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Best proximity points for $(\varphi - \psi)$-weak contractions and some applications

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Abstract. The principal goal of this paper is to express the existence and uniqueness of the best proximity point for a comprehensive contractive non-self mapping in partially ordered metric spaces. The main result covers a lot of former well-known theorems in related to best proximity point. Moreover, as an interesting application, integral versions of main theorem are obtained.

1. Introduction and Preliminaries

In 2011, Raj [18] introduced the concept of best proximity point (in short, bpp) as below and proved some theorems for weakly contractive non-self mappings. Assume that $(\mathcal{X}, \mathsf{d})$ is a metric space, $(\mathcal{E}, \mathcal{F})$ is a pair of nonempty subsets of $\mathcal{X}$, $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ be a non-self mapping and $\mathsf{d}(\mathcal{E}, \mathcal{F}) = \inf\left\{\mathsf{d}(x, y) : x \in \mathcal{E},\ y \in \mathcal{F}\right\}$. An element $x \in \mathcal{E}$ is named a bpp for $\mathcal{T}$ if $\mathsf{d}(x, \mathcal{T}x) = \mathsf{d}(\mathcal{E}, \mathcal{F})$.

Also, assume that

$$\mathcal{E}_0 = \left\{x \in \mathcal{E} : \ \mathsf{d}(x, y) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \text{ for some } y \in \mathcal{F}\right\},$$
$$\mathcal{F}_0 = \left\{y \in \mathcal{F} : \ \mathsf{d}(x, y) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \text{ for some } x \in \mathcal{E}\right\}.$$

Note that if x is a bpp for $\mathcal{T}$, then we have $x \in \mathcal{E}_0$ and $\mathcal{T}x \in \mathcal{F}_0$.

The purpose of bpp theory is to furnish sufficient conditions that assure the existence of such points. Hence, numerous works on this theory were studied by giving sufficient conditions assuring the existence and uniqueness of these points such that several authors have studied different contractions for having the bpp in various metric spaces and partially ordered metric spaces in [1, 3–17, 19, 20] and references therein.

Let $\mathcal{E}$ and $\mathcal{F}$ be nonempty subsets of a metric space $(\mathcal{X}, \mathsf{d})$ with $\mathcal{E}_0 \neq \emptyset$. We say the pair $(\mathcal{E}, \mathcal{F})$ have the P-property [18] if

$$\left.\begin{array}{l} \mathsf{d}(x_1, y_1) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \\ \mathsf{d}(x_2, y_2) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \end{array}\right\} \Longrightarrow \mathsf{d}(x_1, x_2) = \mathsf{d}(y_1, y_2)$$

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for all $x_1, x_2 \in \mathcal{E}_0$ and $y_1, y_2 \in \mathcal{F}_0$.

In the sequel, suppose that $(\mathcal{X}, \mathsf{d}, \preceq)$ is a partially ordered metric spaces (in short, POMS) and $(\mathcal{E}, \mathcal{F})$ is a pair of nonempty closed subsets of $\mathcal{X}$ unless otherwise stated. Also, assume that $\psi, \varphi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ are two functions such that

- ψ is nondecreasing and continuous;
- φ is lower semi-continuous (in short, lsc) on $\mathbb{R}^{\geqslant 0}$ and $\varphi(\mathfrak{t}) = 0$ implies $\mathfrak{t} = 0$.

In this paper, we prove the existence and uniqueness of the bpp for a comprehensive contractive non-self mapping in partially ordered metric spaces. Our theorems contain some former well-known theorems in related to the bpp. Ultimately, integral versions of main theorem are obtained.

2. Main results

First, following the idea of Sadiq Basha [19], we define the concept of an order-proximal mapping.

Definition 2.1. *A non-self mapping $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ is said to be order proximal if*

$$\left.\begin{array}{r} y_1 \preceq y_2 \\ \mathsf{d}(x_1, \mathcal{T} y_1) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \\ \mathsf{d}(x_2, \mathcal{T} y_2) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \end{array}\right\} \text{ implies that } x_1 \preceq x_2$$

for all $x_1, x_2, y_1, y_2 \in \mathcal{E}$.

The main theorem of this article is as below.

Theorem 2.2. *Assume that $(\mathcal{X}, \mathsf{d})$ is a complete POMS and $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ is a mapping such that the following properties are held:*

(*i*) *$\mathcal{T}$ is order proximal, $\mathcal{T}(\mathcal{E}_0) \subseteq \mathcal{F}_0$ and $(\mathcal{E}, \mathcal{F})$ have the P-property;*

(*ii*) *there exist elements $x_0, x_1 \in \mathcal{E}_0$ provided that $x_0 \preceq x_1$ and $\mathsf{d}(x_1, \mathcal{T} x_0) = \mathsf{d}(\mathcal{E}, \mathcal{F})$;*

(*iii*) *$\mathcal{T}$ is continuous on $\mathcal{E}$ and*

$$\psi(\mathsf{d}(\mathcal{T} x, \mathcal{T} y)) \leq \psi\Big(Q_{\mathcal{T}}(x, y)\Big) - \varphi\Big(Q_{\mathcal{T}}(x, y)\Big) \tag{1}$$

for all $x, y \in \mathcal{E}$ with $x \preceq y$, where

$$Q_{\mathcal{T}}(x, y) = \max\Big\{\mathsf{d}(x, y), \mathsf{d}(x, \mathcal{T} x) - \mathsf{d}(\mathcal{E}, \mathcal{F}), \mathsf{d}(y, \mathcal{T} y) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big\}.$$

Then $\mathcal{T}$ has a bpp in $\mathcal{E}$. Moreover, if for any two bpp(s) $\mathsf{u}, \mathsf{v} \in \mathcal{E}$ we have $\mathsf{u} \preceq \mathsf{v}$, then $\mathcal{T}$ has a unique bpp in $\mathcal{E}$.

Proof. From $x_1 \in \mathcal{E}_0$ and $\mathcal{T}(\mathcal{E}_0) \subseteq \mathcal{F}_0$, there exists $x_2 \in \mathcal{E}$ such that $\mathsf{d}(x_2, \mathcal{T} x_1) = \mathsf{d}(\mathcal{E}, \mathcal{F})$. In particular, $x_2 \in \mathcal{E}_0$. Since $\mathsf{d}(x_1, \mathcal{T} x_0) = \mathsf{d}(\mathcal{E}, \mathcal{F})$ and $x_0 \preceq x_1$, by the order proximality of $\mathcal{T}$ we have $x_1 \preceq x_2$. By continuing this process, we can construct a sequence $\{x_\mathfrak{n}\}$ in $\mathcal{E}_0$ provided that

$$x_\mathfrak{n} \preceq x_{\mathfrak{n}+1} \text{ and } \mathsf{d}(x_{\mathfrak{n}+1}, \mathcal{T} x_\mathfrak{n}) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \qquad \mathfrak{n} = 0, 1, \cdots. \tag{2}$$

In particular, $\{x_\mathfrak{n}\}$ is nondecreasing with respect to $\preceq$; that is, $x_0 \preceq x_1 \preceq \cdots$. On the other hand, $(\mathcal{E}, \mathcal{F})$ satisfies the *P*-property. Hence,

$$\left.\begin{array}{r} \mathsf{d}(x_\mathfrak{n}, \mathcal{T} x_{\mathfrak{n}-1}) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \\ \mathsf{d}(x_{\mathfrak{n}+1}, \mathcal{T} x_\mathfrak{n}) = \mathsf{d}(\mathcal{E}, \mathcal{F}) \end{array}\right\} \Longrightarrow \mathsf{d}(x_\mathfrak{n}, x_{\mathfrak{n}+1}) = \mathsf{d}(\mathcal{T} x_{\mathfrak{n}-1}, \mathcal{T} x_\mathfrak{n}) \tag{3}$$

for all $\mathfrak{n} \in \mathbb{N}$. Therefore, the contractive condition (1) yields

$$\psi\Big(\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big) = \psi\Big(\mathsf{d}(\mathcal{T}x_{\mathfrak{n}-1}, \mathcal{T}x_{\mathfrak{n}})\Big) \le \psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}})\Big) - \varphi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}})\Big), \tag{4}$$

which implies $\psi\Big(\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big) \le \psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}})\Big)$. Since ψ is a nondecreasing function, we get

$$\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}) \le Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}), \tag{5}$$

where

$$Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) = \max\Big\{\mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}), \mathsf{d}(x_{\mathfrak{n}-1}, \mathcal{T}x_{\mathfrak{n}-1}) - \mathsf{d}(\mathcal{E}, \mathcal{F}), \mathsf{d}(x_{\mathfrak{n}}, \mathcal{T}x_{\mathfrak{n}}) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big\}.$$

On the other hand, using (2) and (3) as well as the triangle inequality, we have

$$\mathsf{d}(x_{\mathfrak{n}-1}, \mathcal{T}x_{\mathfrak{n}-1}) - \mathsf{d}(\mathcal{E}, \mathcal{F}) \le \mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) + \underbrace{\mathsf{d}(x_{\mathfrak{n}}, \mathcal{T}x_{\mathfrak{n}-1}) - \mathsf{d}(\mathcal{E}, \mathcal{F})}_{=0} = \mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}})$$

and

$$\mathsf{d}(x_{\mathfrak{n}}, \mathcal{T}x_{\mathfrak{n}}) - \mathsf{d}(\mathcal{E}, \mathcal{F}) \le \mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}) + \underbrace{\mathsf{d}(x_{\mathfrak{n}+1}, \mathcal{T}x_{\mathfrak{n}}) - \mathsf{d}(\mathcal{E}, \mathcal{F})}_{=0} = \mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}),$$

which induce that

$$\begin{aligned} Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) &= \max\Big\{\mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}), \mathsf{d}(x_{\mathfrak{n}-1}, \mathcal{T}x_{\mathfrak{n}-1}) - \mathsf{d}(\mathcal{E}, \mathcal{F}), \mathsf{d}(x_{\mathfrak{n}}, \mathcal{T}x_{\mathfrak{n}}) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big\} \\ &\le \max\Big\{\mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}), \mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}), \mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big\}. \end{aligned}$$

If $\mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) < \mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})$, then $\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}) = Q_{T}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) > 0$ and using (1), we have

$$\psi\Big(\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big) \le \psi\Big(\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big) - \varphi\Big(\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\Big),$$

which is a contradiction. Consequently,

$$\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}) \le Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) \le \mathsf{d}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) \tag{6}$$

for all $\mathfrak{n} \ge 1$. Hence, the sequence $\{\mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1})\}$ is monotone nonincreasing and bounded. Thus, there exists $\alpha \ge 0$ such that

$$\lim_{\mathfrak{n}\to\infty} \mathsf{d}(x_{\mathfrak{n}}, x_{\mathfrak{n}+1}) = \lim_{\mathfrak{n}\to\infty} Q_{\mathcal{T}}(x_{\mathfrak{n}-1}, x_{\mathfrak{n}}) = \alpha.$$

Now, suppose that $\alpha > 0$. By the contractive condition (1), the continuity of ψ and the lower semi-continuity of φ on $\mathbb{R}^{\geqslant 0}$, we have

$$\begin{aligned} \psi(\alpha) &\le \lim_{\mathfrak{n}\to\infty} \psi\Big(\mathsf{d}(x_{\mathfrak{n}+1}, x_{\mathfrak{n}})\Big) \\ &= \lim_{\mathfrak{n}\to\infty} \psi\Big(\mathsf{d}(\mathcal{T}x_{\mathfrak{n}}, \mathcal{T}x_{\mathfrak{n}-1})\Big) \\ &\le \lim_{\mathfrak{n}\to\infty} \sup\Big(\psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}}, x_{\mathfrak{n}-1})\Big) - \varphi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}}, x_{\mathfrak{n}-1})\Big)\Big) \\ &\le \lim_{\mathfrak{n}\to\infty} \psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}}, x_{\mathfrak{n}-1})\Big) - \lim_{n\to\infty} \inf \varphi\Big(Q_{\mathcal{T}}(x_{\mathfrak{n}}, x_{\mathfrak{n}-1})\Big)\Big) \\ &\le \psi(\alpha) - \varphi(\alpha), \end{aligned}$$

which yields $\varphi(\alpha) = 0$ and so $\alpha = 0$. Thus, $\lim_{\mathfrak{n}\to\infty} \mathsf{d}(x_{\mathfrak{n}+1}, x_{\mathfrak{n}}) = 0$. Now, we establish that sequence $\{x_{\mathfrak{n}}\}$ is Cauchy. To this end, assume on the contrary that there exist $\varepsilon > 0$ and positive integers $\mathfrak{m}_k$ and $\mathfrak{n}_k$ provided that

$$\mathfrak{m}_k > \mathfrak{n}_k \geq k \quad \text{and} \quad \mathsf{d}(x_{\mathfrak{m}_k}, x_{\mathfrak{n}_k}) \geq \varepsilon \qquad k = 1, 2, \cdots .$$

Keeping the integer $\mathfrak{n}_k$ fixed for sufficiently large k, say $k \geq k_0$, one can assume without loss of generality that $\mathfrak{m}_k$ is the smallest integer greater than $\mathfrak{n}_k$ with $\mathsf{d}(x_{\mathfrak{m}_k}, x_{\mathfrak{n}_k}) \geq \varepsilon$; that is, $\mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k}) < \varepsilon$ for all $k \geq k_0$. Thus, we have

$$\begin{aligned}\varepsilon &\leq \mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\\ &\leq \mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k}) + \mathsf{d}(x_{\mathfrak{n}_k}, x_{\mathfrak{n}_k-1})\\ &< \varepsilon + \mathsf{d}(x_{\mathfrak{m}_k}, x_{\mathfrak{m}_k-1})\end{aligned}$$

for all $k \geq k_0$. Letting $k \to \infty$, we obtain $\mathsf{d}(x_{\mathfrak{m}_k}, x_{\mathfrak{m}_k-1}) \to 0$ and so $\mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1}) \to \varepsilon$. On the other hand,

$$\begin{aligned}\varepsilon &\leq Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\\ &= \max\Big\{\mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1}), \mathsf{d}(x_{\mathfrak{m}_k-1}, \mathcal{T}x_{\mathfrak{m}_k-1}) - \mathsf{d}(\mathcal{E},\mathcal{F}), \mathsf{d}(x_{\mathfrak{n}_k-1}, \mathcal{T}x_{\mathfrak{n}_k-1}) - \mathsf{d}(\mathcal{E},\mathcal{F})\Big\}\\ &\leq \max\Big\{\mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1}), \mathsf{d}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{m}_k}), \mathsf{d}(x_{\mathfrak{n}_k-1}, x_{\mathfrak{n}_k})\Big\},\end{aligned}$$

which by taking limitation induces that $\lim_{k\to\infty} Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1}) = \varepsilon$. Consequently, by the contractive condition (1), the continuity of ψ and the lower semi-continuity of φ on $\mathbb{R}^{\geqslant 0}$, we have

$$\begin{aligned}\psi(\varepsilon) &\leq \lim_{k\to\infty} \psi\Big(\mathsf{d}(x_{\mathfrak{m}_k}, x_{\mathfrak{n}_k})\Big)\\ &= \lim_{k\to\infty} \psi\Big(\mathsf{d}(\mathcal{T}x_{\mathfrak{m}_k-1}, \mathcal{T}x_{\mathfrak{n}_k-1})\Big)\\ &\leq \lim_{k\to\infty} \sup\Big(\psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\Big) - \varphi\Big(Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\Big)\Big)\\ &\leq \lim_{k\to\infty} \psi\Big(Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\Big) - \lim_{k\to\infty} \inf \varphi\Big(Q_{\mathcal{T}}(x_{\mathfrak{m}_k-1}, x_{\mathfrak{n}_k-1})\Big)\\ &\leq \psi(\varepsilon) - \varphi(\varepsilon),\end{aligned}$$

which yields $\varphi(\varepsilon) = 0$ and so $\varepsilon = 0$, which is a contradiction. Therefore, the sequence $\{x_{\mathfrak{n}}\}$ is Cauchy in $\mathcal{E}_0 \subseteq \mathcal{E}$. Since $\mathcal{E}$ is a closed subset of a complete metric space $\mathcal{X}$, there exists a $x^* \in \mathcal{E}$ such that $x_{\mathfrak{n}} \to x^*$. Now, we establish that x^* is a bpp of $\mathcal{T}$. Using the continuity of $\mathcal{T}$ on $\mathcal{E}$, we have $\mathcal{T}x_{\mathfrak{n}} \to \mathcal{T}x^*$. Also, by the joint continuity of d, we conclude that $\mathsf{d}(x_{\mathfrak{n}+1}, \mathcal{T}x_{\mathfrak{n}}) \to \mathsf{d}(x^*, \mathcal{T}x^*)$. On the other hand, it follows by (2) that $\{\mathsf{d}(x_{\mathfrak{n}+1}, \mathcal{T}x_{\mathfrak{n}})\}$ is constant sequence converging to $\mathsf{d}(\mathcal{E},\mathcal{F})$. Since the limit of a sequence is unique, we get $\mathsf{d}(x^*, \mathcal{T}x^*) = \mathsf{d}(\mathcal{E},\mathcal{F})$; that is, x^* is a bpp of $\mathcal{T}$. Moreover, we have $x^* \in \mathcal{E}_0$ and $\mathcal{T}x^* \in \mathcal{F}_0$.

To prove uniqueness, assume that x^{**} is another bpp of $\mathcal{T}$ such that $x^* \preceq x^{**}$. Since $x^*, x^{**} \in \mathcal{E}_0$ and $\mathcal{T}x^*, \mathcal{T}x^{**} \in \mathcal{F}_0$, it follows from the P-property that

$$\left.\begin{array}{r}\mathsf{d}(x^*, \mathcal{T}x^*) = \mathsf{d}(\mathcal{E},\mathcal{F})\\ \mathsf{d}(x^{**}, \mathcal{T}x^{**}) = \mathsf{d}(\mathcal{E},\mathcal{F})\end{array}\right\} \Longrightarrow \mathsf{d}(x^*, x^{**}) = \mathsf{d}(\mathcal{T}x^*, \mathcal{T}x^{**}). \tag{7}$$

Hence,

$$Q_{\mathcal{T}}(x^*, x^{**}) = \max\Big\{\mathsf{d}(x^*, x^{**}), \mathsf{d}(x^*, \mathcal{T}x^*) - \mathsf{d}(\mathcal{E},\mathcal{F}), \mathsf{d}(x^{**}, \mathcal{T}x^{**}) - \mathsf{d}(\mathcal{E},\mathcal{F})\Big\} = \mathsf{d}(x^*, x^{**}). \tag{8}$$

Now, by (1), (7) and (8), we have

$$\begin{aligned}\psi\Big(\mathsf{d}(x^*, x^{**})\Big) &= \psi\Big(\mathsf{d}(\mathcal{T}x^*, \mathcal{T}x^{**})\Big)\\ &\leq \psi\Big(Q_{\mathcal{T}}(x^*, x^{**})\Big) - \varphi\Big(Q_{\mathcal{T}}(x^*, x^{**})\Big)\\ &= \psi\Big(\mathsf{d}(x^*, x^{**})\Big) - \varphi\Big(\mathsf{d}(x^*, x^{**})\Big),\end{aligned}$$

which yields $\varphi\Big(\mathsf{d}(x^*, x^{**})\Big) = 0$ and so $\mathsf{d}(x^*, x^{**}) = 0$; that is, $x^* = x^{**}$. $\square$

Example 2.3. *Consider the same usual metric on $\mathcal{X} = \mathbb{R}^2$ as below*

$$\mathsf{d}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad \textit{where} \qquad (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$$

and suppose

$$\mathcal{E} = \Big\{(x, 1) : x \in [0, 1]\Big\} \ \textit{ and } \ \mathcal{F} = \Big\{(y, 0) : y \in [0, 1]\Big\}.$$

Also, consider $\varphi, \psi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ as below

$$\psi(t) = \begin{cases} t^2, & 0 \leq t < \dfrac{1}{2} \\ \dfrac{t}{2}, & t \geq \dfrac{1}{2} \end{cases} \quad \textit{and} \quad \varphi(t) = \begin{cases} \dfrac{t^2}{4}, & 0 \leq t < \dfrac{1}{2} \\ \dfrac{t}{8}, & t \geq \dfrac{1}{2} \end{cases}.$$

Let the mapping $\mathcal{T} : \mathcal{E} \to \mathcal{F}$, where

$$\mathcal{T}(x, 1) = \begin{cases} (0, 0), & 0 \leq x < 1 \\ (\dfrac{2}{3}, 0), & x = 1 \end{cases}.$$

Note that for the elements $(1, 1)$ and $(\frac{1}{2}, 1)$, we have

$$Q_T\Big((1, 1), (\frac{1}{2}, 1)\Big) = \max\Big\{\frac{1}{2}, \sqrt{\frac{1}{9} + 1} - 1, \sqrt{\frac{1}{4} + 1} - 1\Big\} = \frac{1}{2}$$

and, by (1), *we obtain*

$$\begin{aligned} \psi\Big(\mathsf{d}\Big(\mathcal{T}(1, 1), \mathcal{T}(\frac{1}{2}, 1)\Big)\Big) &= \psi\Big(\mathsf{d}\Big((\frac{2}{3}, 0), (0, 0)\Big)\Big) \\ &= \frac{1}{3} \\ &> \frac{3}{16} \\ &= \psi(\frac{1}{2}) - \varphi(\frac{1}{2}) \\ &= \psi\Big(Q_T\Big((1, 1), (\frac{1}{2}, 1)\Big)\Big) - \varphi\Big(Q_T\Big((1, 1), (\frac{1}{2}, 1)\Big)\Big). \end{aligned}$$

Hence, $\mathcal{T}$ does not satisfy the non-order version of (1).

Now, define a partial order relation $\preceq$ on $\mathbb{R}^2$ by

$$\begin{aligned} (x_1, x_2) \preceq (y_1, y_2) \Longleftrightarrow\ & (x_1 = y_1, x_2 = y_2) \\ & \vee \Big((x_1, x_2) = (0, 1), (y_1, y_2) = (1, 1)\Big) \\ & \vee \Big((x_1, x_2) = (1, 1), (y_1, y_2) = (0, 1)\Big). \end{aligned}$$

Then $(\mathbb{R}^2, \mathsf{d}, \preceq)$ is a complete POMS. Also, $\mathsf{d}(\mathcal{E}, \mathcal{F}) = 1$, $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{F} = \mathcal{F}_0$. Moreover, one can simply prove that $(\mathcal{E}, \mathcal{F})$ have the P-property, $\mathcal{T}$ is ordered proximal and $\mathcal{T}(\mathcal{E}_0) \subseteq \mathcal{F}_0$. Next, assume that $x \in [0, 1]$. Then we have

$$\psi\Big(\mathsf{d}\Big(\mathcal{T}(x, 1), \mathcal{T}(x, 1)\Big)\Big) = \psi(0) \leq \psi\Big(Q_T\Big((x, 1), (x, 1)\Big)\Big) - \varphi\Big(Q_T\Big((x, 1), (x, 1)\Big)\Big).$$

Also, we have

$$\mathcal{Q}_T\Big((0,1),(1,1)\Big) = \max\Big\{1, 0, \sqrt{\frac{1}{9}+1} - 1\Big\} = 1,$$

which yields

$$\begin{aligned}\psi\Big(\mathsf{d}\Big(\mathcal{T}(0,1),\mathcal{T}(1,1)\Big)\Big) &= \psi\Big(\mathsf{d}\Big((0,0),(\frac{2}{3},0)\Big)\Big)\\ &= \frac{1}{3}\\ &\leq \frac{3}{8}\\ &= \psi\Big(\mathcal{Q}_T\Big((0,1),(1,1)\Big)\Big) - \varphi\Big(\mathcal{Q}_T\Big((0,1),(1,1)\Big)\Big).\end{aligned}$$

Thus, $\mathcal{T}$ satisfies in contractive condition (1)*. Hence, all hypotheses of Theorem 2.2 are held. Consequently, $\mathcal{T}$ has a bpp $x^* = (0,1)$. Now, let $x^{**} = (x,1) \in \mathcal{E}$ with $x \in [0,1]$ be another bpp of $\mathcal{T}$. If $x \in [0,1)$, then*

$$\mathsf{d}\Big((x,1),\mathcal{T}(x,1)\Big) = \mathsf{d}\Big((x,1),(0,0)\Big) = \sqrt{x^2+1} > \mathsf{d}(\mathcal{E},\mathcal{F}).$$

Otherwise, if $x = 1$, then

$$\mathsf{d}\Big((1,1),\mathcal{T}(1,1)\Big) = \mathsf{d}\Big((1,1),(\frac{2}{3},0)\Big) = \sqrt{\frac{1}{9}+1} > \mathsf{d}(\mathcal{E},\mathcal{F}),$$

which is a contradiction. Hence, $(0,1)$ *is the unique bpp of $\mathcal{T}$.*

Remark 2.4. *By selecting special types of the function ψ and φ, we can replace the condition* (iii) *of Theorem 2.2 with one of the following conditions:*

(I) $\mathcal{T}$ *is continuous on $\mathcal{E}$ and*

$$\mathsf{d}(\mathcal{T}x,\mathcal{T}y) \leq \mathcal{Q}_{\mathcal{T}}(x,y) - \varphi\Big(\mathcal{Q}_{\mathcal{T}}(x,y)\Big)$$

for all $x, y \in \mathcal{E}$ with $x \preceq y$, where

$$\mathcal{Q}_{\mathcal{T}}(x,y) = \max\Big\{\mathsf{d}(x,y), \mathsf{d}(x,\mathcal{T}x) - \mathsf{d}(\mathcal{E},\mathcal{F}), \mathsf{d}(y,\mathcal{T}y) - \mathsf{d}(\mathcal{E},\mathcal{F})\Big\}.$$

For this case, it is sufficient to put $\psi(\mathsf{t}) = \mathsf{t}$ in Theorem 2.2.

(II) $\mathcal{T}$ *is continuous on $\mathcal{E}$ and there exists $\alpha \in [0,1)$ such that*

$$\mathsf{d}(\mathcal{T}x,\mathcal{T}y) \leq \alpha \cdot \mathcal{Q}_{\mathcal{T}}(x,y)$$

for all $x, y \in \mathcal{E}$ with $x \preceq y$, where

$$\mathcal{Q}_{\mathcal{T}}(x,y) = \max\Big\{\mathsf{d}(x,y), \mathsf{d}(x,\mathcal{T}x) - \mathsf{d}(\mathcal{E},\mathcal{F}), \mathsf{d}(y,\mathcal{T}y) - \mathsf{d}(\mathcal{E},\mathcal{F})\Big\}.$$

For this case, it is sufficient to put $\psi(\mathsf{t}) = \mathsf{t}$ and $\varphi(\mathsf{t}) = (1-\alpha)\mathsf{t}$ in Theorem 2.2.

Remark 2.5. (ψ,φ)*-weak contractions involves a lot of contractions defined until now. So we can obtain Theorem 2.2 for various contractions and in a complete POMS. For all $(x,y) \in \mathcal{E}$ with $x \preceq y$, some of the contractions obtained from* (ψ,φ)*-weak contractions as below:*

- *There exists $\alpha \in [0,1)$ such that $\mathsf{d}(\mathcal{T}x,\mathcal{T}y) \leq \alpha\mathsf{d}(x,y)$. In this case, $\mathcal{T}$ is automatically continuous on $\mathcal{E}$;*

- *There exists* $\alpha \in [0, \frac{1}{2})$ *provided that*

$$\mathsf{d}(\mathcal{T}x, \mathcal{T}y) \leq \alpha\Big(\mathsf{d}(x, \mathcal{T}x) + \mathsf{d}(y, \mathcal{T}y)\Big) - 2\alpha\mathsf{d}(\mathcal{E}, \mathcal{F});$$

- *There exist* $\alpha, \beta \in [0, \frac{1}{2})$ *provided that*

$$\mathsf{d}(\mathcal{T}x, \mathcal{T}y) \leq \alpha\mathsf{d}(x, \mathcal{T}x) + \beta\mathsf{d}(y, \mathcal{T}y) - (\alpha + \beta)\mathsf{d}(\mathcal{E}, \mathcal{F});$$

- *There exist* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma < 1$ *provided that*

$$\mathsf{d}(\mathcal{T}x, \mathcal{T}y) \leq \alpha\mathsf{d}(x, y) + \beta\Big((\mathsf{d}(x, \mathcal{T}y) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big) + \gamma\Big(\mathsf{d}(y, \mathcal{T}x) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big);$$

- *There exist functions* $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{\geqslant 0}$ *with*

$$\sup\Big\{\alpha(x, y) + \beta(x, y) + \gamma(x, y) : (x, y) \in \mathcal{X} \times \mathcal{X}\Big\} = \lambda < 1$$

such that

$$\mathsf{d}(\mathcal{T}x, \mathcal{T}y) \leq \alpha(x, y)\mathsf{d}(x, y) + \beta(x, y)\Big(\mathsf{d}(x, \mathcal{T}x) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big) + \gamma(x, y)\Big(\mathsf{d}(y, \mathcal{T}y) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big).$$

3. Application

In 2002, the idea of using Lebesgue integrals in metric fixed point theory was introduced by Branciari [2]. In fact, he considered mapping T from a complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx, Ty)} \varphi(t)dt \leq \alpha \int_0^{d(x,y)} \varphi(t)dt$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable function whose Lebesgue-integral is finite on each compact subset of $[0, +\infty)$ and satisfies $\int_0^\varepsilon \varphi(t)dt > 0$ for all $\varepsilon > 0$. Then he investigated the existence and uniqueness of fixed points for mappings satisfying such integrally contraction.

Now, let λ be the Lebesgue measure on the Borel σ-algebra of the metric subspace $\mathbb{R}^{\geqslant 0}$ of $\mathbb{R}$, $E = [a, b]$ be a Borel set and $\int_a^b \varphi(\mathrm{t})dt$ be Lebesgue integral of a function φ on E. Also, let Φ be a class of all functions $\varphi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ satisfying the following properties:

(Φ1) φ is Lebesgue-integrable on $\mathbb{R}^{\geqslant 0}$;

(Φ2) $\int_0^\varepsilon \varphi(\mathrm{t})d\mathrm{t} > 0$ and finite for all $\varepsilon > 0$.

Theorem 3.1. *Assume that* $(\mathcal{X}, \mathsf{d})$ *is a complete POMS and* $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ *is a mapping such that the following properties hold:*

(*i*) $\mathcal{T}$ *is order proximal,* $\mathcal{T}(\mathcal{E}_0) \subseteq \mathcal{F}_0$ *and* $(\mathcal{E}, \mathcal{F})$ *have the P-property;*

(*ii*) *There exist elements* $x_0, x_1 \in \mathcal{E}_0$ *provided that* $x_0 \preceq x_1$ *and* $\mathsf{d}(x_1, \mathcal{T}x_0) = \mathsf{d}(\mathcal{E}, \mathcal{F})$*;*

(*iii*) $\mathcal{T}$ *is continuous on* $\mathcal{E}$ *and there exist* $\mu, \gamma \in \Phi$ *provided that*

$$\int_0^{\mathsf{d}(\mathcal{T}x, \mathcal{T}y)} \mu(\mathrm{t})dt \leq \int_0^{Q_{\mathcal{T}}(x,y)} \mu(\mathrm{t})d\mathrm{t} - \int_0^{Q_{\mathcal{T}}(x,y)} \gamma(\mathrm{t})d\mathrm{t}$$

for all $x, y \in \mathcal{E}$ *with* $x \preceq y$*, where*

$$Q_{\mathcal{T}}(x, y) = \max\Big\{\mathsf{d}(x, y), \mathsf{d}(x, \mathcal{T}x) - \mathsf{d}(\mathcal{E}, \mathcal{F}), \mathsf{d}(y, \mathcal{T}y) - \mathsf{d}(\mathcal{E}, \mathcal{F})\Big\}.$$

Then $\mathcal{T}$ has a bpp in $\mathcal{E}$. Furthermore, if for any two bpp(s) $\mathsf{u}, \mathsf{v} \in \mathcal{E}$ we have $\mathsf{u} \preceq \mathsf{v}$, then $\mathcal{T}$ has a unique bpp in $\mathcal{E}$.

Proof. In Theorem 2.2, we set

$$\psi(s) = \int_0^s \mu(\mathsf{t})d\mathsf{t} \qquad \text{and} \qquad \varphi(s) = \int_0^s \gamma(\mathsf{t})d\mathsf{t}.$$

Clearly, ψ is nondecreasing and continuous, and φ is lsc on $\mathbb{R}^{\geqslant 0}$ and $\varphi(s) = 0$ implies $s = 0$. Then the result follows from Theorem 2.2. □

If we set $\mathcal{E} = \mathcal{F} = \mathcal{X}$ in Theorem 3.1, then we get the following corollary.

Corollary 3.2. *Let $(\mathcal{X}, \mathsf{d})$ be a complete POMS and $\mathcal{T}$ be a self-mapping on $\mathcal{X}$ satisfying the following conditions:*

(*i*) *$\mathcal{T}$ is order preserving; that is, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $(x, y) \in \mathcal{X}$;*

(*ii*) *There exist $x_0, x_1 \in \mathcal{E}_0$ so that $x_0 \preceq \mathcal{T}x_0$;*

(*iii*) *$\mathcal{T}$ is continuous on $\mathcal{X}$ and there exist $\mu, \gamma \in \Phi$ such that*

$$\int_0^{\mathsf{d}(\mathcal{T}x,\mathcal{T}y)} \mu(\mathsf{t})d\mathsf{t} \leq \int_0^{\max\left\{\mathsf{d}(x,y),\mathsf{d}(x,\mathcal{T}x),\mathsf{d}(y,\mathcal{T}y)\right\}} \mu(\mathsf{t})d\mathsf{t} - \int_0^{\max\left\{\mathsf{d}(x,y),\mathsf{d}(x,\mathcal{T}x),\mathsf{d}(y,\mathcal{T}y)\right\}} \gamma(\mathsf{t})d\mathsf{t}$$

for all $x, y \in \mathcal{E}$ with $x \preceq y$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{E}$. Furthermore, if for any two fixed points $\mathsf{u}, \mathsf{v} \in \mathcal{E}$ we have $\mathsf{u} \preceq \mathsf{v}$, then $\mathcal{T}$ has a unique fixed point in $\mathcal{E}$.

References

[1] H. Aydi, A. Felhi, E. Karapınar, On common best proximity points for generalized α-ψ-proximal contractions. J. Nonlinear Sci. Appl. 9 (5) (2016) 2658–2670.

[2] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Sci. 29 (9) (2002) 531–536.

[3] K. Fallahi, H. Ghahramani, G. Soleimani Rad, Integral type contractions in partially ordered metric spaces and best proximity point, Iran. J. Sci. Technol. Trans. Sci. 44 (2020) 177–183.

[4] K. Fallahi, G. Soleimani Rad, Best proximity points theorem in b-metric spaces endowed with a graph, Fixed Point Theory. 21 (2) (2020) 465–474.

[5] F. Fouladia, A Abkara, E. Karapınar, A discussion on the coincidence quasi-best proximity points, Filomat. 35 (6) (2021) 2107–2119.

[6] L. Gholizadeh, E. Karapınar, Best proximity point results in dislocated metric spaces via R-functions, RACSAM. 112 (4) (2018) 1391-–1407.

[7] G. Hiranmoy, E. Karapınar, L. K. Dey, Best proximity point results for contractive and cyclic contractive type mappings, Numer. Funct. Anal. Optim. 42 (7) (2021) 849–864.

[8] M. Jleli, E. Karapınar, B. Samet, Best proximity points for generalized-proximal contractive type mappings, J. Appl. Math. (2013) 2013:514327.

[9] M. Jleli, B. Samet, Best proximity points for α-ψ-proximal contractive type mappings and applications, Bulletin des Sciences Mathématiques. 137 (8) (2013) 977–995.

[10] E. Karapınar, Best proximity points Of cyclic mappings, Appl. Math. Lett. 25 (11) (2012) 1761–1766.

[11] E. Karapınar, On best proximity point of ψ-Geraghty contractions, Fixed Point Theory Appl. (2013) 2013:1.

[12] E. Karapınar, C. M. Chen, C. T. Lee, Best proximity point theorems for two weak cyclic contractions on metric-like spaces, Mathematics. 7 (2019) 7:349.

[13] E. Karapınar, S. Karpagam, P. Mahadevan, B. Zlatanov, On Ω class of mappings in a p-cyclic complete metric space, Symmetry. 11 (2019) 11:534.

[14] E. Karapınar, F. Khojasteh, An approach to best proximity points results via simulation functions, J. Fixed Point Theory Appl. 19 (3) (2017) 1983–1995.

[15] E. Karapınar, S. Romaguera, K. Tas, Fixed points for cyclic orbital generalized contractions on complete metric spaces, Cent. Eur. J. Math. 11 (3) (2013) 552–560.

[16] A. Kostic, E. Karapınar, V. Rakocević, Best proximity points and fixed points with R-functions in the framework of w-distances, Bull. Australian Math. Soc. 99 (3) (2019) 497–507.

[17] P. Magadevan, S. Karpagam, E. Karapınar, Existence of fixed point and best proximity point of p-cyclic orbital ϕ-contraction map, Nonlinear Anal. 7 (1) (2022) 91–101.

[18] VS. Raj, A best proximity point theorem for weakly contractive non-self mappings, Nonlinear Anal. 74 (2011) 4804–4808.

[19] S. Sadig Basha, Best proximity point theorems in the frameworks of fairly and proximally complete spaces, J. Fixed Point Theory Appl. 19 (3) (2017) 1939–1951.

[20] S. Sadiq Basha, Discrete optimization in partially ordered sets, J. Glob. Optim. 54 (2012) 511–517.