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Several new integral inequalities via Caputo fractional integral operators

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Abstract. In this paper, we establish several new integral inequalities including Caputo fractional derivatives for quasi–convex, s−Godunova–Levin convex. In order to obtain our results, we have used fairly elementary methodology by using the classical inequalities such that Hölder inequality, Power mean inequality and Weighted Hölder inequality. This work is motivated by Farid et al in [17]. Especially we aim to obtain inequalities involving only right-sided Caputo-fractional derivative of order α.

1. Introduction

The following definitions are well known in the literature:

Definition 1. *The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. A huge amount of the researchers interested in this definition and there are several papers based on convexity. Many important inequalities are established for the class of convex functions, but one of the most important is so-called Hermite–Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows in literature: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is in the reversed direction if f is concave.

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Definition 2. *[5] Let real function f be defined on some nonempty interval I of real numbers line R. The function f is said to be quasi–convex on I if inequality*

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\} \qquad (QC)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, any convex function is a quasi–convex function but every quasi–convex function is not convex function.

For example the function $f : R^+ \to R,\ f(x) = \ln x,\ x \in R^+$ is quasi-convex but it is not convex.

Definition 3. *[10] we say that the function $f : C \subset X \to [0, \infty)$ is of s – Godunova – Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1 - t)y) \leq t^{-s} f(x) + (1 - t)^{-s} f(y)$$

for all $t \in (0, 1)$ and $x, y \in C$.

In [2] Hudzik and Maligranda considered, amongs others, the class of functions which are $s-$convex in the second sense.

Now we give a necessary definition of fractional calculus theory which is used throughout this paper.

Definition 4. *[1] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having $n-th$ derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:*

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{f^{(n)}(t)}{(x - t)^{\alpha - n + 1}} dt, x > a, \tag{1.1}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_x^b \frac{f^{(n)}(t)}{(t - x)^{\alpha - n + 1}} dt, x < b. \tag{1.2}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{n}_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^{n}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x), \tag{1.3}$$

where $n = 1$ and $\alpha = 0$.

Since the inequalities always have been proved worthy in establishing the mathematical models and their solutions in almost all branches of applied sciences. Especially the convexity takes very important role in the optimization theory. The aim of this paper is to introduce some fractional inequalities for the Caputo fractional derivatives via the convexity property of the functions which have derivatives of any integer order.

We will also use the weighted version of the Hölder inequality well known in the literature see [18] :

$$\left| \int_I f(t)s(t)h(t)dt \right| \leq \left(\int_I \left| f(t) \right|^p h(t)dt \right)^{\frac{1}{p}} \left(\int_I |s(t)|^q h(t)dt \right)^{\frac{1}{q}}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and h is non–negative on I.

In [17], Farid *et al.* proved the following identity and established some inequalities for Caputo fractional integrals.

Lemma 1. *Let $f : [a, b] \to R$, be a differentiable mapping on (a, b) with $a < b$. If $f^{(n+1)} \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{a^+}f\right)(b) + (-1)^n\left({}^{C}D^{\alpha}_{b^-}f\right)(a)\right]$$
$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{n-\alpha} - t^{n-\alpha}\right] f^{(n+1)}(ta + (1-t)b)dt.$$

The main aim of this paper is to establish three new integral identities and by using these equalities to prove some new integral inequalities for quasi–convex and s–Godunova–Levin convex via the Caputo–fractional integral operators.

2. Main Results

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I where $a, b \in I$ with $t \in [0, 1]$. If $f^{(n+1)} \in L[a, b]$,Then for all $a \le x < y \le b$ and $\alpha > 0$ we have:*

$$\frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^{C}D^{\alpha}_{y^-}f)(x) = \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt.$$

Proof. Firstly, by integrating by parts

$$\int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt$$
$$= \frac{1}{y-x}f^{(n)}(y) - \frac{(n-\alpha)}{y-x}\int_0^1 (1-t)^{n-\alpha-1} f(tx + (1-t)y)\,dt.$$

Secondly, by applying the change of the variable $u = tx + (1-t)y$ to the above integrals, we get

$$\frac{1}{y-x}f^{(n)}(y) - \frac{(n-\alpha)}{(y-x)^{n-\alpha+1}}\int_x^y (u-x)^{n-\alpha-1} f(u)du$$
$$= \frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n}{(y-x)^{n-\alpha+1}}\Gamma(n-\alpha+1)({}^{C}D^{\alpha}_{y^-}f)(x).$$

This completes the proof. ◄ □

I̊f we choose $x = a$ and $y = b$ in Lemma 2, we obtain

$$\frac{1}{b-a}f^{(n)}(b) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}({}^{C}D^{\alpha}_{y^-}f)(a) = \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(ta + (1-t)b)dt.$$

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, $I \subset [0, \infty)$,be a differentiable function on I such that $f^{(n+1)} \in L[a, b]$ where $a, b \in I$ with $a \le x < y \le b$. If $f^{(n+1)}$is quasi-convex on $[x, y]$ for $t \in [0, 1]$. Then for all $\alpha > 0$ we have*

$$\frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^{C}D^{\alpha}_{y^-}f)(x) \le \frac{1}{(n-\alpha+1)}\max\left\{f^{(n+1)}(x), f^{(n+1)}(y)\right\},$$

where $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$.

Proof. Since $f^{(n+1)}(tx+(1-t)y) \leq \max\left\{f^{(n+1)}(x), f^{(n+1)}(y)\right\}$ for $t \in [0,1]$ and from Lemma 2, we obtain

$$\begin{aligned}\frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^CD^{\alpha}_{y^-}f)(x) &= \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx+(1-t)y)dt \\ &\leq \max\left\{f^{(n+1)}(x), f^{(n+1)}(y)\right\}\int_0^1 (1-t)^{n-\alpha}dt \\ &= \frac{1}{(n-\alpha+1)}\max\left\{f^{(n+1)}(x), f^{(n+1)}(y)\right\},\end{aligned}$$

which completes the proof of the Theorem. ◄ □

Corollary 1. *If we choose $x = a$ and $y = b$ in Theorem 1, with increasing of $f^{(n+1)}$ we obtain*

$$\begin{aligned}\frac{1}{b-a}f^{(n)}(b) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}({}^CD^{\alpha}_{b^-}f)(a) &\leq \frac{1}{(\alpha+1)}\max\left\{f^{(n+1)}(a), f^{(n+1)}(b)\right\} \\ &\leq \left\|f^{(n+1)}\right\|_{\infty}\frac{1}{n-\alpha+1}.\end{aligned}$$

Corollary 2. *In inequality of corollary 2.3, if we choose $\alpha = 1$, we have*

$$\frac{f^{(n)}(b)}{b-a} - \frac{n-1}{(b-a)^n}\int_a^b (u-a)^{n-2} f^{(n)}(u)du \leq \frac{1}{n}\max\left\{f^{(n+1)}(a), f^{(n+1)}(b)\right\}.$$

Particularly for $n = 2$, we get

$$f''(b) - \frac{1}{(b-a)}\int_a^b f''(u)du \leq \frac{b-a}{2}\max\left\{f'''(a), f'''(b)\right\}.$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I such that $f^{(n+1)} \in L[a,b]$ where with $a, b \in I$, $a \leq x < y \leq b$. If $\left|f^{(n+1)}\right|^q$ is quasi-convex on $[x,y]$ for $t \in [0,1]$, $q > 1$, $p = \frac{q}{q-1}$, then for all $\alpha > 0$, we have*

$$\left|\frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^CD^{\alpha}_{y^-}f)(x)\right| \leq \left(\frac{1}{n-\alpha+1}\right)\max\left\{\left|f^{(n+1)}(x)\right|^q, \left|f^{(n+1)}(y)\right|^q\right\}^{\frac{1}{q}}. \tag{2.1}$$

Proof. First of all, we know that is

$$\left|f^{(n+1)}(tx+(1-t)y)\right|^q \leq \left[\max\left\{\left|f^{(n+1)}(x)\right|^q, \left|f^{(n+1)}(y)\right|^q\right\}\right].$$

Using well known Hölder's inequality for n-th derivative, properties of modulus and from Lemma 2, we obtain

$$\begin{aligned}&\left|\frac{1}{y-x}f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^CD^{\alpha}_{y^-}f)(x)\right| \\ &= \left|\int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx+(1-t)y)\,dt\right| \\ &\leq \int_0^1 \left|(1-t)^{n-\alpha}\right|\left|f^{(n+1)}(tx+(1-t)y)\right|dt \\ &= \int_0^1 (1-t)^{(n-\alpha)\left(1-\frac{1}{q}\right)}(1-t)^{(n-\alpha)\frac{1}{q}}\left|f^{(n+1)}(tx+(1-t)y)\right|dt \\ &\leq \left(\int_0^1 (1-t)^{(n-\alpha)}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{(n-\alpha)}\left|f^{(n+1)}(tx+(1-t)y)\right|^q dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{(n-\alpha+1)}\right)\max\left\{\left|f^{(n+1)}(x)\right|^q, \left|f^{(n+1)}(y)\right|^q\right\}^{\frac{1}{q}},\end{aligned}$$

which completes the proof of desired inequality. ◂ □

Corollary 3. *If we choose $x = a$, $y = b$ and $\alpha = 1$ in inequality (2.1), then*

$$\left|\frac{f^{(n)}(b)}{b-a} - \frac{n-1}{(b-a)^n}\int_a^b (u-a)^{n-2} f^{(n)}(u)du\right| \leq \frac{1}{n}\max\left\{\left|f^{(n+1)}(a)\right|^q, \left|f^{(n+1)}(b)\right|^q\right\}^{\frac{1}{q}}.$$

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I where $a, b \in I$ with $t \in [0,1]$. If $f^{(n+1)} \in L[a,b]$, then, for all $a \leq x < y \leq b$ and $\alpha > 0$, we have*

$$\frac{1}{x-y}\left[f^{(n)}(x) - f^{(n)}(y)\right] + \frac{\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left[({}^C D^\alpha_{x^+} f)(y) - (-1)^n ({}^C D^\alpha_{y^-} f)(x)\right]$$
$$= \int_0^1 t^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt + \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx + (1-t)\,y)\,dt.$$

Proof. By using integration by parts we can write the above integrals as follows

$$\int_0^1 t^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt$$
$$= \frac{1}{x-y} f^{(n)}(x) + \frac{n-\alpha}{(y-x)^{n-\alpha+1}}\int_x^y (y-u)^{n-\alpha-1} f(u)du, \quad u = tx + (1-t)y$$
$$= \frac{1}{x-y}^{n} f(x) + \frac{\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^C D^\alpha_{x^+} f)(y)$$

and

$$\int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx + (1-t)\,y)\,dt$$
$$= -\frac{1}{x-y} f^{(n)}(y) - \frac{n-\alpha}{(y-x)^{n-\alpha+1}}\int_x^y (u-x)^{n-\alpha-1} f(u)du, \quad u = tx + (1-t)y$$
$$= -\frac{1}{x-y} f^{(n)}(y) - \frac{(-1)^n\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}({}^C D^\alpha_{y^-} f)(x).$$

Adding the above integral equalities we get required inequality. ◂ □

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I where $a, b \in I$ with $t \in [0,1]$. If $f^{(n+1)} \in L[a,b]$ and $\left|f^{(n+1)}\right|$ is $s-$Godunova-Levin type function. Then for all $a \leq x < y \leq b$ and $\alpha > 0$, $s \in [0,1)$ we have*

$$\left|\frac{1}{x-y}\left[f^{(n)}(x) - f^{(n)}(y)\right] + \frac{\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left[({}^C D^\alpha_{x^+} f)(y) - (-1)^n ({}^C D^\alpha_{y^-} f)(x)\right]\right|$$
$$\leq \left[\beta(n-\alpha+1, 1-s) + \frac{1}{n-\alpha-s+1}\right]\left|f^{(n+1)}(x)\right| + \left|f^{(n+1)}(y)\right|,$$

where $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, $x > 1$, $y > 0$ is modify of β.

Proof. From Lemma 3 and with properties of modulus

$$\left|\int_0^1 t^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt + \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(tx + (1-t)y)\,dt\right|$$
$$\leq \int_0^1 t^{n-\alpha}\left|f^{(n+1)}(tx + (1-t)\,y)\right|dt + \int_0^1 (1-t)^{n-\alpha}\left|f^{(n+1)}(tx + (1-t)y)\right|dt.$$

Since $\left|f^{(n+1)}\right|$ is $s-$Godunova-Levin type function, applying integration by parts to every integral, respectively, we get

$$\begin{aligned}\int_0^1 t^{n-\alpha}\left|f^{(n+1)}(tx+(1-t)y)\right|dt &\le \left|f^{(n+1)}(x)\right|\int_0^1 t^{n-\alpha-s}dt+\left|f^{(n+1)}(y)\right|\int_0^1 t^{n-\alpha}(1-t)^{-s}dt\\ &= \left[\frac{1}{n-\alpha-s+1}\left|f^{(n+1)}(x)\right|+\left|f^{(n+1)}(y)\right|\beta(n-\alpha+1,1-s)\right]\end{aligned}$$

and

$$\int_0^1 (1-t)^{n-\alpha}\left|f^{(n+1)}(tx+(1-t)y)\right|dt=\left|f^{(n+1)}(x)\right|\beta(1-s,n-\alpha+1)+\left|f^{(n+1)}(y)\right|\frac{1}{n-\alpha-s+1}.$$

Finally, since $\beta(x,y)=\beta(y,x)$, we have

$$\begin{aligned}&\left|\frac{1}{x-y}\left[f^{(n)}(x)-f^{(n)}(y)\right]+\frac{\Gamma(n-\alpha+1)}{(y-x)}\left[({}^{C}D^{\alpha}_{x^+}f)(y)-(-1)^n({}^{C}D^{\alpha}_{y^-}f)(x)\right]\right|\\ &\le\left[\beta(n-\alpha+1,1-s)+\frac{1}{n-\alpha-s+1}\right]\left|f^{(n+1)}(x)\right|+\left|f^{(n+1)}(y)\right|\end{aligned}$$

□

Lemma 4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I where $a,b\in I$ with $t\in[0,1]$. If $f^{(n+2)}\in L[a,b]$, Then for all $a<b$ and $\alpha-1>0$, with properties of Gamma function we have*

$$\begin{aligned}&\frac{2^{n-\alpha-2}(n-\alpha-1)}{(b-a)^{n-\alpha-1}}\Gamma(n-\alpha-1)\left[({}^{C}D^{\alpha-1}_{\frac{a+b}{2}^+}f)(b)+(-1)^n({}^{C}D^{\alpha-1}_{\frac{a+b}{2}^-}f)(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right)\\ &=\frac{(b-a)^2}{(n-\alpha)2^{2-n+\alpha}}\left[\int_0^{\frac12}t^{n-\alpha}f^{(n+2)}(ta+(1-t)b)dt+\int_{\frac12}^1(1-t)^{n-\alpha}f^{(n+2)}(ta+(1-t)b)dt\right].\end{aligned}$$

Proof. Making repeated applications of integration by parts, we obtain the following equalities:

$$\begin{aligned}&\int_0^{\frac12}t^{n-\alpha}f^{(n+2)}(ta+(1-t)b)\,dt\\ &=\frac{1}{(a-b)2^{n-\alpha}}f^{(n+1)}\left(\frac{a+b}{2}\right)-\frac{n-\alpha}{(a-b)^2 2^{n-\alpha-1}}f^{(n)}\left(\frac{a+b}{2}\right)\\ &+\frac{(n-\alpha)(n-\alpha-1)}{(a-b)^2}\int_0^{\frac12}t^{n-\alpha-2}f^{(n)}(ta+(1-t)b)dt\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac12}^1(1-t)^{n-\alpha}f^{(n+2)}(ta+(1-t)b)\,dt\\ &=-\frac{1}{(a-b)2^{n-\alpha}}f^{(n+1)}\left(\frac{a+b}{2}\right)-\frac{(n-\alpha)}{(a-b)^2 2^{n-\alpha-1}}f^{(n)}\left(\frac{a+b}{2}\right)\\ &+\frac{(n-\alpha)(n-\alpha-1)}{(a-b)^2}\int_{\frac12}^1(1-t)^{n-\alpha-2}f^{(n)}(ta+(1-t)b)\,dt\end{aligned}$$

Now, using change of variable $u=ta+(1-t)b$ for every integral, we have

$$\begin{aligned}\int_0^{\frac12}t^{n-\alpha-2}f^{(n)}(ta+(1-t)b)\,dt &=\frac{1}{(b-a)^{n-\alpha-1}}\int_{\frac{a+b}{2}}^b(b-u)^{n-\alpha-2}f(u)du\\ &=\frac{\Gamma(n-\alpha-1)}{(b-a)^{n-\alpha-1}}({}^{C}D^{\alpha-1}_{\frac{a+b}{2}^+}f)(b)\end{aligned}$$

and

$$\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha-2}f^{(n)}(ta+(1-t)b)\,dt=\frac{1}{(b-a)^{n-\alpha-1}}\int_{a}^{\frac{a+b}{2}}(u-a)^{n-\alpha-2}f(u)du$$
$$=\frac{\Gamma(n-\alpha-1)}{(b-a)^{n-\alpha-1}}(-1)^{n}({}^{C}D^{\alpha-1}_{\frac{a+b}{2}-}f)(a).$$

By adding these inequalities and multiplying by $\frac{n-\alpha}{(a-b)^2 2^{n-\alpha-1}}$ we get the required inequality. □

Theorem 4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I such that $f^{(n+2)}\in L[a,b]$,where $a,b\in I$, $a<b$ with $t\in[0,1]$. If $\left|f^{(n+2)}\right|^{q}$ is quasi-convex on $[a,b]\subset I$ and $q\geq 1$, Then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{n-\alpha-2}(n-\alpha-1)}{(b-a)^{n-\alpha-1}}\Gamma(n-\alpha-1)\left[({}^{C}D^{\alpha-1}_{\frac{a+b}{2}+}f)(b)+(-1)^{n}({}^{C}D^{\alpha-1}_{\frac{a+b}{2}-}f)(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\left(\frac{b-a}{2}\right)^{2}\frac{1}{n-\alpha(n-\alpha+1)}\left[\max\left(\left|f^{(n+2)}(a)\right|^{q},\left|f^{(n+2)}(b)\right|^{q}\right)\right]^{\frac{1}{q}},$$

where $\alpha-1>0$.

Proof. From Lemma 4 and using power–mean inequality with properties of modulus, we can write

$$\frac{2^{n-\alpha-2}(n-\alpha-1)}{(b-a)^{n-\alpha-1}}\Gamma(n-\alpha-1)\left[({}^{C}D^{\alpha-1}_{\frac{a+b}{2}+}f)(b)+(-1)^{n}({}^{C}D^{\alpha-1}_{\frac{a+b}{2}-}f)(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right)=U$$

$$|U|\leq\frac{(b-a)^{2}}{(n-\alpha)2^{2+\alpha-n}}\left[\begin{array}{c}\int_{0}^{\frac{1}{2}}t^{n-\alpha}\left|f^{(n+2)}(ta+(1-t)b)\right|dt\\+\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha}\left|f^{(n+2)}(ta+(1-t)b)\right|dt\end{array}\right]$$
$$\leq\frac{(b-a)^{2}}{(n-\alpha)2^{2-\alpha}}\left[\begin{array}{c}\left(\int_{0}^{\frac{1}{2}}t^{n-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}t^{n-\alpha}\left|f^{(n+2)}(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha}\left|f^{(n+2)}(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\end{array}\right]$$
$$=\frac{(b-a)^{2}}{(n-\alpha)2^{2+\alpha-n}}\frac{1}{2^{n-\alpha}(n-\alpha+1)}\left[\max\left(\left|f^{(n+2)}(a)\right|^{q},\left|f(n+2)(b)\right|^{q}\right)\right]^{\frac{1}{q}}$$
$$=\left(\frac{b-a}{2}\right)^{2}\frac{1}{(n-\alpha)(n-\alpha+1)}\left[\max\left(\left|f^{(n+2)}(a)\right|^{q},\left|f^{(n+2)}(b)\right|^{q}\right)\right]^{\frac{1}{q}},$$

which completes the proof. □

Here we used the quasi-convex of $\left|f''\right|^{q}$ on $[a,b]$ and it can be easily checked that

$$\int_{0}^{\frac{1}{2}}t^{n-\alpha}dt=\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha}dt=\frac{1}{2^{n-\alpha+1}(n-\alpha+1)}.$$

Theorem 5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$, $I\subset[0,\infty)$,be a differentiable function on I such that $f,g\in L[a,b]$ and $a,b\in I$, $0\leq a<b$. $\left|f\right|^{p}$ and $\left|g\right|^{q}$ are quasi–convex on $[a,b]$ for $t\in[0,1]$, $q>1$. Then for all $x\in[a,b]$, $\alpha+1>0$, $\frac{1}{p}+\frac{1}{q}=1$; if The functions $\left|f\right|^{p}$ and $\left|g\right|^{q}$are increasing on $[a,b]\subset I$.The following inequality hold:*

$$
\begin{aligned}
&\frac{1}{b-a}\left|\int_a^b f^{(n)}(x)g^{(n)}(x)\,h^{(n)}(x)dx\right| \\
&\le \frac{\left\|f^{(n)}\right\|_\infty \left\|g^{(n)}\right\|_\infty}{2}\left[\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}\left[({}^{C}D^{\alpha}_{a^+}f)(b)+(-1)^n({}^{C}D^{\alpha}_{b^-}f)(a)\right]-\frac{1}{b-a}\int_a^b f^{(n)}(x)dx\right],
\end{aligned}
$$

where $h^{(n)}(ta+(1-t)b)=\left[(1-t)^{n-\alpha}+(t^{n-\alpha}-1)\right]f^{(n+1)}(ta+(1-t)b)\ge 0$ *for all* $t\in[0,1]$ *and* $\alpha\in[0,1]$.

Proof. We will use the weighted Hölder inequality. Since

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f^{(n)}(x)g^{(n)}(x)h^{(n)}(x)\,dx \\
&=\int_0^1 f^{(n)}(ta+(1-t)b)g^{(n)}(ta+(1-t)b)h^{(n)}(ta+(1-t)b)dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{b-a}\left|\int_a^b f^{(n)}(x)g^{(n)}(x)\,h^{(n)}(x)dx\right| \\
&\le\left(\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^p h^{(n)}(ta+(1-t)b)dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left|g^{(n)}(ta+(1-t)b)\right|^q h^{(n)}(ta+(1-t)b)dt\right)^{\frac{1}{q}} \\
&\le\left[\max\left\{\left|f^{(n)}(a)\right|^p,\left|f^{(n)}(b)\right|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{\left|g^{(n)}(a)\right|^q,\left|g^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left(\int_0^1 h^{(n)}(ta+(1-t)b)dt\right)^{\frac{1}{p}}\left(\int_0^1 h^{(n)}(ta+(1-t)\,b)\,dt\right)^{\frac{1}{q}} \\
&=\left[\max\left\{\left|f^{(n)}(a)\right|^p,\left|f^{(n)}(b)\right|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{\left|g^{(n)}(a)\right|^q,\left|g^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left(\int_0^1 h^{(n)}(ta+(1-t)b)dt\right)^{\frac{1}{p}+\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&=\left[\max\left\{\left|f^{(n)}(a)\right|^p,\left|f^{(n)}(b)\right|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{\left|g^{(n)}(a)\right|^q,\left|g^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left(\int_0^1 h^{(n)}(ta+(1-t)b)dt\right) \\
&=\left\|f^{(n)}\right\|_\infty\left\|g^{(n)}\right\|_\infty\left(\int_0^1 h^{(n)}(ta+(1-t)b)dt\right) \\
&=\left\|f^{(n)}\right\|_\infty\left\|g^{(n)}\right\|_\infty\left(\int_0^1\left[(1-t)^{n-\alpha}+(t^{n-\alpha}-1)\right]f^{(n+1)}(ta+(1-t)b)dt\right) \\
&=\frac{\left\|f^{(n)}\right\|_\infty\left\|g^{(n)}\right\|_\infty}{2}\left[\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}\left[(-1)^n({}^{C}D^{\alpha}_{b^-}f)(a)+({}^{C}D^{\alpha}_{a^+}f)(b)\right]-\frac{1}{b-a}\int_a^b f^{(n)}(x)dx\right],
\end{aligned}
$$

which completes the proof. □

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$, be a differentiable function on (a, b) with $a < b$. If $\left|f^{(n+1)}\right|^q$ is quasi-convex on $[a, b]$ for $t \in [0, 1]$, and $\left|f^{(n+1)}\right|^q$ is increasing $q > 1$, $x \in [a, b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$. Then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Gamma(n-\alpha+1)\left(({}^{C}D^{\alpha}_{a^+}f)(b), (-1)^n({}^{C}D^{\alpha}_{b^-}f)(a)\right)_{\alpha\in[0,1]}\\&\leq \frac{b-a}{2}\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}\left(\frac{1}{(n-\alpha)(1-p)+1}\right)^{\frac{1}{q}}\left\|f^{(n+1)}\right\|_{\infty},\end{aligned}$$

where

$$\begin{aligned}&\Gamma(n-\alpha+1)\left(({}^{C}D^{\alpha}_{a^+}f)(b), (-1)^n({}^{C}D^{\alpha}_{b^-}f)(a)\right)_{\alpha\in[0,1]}\\&= \left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{a^+}f)(b) + (-1)n({}^{C}D^{\alpha}_{b^-}f)(a)\right]\right|.\end{aligned}$$

Proof. Using the lemma 1 and with properties of modulus

$$\begin{aligned}&\Gamma(n-\alpha+1)\left(({}^{C}D^{\alpha}_{a^+}f)(b), (-1)^{n C}D^{\alpha}_{b^-}f)(a)\right)_{\alpha\in[0,1]}\\&\leq \frac{b-a}{2}\left[\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(ta+(1-t)b)\right|dt\right].\end{aligned}$$

We know that that for $\alpha \in [0, 1]$ and $\forall t_1, t_2 \in [0, 1]$,

$$\left|t_1^{n-\alpha} - t_2^{n-\alpha}\right| \leq |t_1 - t_2|^{n-\alpha} \ ,$$

That is

$$\int_0^1\left|(1-t^{n-\alpha})-t^{n-\alpha}\right|dt \leq \int_0^1 |1-2t|^{n-\alpha}\,dt.$$

On the other hand, using the power mean inequalitiy to the right hand of elementary integral inequality we have

$$
\begin{aligned}
&\Gamma(n-\alpha+1)\left(\left({}^{C}D_{a^+}^{\alpha}f\right)(b),(-1)^n\left({}^{C}D_{b^-}^{\alpha}f\right)(a)\right)_{\alpha\in[0,1]}\\
&\leq \frac{b-a}{2}\left[\int_0^1 |1-2t|^{n-\alpha}\left|f^{(n+1)}(ta+(1-t)b)\right|dt\right]\\
&=\frac{b-a}{2}\left[\int_0^1 |1-2t|^{(n-\alpha)p}\left|f^{(n+1)}(ta+(1-t)b)\right||1-2t|^{(n-\alpha)(1-p)}dt\right]\\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^{(n-\alpha)p^2}|1-2t|^{(n-\alpha)(1-p)}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 \left|f^{(n+1)}(ta+(1-t)b)\right|^q|1-2t|^{(n-\alpha)(1-p)}dt\right)^{\frac{1}{q}}\\
&=\frac{b-a}{2}\left(\int_0^1 |1-2t|^{(n-\alpha)p^2+(n-\alpha)(1-p)}dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|^{(n-\alpha)(1-p)}dt\right)^{\frac{1}{q}}\\
&\times\left(\max\left\{\left|f^{(n+1)}(a)\right|^q,\left|f^{(n+1)}(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
&=\frac{b-a}{2}\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}\left(\frac{1}{(n-\alpha)(1-p)+1}\right)^{\frac{1}{q}}\\
&\times\left(\max\left\{\left|f^{(n+1)}(a)\right|^q,\left|f^{(n+1)}(b)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\leq \frac{b-a}{2}\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}\left(\frac{1}{(n-\alpha)(1-p)+1}\right)^{\frac{1}{q}}\left\|f^{(n+1)}\right\|_\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 |1-2t|^{(n-\alpha)p^2+(n-\alpha)(1-p)}dt &= \int_0^{\frac{1}{2}}(1-2t)^{(n-\alpha)p^2+(n-\alpha)(1-p)}dt+\int_{\frac{1}{2}}^1(2t-1)^{(n-\alpha)p^2+(n-\alpha)(1-p)}dt\\
&=\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 |1-2t|^{(n-\alpha)(1-p)}dt &= \int_0^{\frac{1}{2}}(1-2t)^{(n-\alpha)(1-p)}dt+\int_{\frac{1}{2}}^1(2t-1)^{(n-\alpha)(1-p)}dt\\
&=\left(\frac{1}{(n-\alpha)(1-p)+1}\right),
\end{aligned}
$$

which completes the required proof. □

Corollary 4. *For $p\in(1,\infty)$ we have the following limits*

$$
\lim_{p\to 1^+}\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}=\frac{1}{(n-\alpha)+1}<1,
$$

$$
\lim_{p\to\infty}\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}=1,
$$

$$
\frac{1}{(n-\alpha)+1}<\left(\frac{1}{(n-\alpha)p^2+(n-\alpha)(1-p)+1}\right)^{\frac{1}{p}}<1,
$$

and

$$\lim_{p \to 1^+} \left(\frac{1}{(n-\alpha)(1-p)+1} \right)^{\frac{1}{q}} = 1 \ , \qquad \lim_{p \to \infty} \left(\frac{1}{(n-\alpha)(1-p)+1} \right)^{1-\frac{1}{p}} = 1.$$

This means that we can make the decision which estimation is least upper bound. Because it becomes better as p increases.

Thus we can rewrite inequality in Theorem with increasing of $f^{(n+1)}$ as following

$$\left| \frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[({}^{C}D^{\alpha}_{a^+} f)(b) + (-)^n ({}^{C}D^{\alpha}_{b^-} f)(a) \right] \right| \leq \frac{b-a}{2} \left\| f^{(n+1)} \right\|_{\infty}.$$

Conclusion: In this paper we established three new integral identities and by using these identities we proved some new integral inequalities for quasi–convex and s–Godunova–Levin convex via the Caputo–fractional integral operators. The results offer new estimations for integral inequalities. Many particular cases can be revealed by using the findings. The interested researchers can investigate different inequalities using the main Lemmas.

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