<1,\ \delta\geq 0,\ \eta\in[-1,1),\ \eta+\delta\geq 0).$$

respectively, where $r, s \in \mathbb{N}_0$, $r \leq s+1$, $m \in \mathbb{N}_0$, $\tau \geq 0$, $\ell \geq 0$ and $f(z) \in \mathcal{A}$. From (13) and (14), it follows that

$$\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z) \in \mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) \Longleftrightarrow zD_q(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)) \in \mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta), \tag{15}$$

and

$$\mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) \subset \mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta).$$

Note that:

(i) $\mathfrak{Y}^{m,1,2}_{q,\tau,\ell}(q,q;q;\eta,0) = \mathcal{S}^*_{q,m}(\tau,\ell,\eta)$ (see [4]);

(ii) $\mathfrak{Y}^{1,1,2}_{q,0,0}(q^{\lambda+1},q;q;\eta,\delta) = \mathcal{ST}(\lambda,\delta,\eta,q)$ $(\lambda > -1)$ (see [21]);

(iii) $\mathfrak{Y}^{1,1,2}_{q,0,0}(q,q;q^{n+1};\eta,0) = \mathcal{Q}(n,\eta,q)$ $(n > -1)$ (see [5, with $A = 1-2\eta$ and $B = -1$]);

(iii) $\mathfrak{Y}^{m,1,2}_{q,1,0}(q,q;q;\eta,\delta) = \mathcal{S}_q(\eta,\delta,m)$ (see [17]);

(iv) $\lim_{q\to 1^-} \mathfrak{Y}^{m,s,r}_{\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) = \mathcal{SP}^{m,s,r}_{\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ and $\lim_{q\to 1^-} \mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) = \mathcal{UCV}^{m,s,r}_{\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ (see [13]);

(v) For $q \to 1^-$ and different choices of the parameters r, s, ξ_1, ζ_1, ℓ, τ, η, δ and m, we will obtain special subclasses which studied by various authors (see [3, 14, 35, 36, 41, 44]).

From geometric interpretation, (13) and (14), $f(z) \in \mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ and $f(z) \in \mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ if and only if $\mathcal{P}(z) = \frac{D_q\left(zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)\right)}{D_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)}$ and $\mathcal{P}(z) = \frac{zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)}{\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)}$, respectively, take all values in the conic domain $R_{\delta,\eta}$ given in (2) which is included in right half plane, we may rewrite the conditions (13) and (14) in the form

$$\mathcal{P} \prec P_{\delta,\eta}$$

where the functions $P_{\delta,\eta}$ given in (3).

By virtue of (13), (14) and the properties of the domain $P_{\delta,\eta}$, we have, respectively

$$\operatorname{Re}\left\{\frac{D_q\left(zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)\right)}{D_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)}\right\} > \frac{\delta+\eta}{1+\delta} \quad (z \in \mathbb{U}),$$

and

$$\operatorname{Re}\left\{\frac{zD_q\left(\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)\right)}{\mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z)}\right\} > \frac{\delta+\eta}{1+\delta} \quad (z \in \mathbb{U}),$$

which means that

$$f(z) \in \mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) \Rightarrow \mathfrak{D}^{m,s,r}_{q,\tau,\ell} f(z) \in \mathcal{ST}\left(\frac{\delta+\eta}{1+\delta}\right) \subseteq \mathcal{ST}.$$

and

$$f(z) \in \mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta) \Rightarrow \mathfrak{K}^{m,s,r}_{q,\tau,\ell} f(z) \in \mathcal{CV}\left(\frac{\delta+\eta}{1+\delta}\right) \subseteq \mathcal{CV},$$

Definition 1.5. (Subordinating Factor Sequence). *An infinite sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \qquad (z \in \mathbb{U};\, a_1 = 1). \tag{16}$$

A finite sequence $\{c_k\}_{k=1}^{n}$ is said to be a subordinating factor sequence if (1) implies (16) whenever $c_{n+1} = c_{n+2} = \ldots = 0$. The class of such infinite sequences, will be denote by $\mathcal{F}$, and the class of sequences of length n by $\mathcal{F}_n$.

Lemma 1.6. [47, p. 690, Theorem 2] *The sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is a subordinating factor sequence if and only if*

$$Re\left(1 + 2\sum_{k=1}^{\infty} c_k z^k\right) > 0 \qquad (z \in \mathbb{D}).$$

2. Main Results

Unless otherwise mentioned we shall assume throughout the paper that $0 < q < 1$, $-1 \leq \eta < 1$, $\delta \geq 0$, $\delta + \eta \geq 0$, $\tau, \ell \geq 0$, r, s, $m \in N_0$, $r \leq s+1$, $\xi_i (i = 1, 2, .., r)$, and $\zeta_j (j = 1, 2, ..., s)$ are positive and real.

First, we obtain sufficient conditions for a function to belong to the classes $\mathfrak{Y}_{q,\tau,\ell}^{m,s,r}(\xi_1; \zeta_1; \eta, \delta)$ and $\mathfrak{K}_{q,\tau,\ell}^{m,s,r}(\xi_1; \zeta_1; \eta, \delta)$.

Theorem 2.1. *A function $f(z)$ of the form* (1) *is in $\mathfrak{Y}_{q,\tau,\ell}^{m,s,r}(\xi_1; \zeta_1; \eta, \delta)$ if*

$$\sum_{k=2}^{\infty} \Big([k]_q(1+\delta) - (\delta+\eta)\Big) \left|\Theta_{q,k}^{m}(\tau, \ell, \xi_1)\right| |a_k| \leq 1 - \eta, \tag{17}$$

where $\Theta_{q,k}^{m}(\tau, \ell, \xi_1)$ is defined by (12).

Proof. It suffices to show that

$$\delta \left| \frac{zD_q\left(\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)\right)}{\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)} - 1 \right| - \mathrm{Re}\left(\frac{zD_q\left(\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)\right)}{\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)} - 1 \right) < 1 - \eta.$$

We have

$$\begin{aligned}
\delta \left| \frac{zD_q\left(\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)\right)}{\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)} - 1 \right| \quad &- \quad \mathrm{Re}\left(\frac{zD_q\left(\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)\right)}{\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)} - 1 \right) \\
&\leq \quad (1+\delta) \left| \frac{zD_q\left(\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)\right)}{\mathfrak{D}_{q,\tau,\ell}^{m,s,r} f(z)} - 1 \right| \\
&\leq \quad \frac{(1+\delta)\sum_{k=2}^{\infty}([k]_q - 1)\left|\Theta_{q,k}^{m}(\tau, \ell, \xi_1)\right| |a_k||z|^{k-1}}{1 - \sum_{k=2}^{\infty}\left|\Theta_{q,k}^{m}(\tau, \ell, \xi_1)\right| |a_k||z|^{k-1}} \\
&< \quad \frac{(1+\delta)\sum_{k=2}^{\infty}([k]_q - 1)\left|\Theta_{q,k}^{m}(\tau, \ell, \xi_1)\right| |a_k|}{1 - \sum_{k=2}^{\infty}\left|\Theta_{q,k}^{m}(\tau, \ell, \xi_1)\right| |a_k|}.
\end{aligned}$$

The last expression is bounded above by $(1-\eta)$ if (17) is satisfied. □

By virtue of (15) and Theorem 2.1, we have

Corollary 2.2. *A function $f(z)$ of the form* (1) *is in* $\mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ *if*

$$\sum_{k=2}^{\infty}[k]_q\left[[k]_q(1+\delta)-(\delta+\eta)\right]\left|\Theta^m_{q,k}(\tau,\ell,\xi_1)\right||a_k|\leq 1-\eta, \tag{18}$$

where $\Theta^m_{q,k}(\tau,\ell,\xi_1)$ *is defined by* (12).

Remark 2.3.

(i) Taking $m=1$, $r=2$, $s=1$, $\xi_1=q^{\lambda+1}$ $(\lambda>-1)$, $\xi_2=\zeta_1=q$ and $\ell=\tau=0$ in Theorem 2.1, we obtain the results obtained by Kanas and Raducanu [21];

(ii) Taking $q\to 1^-$ in Theorem 2.1 and Corollary 2.2, we obtain the results obtained by El-Ashwah et al. [13].

Second, In view of Theorem 2.1 and Corollary 2.2, we define $\widehat{\mathfrak{Y}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)\subset\mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ and $\widehat{\mathfrak{K}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)\subset\mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ which consists of functions $f\in A$ whose coefficients satisfy the inequalities (17) and (18), respectively. Now, we investigate some subordination results for the functions in the classes $\widehat{\mathfrak{Y}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ and $\widehat{\mathfrak{K}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ employing the technique used earlier by Attiya [6] and Srivastava and Attiya [40].

Theorem 2.4. *Let the function $f(z)\in A$ defined by (1) be in the class $\widehat{\mathfrak{Y}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$. Then*

$$\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in\mathbb{U};\phi\in\mathcal{CV}), \tag{19}$$

and

$$Re\{f(z)\}>-\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}\quad(z\in\mathbb{U}). \tag{20}$$

The constant $\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}$ *is the best estimate.*

Proof. Let $f(z)\in\mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$, and let $\phi(z)=z+\sum\limits_{k=2}^{\infty}c_kz^k$ be any function in the class $\mathcal{CV}$. Then

$$\begin{aligned}&\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}(f*\phi)(z)\\&\qquad=\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}\left(z+\sum_{k=2}^{\infty}a_kc_kz^k\right)\end{aligned}$$

Thus, by Definition 1.5, the assertion of the theorem will hold if the sequence

$$\left\{\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}a_k\right\}_{k=1}^{\infty}$$

is a subordination factor sequence, with $a_1 = 1$. In view of Lemma 1.6, this equivalent to

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}a_k z^k\right\}>0 \ \ (z\in\mathbb{U}). \tag{21}$$

Since $\Phi(\kappa) = \left([k]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,k}^{m}(\tau,\ell,\xi_1)\right|$ is an increasing function of $k\,(k>2)$. Now

$$\begin{aligned}
&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}a_k z^k\right\}\\
&=\operatorname{Re}\left\{1+\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)+(1-\eta)\right|}z\right.\\
&\qquad\left.+\sum_{k=2}^{\infty}\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}a_k z^k\right\}\\
&\geq 1-\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}r\\
&\qquad-\sum_{k=2}^{\infty}\frac{\left([k]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,k}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}|a_k|r^k\\
&>1-\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}r\\
&\qquad-\frac{(1-\eta)}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)}r=1-r>0 \qquad (|z|=r).
\end{aligned}$$

Thus (21) holds true in $\mathbb{U}$. This prove (19), (20) follows by taking $\phi(z)=\dfrac{z}{1-z}$ in (19).

Now we consider the function $f_0(z)\in\widehat{\mathfrak{Y}}_{q,\tau,\ell}^{m,s,r}(\xi_1;\zeta_1;\eta,\delta)$ given by

$$f_0(z)=z-\frac{(1-\eta)}{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}z^2 \quad (-1\leq\eta<1;\delta\geq 0)$$

which is a member of the class $\widehat{\mathfrak{Y}}_{q,\tau,\ell}^{m,s,r}(\xi_1;\zeta_1;\eta,\delta)$, then by using (19), we have

$$\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}f_0(z)\prec\frac{z}{1-z}.$$

It can be easily verified that

$$\min_{|z|\leq 1}\operatorname{Re}\left[\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta_{q,2}^{m}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}f_0(z)\right]=-\frac{1}{2} \ \ (z\in\mathbb{U}),$$

then the constant $\frac{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}$ is the best possible. This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let the function $f(z) \in A$ defined by (1) be in the class $\widehat{\mathfrak{K}}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$. Then*

$$\frac{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}(f*\phi)(z) \prec \phi(z) \quad (z\in\mathbb{U};\phi\in\mathcal{CV}),$$

and

$$Re\{f(z)\} > -\frac{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)}{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|} \quad (z\in\mathbb{U}).$$

The constant $\frac{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|}{2\left\{[2]_q\left([2]_q(1+\delta)-(\delta+\eta)\right)\left|\Theta^m_{q,2}(\tau,\ell,\xi_1)\right|+(1-\eta)\right\}}$ is the best estimate.

Remark 2.6.

(i) Putting $q \to 1^-$ in Theorems 2.4 and 2.5, we obtain the results which studied by El-Ashwah et al. [13];

(ii) For different choices on $f \in A, r, s, \xi_i (i = 1, 2, .., r)$ and $\zeta_j (j = 1, 2, ..., s), \ell, \tau, m$ and q, we will obtain several results analogous to special cases of the operator mentioned in Remark 1.4 and the classes given by (13) and (14) (see [13]).

3. Conclusion

In our investigation, we generalized the fractional q-calculus and q-Hypergeometric function to define the linear convolution q-Dziok-Srivastava-Catas operator $\mathfrak{D}^{m,s,r}_{q,\tau,\ell}(m \in \mathbb{N}_0 = \{0, 1, ..\}, r \leq s+1; r, s \in \mathbb{N}_0, 0 < q < 1, \tau \geq 0, \ell \geq 0)$. Using this operator we defined and study the subclass $\mathfrak{Y}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ $(\delta \geq 0,\ \eta \in [-1, 1),\ \eta + \delta \geq 0)$ of q-uniformly starlike functions of order η and type δ in $\mathbb{U}$ and the subclass $\mathfrak{K}^{m,s,r}_{q,\tau,\ell}(\xi_1;\zeta_1;\eta,\delta)$ $(\delta \geq 0,\ \eta \in [-1, 1),\ \eta + \delta \geq 0)$ of q-uniformly convex functions of order η and type δ in $\mathbb{U}$. We have derived their associated coefficient estimates. For these function classes, we establish subordination theorems and also, point out some new and known consequences of the results. There are some obvious connection between the classical q-analysis, which we used here, and the so-called (p, q)-analysis. Specifically, we can see that the results for the $q-$analogues, which we have considered in this article for $0 < q < 1$, can easily be translated into the corresponding results for the $(p, q)-$analogues (with $0 < p, q < 1$) by applying some obvious parametric and argument variations, for details about the fractional (p, q)-calculus see [37, p. 340] and [38, p. 511-512].

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