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On strongly partial-quasi k-metric spaces

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Abstract. In this paper, we introduce the concepts of partial-quasi k-metric spaces and strongly partial-quasi k-metric spaces, and their relationship to k-metric spaces and partial-quasi metric spaces are studied. Furthermore, we obtain some results on fixed point theorems in strongly partial-quasi k-metric spaces.

1. Introduction and preliminaries

Since Wilson introduced the notion of quasi-metric spaces in 1931 [1], several generalizations of metric spaces were studied by topological researchers [2–22]. For instance, Bakbtin introduced the notion of b-metric spaces in 1989 [2] (see also Czerwik in 1993 [3]). From a different point of view, Matthews gave the concept of the partial metric spaces in 1994 [4]. As a generalization of Matthews, Heckmann defined the concept of weak partial metric spaces in 1999 [5]. Later, Künzi et al and Karapinar studied another two variants of partial metric spaces, namely partial-quasi metric spaces in 2006 [6] and quasi-partial metric spaces in 2013 [7], respectively. In the past years, Shukla introduced the notion of partial b-metric spaces in 2013 [8], which combines the b-metric space with partial metric space. Further, Gupta introduced the notion of quasi-partial b-metric spaces and studied some fixed point theorems on these spaces in 2015 [9].

In addition, the term "b-metric" has no justification for the letter "b" and says nothing about the constant "k" laid in the basis of the definition of such "metrics". In this paper, we continue to generalize the concept of k-metric and partial-quasi metrics by introducing the partial-quasi k-metric. Also we give some fixed point theorems in these spaces.

First, we recall some basic notions and results that will be used in the following sections (see more details in [4–13]).

Throughout this paper, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}^+$ always denote the set of real numbers, of all positive real numbers and of positive integers, respectively.

Definition 1.1. *[1] A* quasi-metric *d is a function $d : X \times X \to [0, +\infty)$ satisfying the following conditions: $\forall x, y, z \in X$,*

(M1) $x = y \Leftrightarrow d(x, y) = 0$;

(M2) $d(x, z) \leq d(x, y) + d(y, z)$.

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A quasi-metric d is called a metric *if it also satisfies*

(M3) $d(x,y) = d(y,x)$.

A (quasi-)metric space is a pair (X,d) *such that d is a (quasi-)metric on X.*

Definition 1.2. *A* partial-quasi k-metric *is function* $p_k : X \times X \to [0,+\infty)$ *satisfying the following conditions:* $\forall x,y,z \in X$,

(PK1) $x = y \Leftrightarrow p_k(x,x) = p_k(x,y)$ *and* $p_k(y,y) = p_k(y,x)$;

(PK2) $p_k(x,x) \leq p_k(x,y) \wedge p_k(y,x)$;

(PK3) $p_k(x,z) \leq k[p_k(x,y) + p_k(y,z)] - p_k(y,y)$.

A partial-quasi k-metric is called a partial k-metric *if it also satisfies*

(PK4) $p_k(x,y) = p_k(y,x)$.

Particularly, a partial(-quasi) 1*-metric (i.e. k=1) is called* partial(-quasi) metric*[6].*

Remark 1.3. *(1) A partial(-quasi) metric p on X is a (quasi-)metric if and only if* $p(x,x) = 0$ *for all* $x \in X$ *[4].*

(2) A partial(-quasi) k-metric p_k *on X is called a* (quasi-)k-metric *if* $p_k(x,x) = 0$ *for all* $x \in X$ *[2].*

Next, we give an example of quasi-partial k-metrics, which is not a partial quasi-metric.

Example 1.4. *Let* $X = [0,+\infty)$ *and define* p_k*:*$X \times X \to [0,+\infty)$ *by*

$$p_k(x,y) = |x-y|^3 + 1$$

for all $x,y \in X$*. It is not difficult to prove that* (X,p_k) *is a partial-quasi k-metric space with* $k = 4$*. But it is not a partial-quasi metric space. To show this, let* $x = 1$*,* $y = 2$ *and* $z = 4$*. Then* $p_k(x,z) = 28$*, it holds that* $p_k(x,y) = 2$*,* $p_k(y,z) = 9$ *and* $p_k(y,y) = 1$*, which implies that*

$$p_k(x,z) > p_k(x,y) + p_k(y,z) - p_k(y,y).$$

Hence, p_k *does not satisfy the condition (PK3) with* $k = 1$.

The following example shows that a partial-quasi k-metric may not be a partial k-metric.

Example 1.5. *Let* $X = [0,+\infty)$ *and define* p_k*:*$X \times X \to [0,+\infty)$*. Set* $p_k(x,y) = |x-y| + x$*. Since* p_k *does not satisfy the condition (PK4), it is not a partial k-metric (hence not a k-metric).*

Next, we verify the conditions (PK1)-(PK3) one by one.

(PK1): Suppose $p_k(x,x) = p_k(x,y)$ *and* $p_k(y,y) = p_k(y,x)$*. Then* $x = |x-y| + x$ *and* $y = |y-x| + y$*, which implies* $|x-y| = 0$*, so* $x = y$*.*

(PK2): For any $x,y \in X$*, since* $p_k(x,x) = x$ *and* $p_k(x,y) = |x-y| + x$*, it holds that* $p_k(x,x) \leq p_k(x,y)$*. In addition, since* $x = |x-y+y| \leq |x-y| + y$*, it holds that* $p_k(x,x) \leq p_k(y,x)$*.*

(PK3): For each $x,y,z \in X$*, we have*

$$\begin{aligned} p_k(x,z) &= |x-y+y-z| + x \\ &\leq (|x-y| + x) + (|y-z| + y) - y \\ &= p_k(x,y) + p_k(y,z) - p_k(y,y). \end{aligned}$$

Hence, (X,p_k) *is a partial-quasi k-metric space with* $k = 1$.

2. The relations among partial-quasi k-metrics, partial quasi-metrics and k-metrics

Proposition 2.1. *Let X be a nonempty set, p a partial-quasi metric and d_k a quasi-k-metric with coefficient $k \geq 1$ on X. Then the function $p_k : X \times X \to [0, +\infty)$ defined by*

$$\forall x, y \in X, \qquad p_k(x, y) = p(x, y) + d_k(x, y).$$

Then p_k is a partial-quasi k-metric on X.

Proof. It is trivial to prove that p_k satisfied (PK1) and (PK2). Next, we verify condition (PK3).

Let $x, y, z \in X$. Since p is a partial-quasi metric, it follows that $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$. Moreover, since d_k is a k-metric, we have that $d_k(x, z) \leq k[d_k(x, y) + d_k(y, z)]$. Thus

$$\begin{aligned}
&p_k(x, z) \\
&= p(x, z) + d_k(x, z) \\
&\leq [p(x, y) + p(y, z) - p(y, y)] + k[d_k(x, y) + d_k(y, z)] \\
&\leq k[p(x, y) + p(y, z) + d_k(x, y) + d_k(y, z)] - p(y, y) \\
&= k[(p(x, y) + d_k(x, y)) + (p(y, z) + d_k(y, z))] - [(p(y, y) + d_k(y, y)] \\
&= k[p_k(x, y) + p_k(y, z)] - p_k(y, y),
\end{aligned}$$

completing the proof. □

From the above Proposition, one can obtain partial-quasi k-metric by adding a partial-quasi metric and aquasi-k-metric. An example is shown as follows.

Example 2.2. *Let $X = \mathbb{R}$ and define a function d_k and p on X as follows: $\forall x, y \in X$,*

$$d_k(x, y) = |x - y|^3 \ \text{ and } \ p(x, y) = |x - y| + 1,$$

respectively. Then it is trivial to check that d_k is a k-quasi-metric and p is a partial metric. Therefore, by Proposition 2.1, $p_k : X \times X \to [0, +\infty)$ defined by

$$p_k(x, y) = |x - y|^3 + |x - y| + 1$$

is a partial-quasi k-metric.

Since Künzi, Pajoohesh and Schellekens investigated the concept of a partial quasi-metric and some of its applications by dropping the symmetry condition in the definition of a partial metric given by Matthews, roughly speaking a partial quasi metric is a partial metric with does not satisfy the symmetry property. On the other hand, Mustafa, Roshan, Parvaneh and Kadelburg modified the condition (P_{b4}) in the definition of a partial-b metric given by Shukla, and introduced another variant concept of a partial-b metric. In the following definition, we modify Definition 1.2 in order to obtain that each strongly partial-quasi k-metric p_k^s generates a k-metric.

Definition 2.3. *A* strongly partial-quasi k-metric *is a function $p_k^s : X \times X \to [0, +\infty)$ satisfying the following conditions: for some number $k \geq 1$, $\forall x, y, z \in X$,*

- *(SPK1)* $p_k^s(x, x) = p_k^s(x, y) = p_k^s(y, y) \Leftrightarrow x = y$;
- *(SPK2)* $p_k^s(x, x) \leq p_k^s(x, y) \wedge p_k^s(y, x)$;
- *(SPK3)* $p_k^s(x, z) \leq k[p_k^s(x, y) + p_k^s(y, z)] - \dfrac{k-1}{2}[p_k^s(x, x) + p_k^s(z, z)] - kp_k^s(y, y)$.

A strongly partial-quasi k-metric space *is a pair (X, p_k^s) such that p_k^s is a strongly partial-quasi k-metric on X, the number k is called the* coefficient *of (X, p_k^s).*

Remark 2.4. *(1) By Definition 2.3, every strongly partial-quasi k-metric space with coefficient $k = 1$ is a partial-quasi metric space;*

(2) Every strongly partial-quasi k-metric space is a partial-quasi k-metric space. Indeed, we have

$$\begin{aligned} p_k^s(x,z) &\le k[p_k^s(x,y) + p_k^s(y,z)] - \frac{k-1}{2}[p_k^s(x,x) + p_k^s(z,z)] - kp_k^s(y,y) \\ &\le k[p_k^s(x,y) + p_k^s(y,z)] - p_k^s(y,y). \end{aligned}$$

But we can show the reverse may not be true in the following:

Let $X = X_1 \cup X_2$, where $X_1 = (-\infty, 0)$ and $X_2 = (0, +\infty)$. We define p_k: $X \times X \to [0, +\infty)$ as follows:

$$p_k(x,y) = \begin{cases} 1, & x = y; \\ 2q, & x, y \in X_1; \\ \frac{3}{2}, & \text{otherwise.} \end{cases}$$

for all $x, y \in X$, where $q > 1$.

It is trivial to verify that (X, p_k) is a partial-quasi k-metric space with coefficient $k = q$.

Indeed, let $x = -1, z = -2$ and $y = 1$. We have $p_k(-1,-2) = 2q$, $p_k(-1,1) = p_k(1,-2) = \frac{3}{2}$, and $p_k(-1,-1) = p_k(1,1) = p_k(-2,-2) = 1$. It follows that $p_k(-1,-2) > q[p_k(-1,1) + p_k(1,-2)] - \frac{q-1}{2}[p_k(-1,-1) + p_k(-2,-2)] - qp_k(1,1)$ for all $q > 1$. Hence, it is not a strongly partial-quasi k-metric.

Example 2.5. *Let $X = [0, +\infty)$, and define a function p_k^s:$X \times X \to [0, +\infty)$ by*

$$p_k^s(x,y) = |x-y|^3 + 3$$

for all $x, y \in X$. Then (X, p_k^s) is a strongly partial-quasi k-metric with coefficient $k = 4$.

In fact, it is trivial that p_k^s satisfies (SPK1) and (SPK2). Condition (SPK3) can be obtained by

$$\begin{aligned} p_k^s(x,z) &\le 2^{3-1}[|x-y|^3 + |y-z|^3] + 3 \\ &= 4[(|x-y| + 3) + (|y-z| + 3)] - 21 \\ &= 4[p_k^s(x,y) + p_k^s(y,z)] - 21 \\ &= 4[p_k^s(x,y) + p_k^s(y,z)] - \frac{4-1}{2}[p_k^s(x,x) + p_k^s(z,z)] - 4p_k^s(y,y), \end{aligned}$$

completing the proof.

Proposition 2.6. *Let (X, p_k^s) be a strongly partial-quasi k-metric. The following statements hold.*

(1) If $p_k^s(x,y) = 0$, then $x = y$.

(2) The set of all p_k^s-balls $\mathbb{B}_{p_k^s}(x,r)$ in (X, p_k^s) forms a base for a topology, denoted by $\mathcal{T}(p_k^s)$, where $\mathbb{B}_{p_k^s}(x,r) = \{y \in X : p_k^s(x,y) < p_k^s(x,x) + r\}$ for any $x \in X$ and $r > 0$.

(3) For any $x, y \in X$, define

$$\hat{p}_k^s(x,y) = p_k^s(x,y) + p_k^s(y,x) - p_k^s(x,x) - p_k^s(y,y).$$

Then $\hat{p}_k^s$ is a k-metric.

Proof. (1) Suppose $p_k^s(x,y) = 0$. Since $p_k^s(x,x) \le p_k^s(x,y) = 0$ and $p_k^s(y,y) \le p_k^s(x,y) = 0$ by (SPK2), we have that $p_k^s(x,x) = p_k^s(y,y) = 0$. From (SPK1), it follows that $x = y$.

(2) It is trivial by Theorem 3.3 in [4].

(3) We show that $\hat{p}_k^s$ satisfies the rules (M1), (M3) and (PK3) one by one.

First, suppose $\hat{p}_k^s(x,y) = 0$. Then $p_k^s(x,y) + p_k^s(y,x) = p_k^s(x,x) + p_k^s(y,y)$. Furthermore, $p_k^s(x,x) \le p_k^s(y,x)$ by (SPK2). Then we have that

$$p_k^s(x,y) + p_k^s(x,x) \le p_k^s(x,y) + p_k^s(y,x) = p_k^s(x,x) + p_k^s(y,y),$$

which implies that $p_k^s(x,y) \le p_k^s(y,y)$. Thus we have $p_k^s(x,y) = p_k^s(y,y)$. Similarly, $p_k^s(x,y) = p_k^s(x,x)$. Hence, $p_k^s(x,y) = p^s(y,y) = p_k^s(x,x)$, which implies $x = y$ by (SPK1).

(M3) is trivial.

(PK3): For any $x, y, z \in X$, we have

$$\begin{aligned}
&\hat{p}_k^s(x,z) \\
&= p_k^s(x,z) + p_k^s(z,x) - p_k^s(x,x) - p_k^s(z,z) \\
&\le k[p_k^s(x,y) + p_k^s(y,z)] - \frac{k-1}{2}[p_k^s(x,x) + p_k^s(z,z)] - kp_k^s(y,y) + k[p_k^s(z,y) + p_k^s(y,x)] \\
&- \frac{k-1}{2}[p_k^s(z,z) + p_k^s(x,x)] - kp_k^s(y,y) - p_k^s(x,x) - p_k^s(z,z) \\
&= k[p_k^s(x,y) + p_k^s(y,x) - p_k^s(x,x) - p_k^s(y,y)] + k[p_k^s(y,z) + p_k^s(z,y) - p_k^s(y,y) - p_k^s(z,z)] \\
&= k[\hat{p}_k^s(x,y) + \hat{p}_k^s(y,z)].
\end{aligned}$$

Therefore, $\hat{p}_k^s$ is a k-metric. □

3. Fixed point theorem on strongly partial-quasi k-metric spaces

In this section we give some fixed point results on strongly partial-quasi k-metric spaces. We begin by giving some basic notions that will be used in the following.

Definition 3.1. *Let (X, p_k^s) be a strongly partial-quasi k-metric space and $\{x_n\}$ a sequence in X.*

(1) A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $p_k^s(x,x) = \lim_{n\to+\infty} p_k^s(x,x_n)$;

(2) A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to+\infty} p_k^s(x_n,x_m)$ exists and is finite;

(3) (X, p_k^s) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\lim_{n,m\to+\infty} p_k^s(x_n,x_m) = \lim_{n\to+\infty} p_k^s(x_n,x) = p_k^s(x,x)$.

The limit of convergent sequence $\{x_n\}$ in a strongly partial-quasi k-metric space may not be unique, as shown in the following example.

Example 3.2. *Let $X = [0,+\infty)$ and define a function $p_k^s : X \times X \to [0,+\infty)$ by*

$$p_k^s(x,y) = x^2 \vee y^2 + 3$$

for all $x, y \in X$. It is not difficult to prove that (X, p_k^s) is a strongly partial-quasi k-metric with coefficient $k = 3$. Given a sequence $\{x_n\}$, where $x_n = 1$ for all $n \in \mathbb{N}^+$. For all $x \ge 1$, we have that $p_k^s(x_n,x) = x^2 + 3$ and $p_k^s(x,x) = x^2 + 3$, which implies that $\lim_{n\to+\infty} p_k^s(x_n,x) = p_k^s(x,x)$.

Lemma 3.3. *Let (X, p_k^s) be a strongly partial-quasi k-metric space, $\{x_n\}$ a sequence in X and $(X, \hat{p}_k^s)$ the corresponding k-metric space defined in Proposition 2.6, i.e., where $\hat{p}_k^s(x,y) = p_k^s(x,y) + p_k^s(y,x) - p_k^s(x,x) - p_k^s(y,y)$. Then the following statements hold.*

(1) A sequence is a Cauchy sequence in (X, p_k^s) if and only if it is a Cauchy sequence in $(X, \hat{p}_k^s)$.

(2) (X, p_k^s) is complete if and only if $(X, \hat{p}_k^s)$ is complete.
Furthermore, $\lim_{n\to+\infty} \hat{p}_k^s(x_n,x) = 0$ if and only if $p_k^s(x,x) = \lim_{n\to+\infty} p_k^s(x_n,x) = \lim_{n,m\to+\infty} p_k^s(x_n,x_m)$.

Proof. (1) ($\Rightarrow$) Let $\{x_n\}$ be a Cauchy sequence in (X, p_k^s). There exists $a \in [0, +\infty)$ such that $\lim_{n,m\to+\infty} p_k^s(x_n, x_m) = a$. Then for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^+$ such that

$$|p_k^s(x_n, x_m) - a| < \varepsilon/4, \ \forall n, m > n_0.$$

Then we have that

$$\begin{aligned}
& |\hat{p}_k^s(x_n, x_m)| \\
&= |p_k^s(x_n, x_m) + p_k^s(x_m, x_n) - p_k^s(x_n, x_n) - p_k^s(x_m, x_m)| \\
&= |(p_k^s(x_n, x_m) - a) + (p_k^s(x_m, x_n) - a) - (p_k^s(x_n, x_n) - a) - (p_k^s(x_m, x_m) - a)| \\
&\leq |p_k^s(x_n, x_m) - a| + |p_k^s(x_m, x_n) - a| + |p_k^s(x_n, x_n) - a| + |p_k^s(x_m, x_m) - a| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $(X, \hat{p}_k^s)$.

($\Leftarrow$) Suppose $\{x_n\}$ is a Cauchy sequence in $(X, \hat{p}_k^s)$ and let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}^+$ such that

$$\hat{p}_k^s(x_n, x_m) < \frac{\varepsilon}{2}, \ \forall n, m \geq n_\varepsilon.$$

Set $\varepsilon = 1$. Then there exists $n_0 \in \mathbb{N}^+$ such that

$$\hat{p}_k^s(x_n, x_m) < \frac{1}{2}, \ \forall n, m \geq n_0.$$

Step 1: Since $\hat{p}_k^s(x, y) = p_k^s(x, y) + p_k^s(y, x) - p_k^s(x, x) - p_k^s(y, y)$, we have that

$$\begin{aligned}
& p_k^s(x, y) - p_k^s(y, y) \\
= \ & \hat{p}_k^s(x, y) - [p_k^s(y, x) - p_k^s(x, x)] \\
\leq \ & \hat{p}_k^s(x, y),
\end{aligned}$$

which shows that $p_k^s(x, y) \leq \hat{p}_k^s(x, y) + p_k^s(y, y)$ for all $x, y \in X$. Now we can deduce that $p_k^s(x_n, x_{n_0}) \leq \hat{p}_k^s(x_n, x_{n_0}) + p_k^s(x_{n_0}, x_{n_0})$. From (SPK2), it follows that

$$\begin{aligned}
p_k^s(x_n, x_n) &\leq p_k^s(x_n, x_{n_0}) \\
&\leq \hat{p}_k^s(x_n, x_{n_0}) + p_k^s(x_{n_0}, x_{n_0}) \\
&< \tfrac{1}{2} + p_k^s(x_{n_0}, x_{n_0})
\end{aligned}$$

for all $n \geq n_0$, which implies that the sequence $\{p_k^s(x_n, x_n)\}$ is bounded in $\mathbb{R}$. Hence the sequence $\{p_k^s(x_n, x_n)\}$ exists a subsequence $\{p_k^s(x_{n_k}, x_{n_k})\}$ that is convergent and we denote $\lim_{n_k\to+\infty} p_k^s(x_{n_k}, x_{n_k}) = a$.

Step 2: By step 1, we have

$$p_k^s(x_n, x_n) - p_k^s(x_m, x_m) \leq p_k^s(x_n, x_m) - p_k^s(x_m, x_m) \leq \hat{p}_k^s(x_n, x_m)$$

and $\lim_{n\to+\infty} p_k^s(x_n, x_n) = \lim_{m\to+\infty} p_k^s(x_m, x_m) = a$ for all $n, m \geq n_0$. Then there exists $n_1 \geq n_0$ such that

$$|p_k^s(x_m, x_m) - a| < \frac{\varepsilon}{2}, \ \forall n, m \geq n_1.$$

Then for any $m, n \geq n_1$, we have that

$$\begin{aligned}
|p_k^s(x_n, x_m) - a| &= |(p_k^s(x_n, x_m) - p_k^s(x_m, x_m)) + (p_k^s(x_m, x_m) - a)| \\
&\leq [p_k^s(x_n, x_m) - p_k^s(x_m, x_m)] + |p_k^s(x_m, x_m) - a| \\
&\leq \hat{p}_k^s(x_n, x_m) + |p_k^s(x_m, x_m) - a| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

From step 1 and step 2, we obtain that $lim_{n,m\to+\infty}p_k^s(x_n,x_m) = a$, which implies that $\{x_n\}$ is a Cauchy sequence in (X, p_k^s).

(2)($\Leftarrow$) Let $\{x_n\}$ be a Cauchy sequence in (X, p_k^s), it is clear that $\{x_n\}$ is a Cauchy sequence in $(X, \hat{p}_k^s)$ by Lemma 3.3(1). Since $(X, \hat{p}_k^s)$ is complete, there exists $x \in X$ such that $\lim_{n\to+\infty} \hat{p}_k^s(x, x_n) = 0$. This shows that $\{x_n\}$ is a convergent sequence in $(X, \hat{p}_k^s)$, and we have

$$\lim_{n\to+\infty}[p_k^s(x,x_n) + p_k^s(x_n,x) - p_k^s(x,x) - p_k^s(x_n,x_n)] = 0.$$

As

$$p_k^s(x_n,x) - p_k^s(x,x) \le \hat{p}_k^s(x_n,x) = \hat{p}_k^s(x,x_n)$$

and

$$p_k^s(x,x_n) - p_k^s(x_n,x_n) \le \hat{p}_k^s(x,x_n),$$

we have

$$\lim_{n\to+\infty}[p_k^s(x_n,x) - p_k^s(x,x)] = 0$$

and

$$\lim_{n\to+\infty}[p_k^s(x,x_n) - p_k^s(x_n,x_n)] = 0.$$

Thus $p_k^s(x,x) = \lim_{n\to+\infty} p_k^s(x_n,x) = \lim_{n\to+\infty} p_k^s(x,x_n) = \lim_{n\to+\infty} p_k^s(x_n,x_n)$.

In addition, by (SPK3) we have

$$p_k^s(x_n,x_m) \le k[p_k^s(x_n,x) + p_k^s(x,x_m)] - \frac{k-1}{2}[p_k^s(x_n,x_n) + p_k^s(x_m,x_m)] - kp_k^s(x,x).$$

Then

$$\lim_{n,m\to+\infty} p_k^s(x_n,x_m) \le p_k^s(x,x).$$

Moreover $p_k^s(x_n,x_n) \le p_k^s(x_n,x_m)$, thus we have

$$p_k^s(x,x) \le \lim_{n,m\to+\infty} p_k^s(x_n,x_m).$$

Hence $\lim_{n,m\to+\infty} p_k^s(x_n,x_m) = p_k^s(x,x)$. This implies (X, p_k^s) is complete.

($\Rightarrow$) Let $\{x_n\}$ be a Cauchy sequence in $(X, \hat{p}_k^s)$. Then $\{x_n\}$ is a Cauchy sequence in (X, p_k^s) by Lemma 3.3(1). Since (X, p_k^s) is complete, there exists $x \in X$ such that

$$\lim_{n,m\to+\infty} p_k^s(x_n,x_m) = \lim_{n\to+\infty} p_k^s(x_n,x) = p_k^s(x,x).$$

In addition, by (SPK2) we have $p_k^s(x_n,x_n) - p_k^s(x,x) \le p_k^s(x_n,x_m) - p_k^s(x,x)$. which implies $\lim_{n\to+\infty} p_k^s(x_n,x_n) = p_k^s(x,x)$. Moreover, by (SPK3) we have

$$p_k^s(x_n,x) \le k[p_k^s(x_n,x_m) + p_k^s(x_m,x)] - \frac{k-1}{2}[p_k^s(x_n,x_n + p_k^s(x,x))] - kp_k^s(x_m,x_m),$$

which implies $\lim_{n\to+\infty} p_k^s(x_n,x) = p_k^s(x,x)$. Since

$$\hat{p}_k^s(x_n,x) = p_k^s(x_n,x) + p_k^s(x,x_n) - p_k^s(x_n,x_n) - p_k^s(x,x),$$

we can deduce $\lim_{n\to+\infty} \hat{p}_k^s(x_n,x) = 0$. Hence $(X, \hat{p}_k^s)$ is complete.

Finally, it is simple matter to check that $\lim_{n\to+\infty} \hat{p}_k^s(x_n,x) = 0$ if and only if $p_k^s(x,x) = \lim_{n\to+\infty} p_k^s(x_n,x) = \lim_{n,m\to+\infty} p_k^s(x_n,x_m)$. $\square$

Theorem 3.4. *Let (X, p_k^s) be complete strongly partial-quasi k-metric space with coefficient $k \ge 1$, and $T : X \to X$ a function satisfying $p_k^s(Tx,Ty) \le \lambda p_k^s(x,y)$ for all $x, y \in X$, where $\lambda \in [0,1)$. Then T has a unique fixed point $x^* \in X$, and $p_k^s(x^*,x^*) = 0$.*

Proof. By assumption, we have $p_k^s(Tx,Ty) \le \lambda p_k^s(x,y)$ for all $x,y \in X$, where $\lambda \in [0,1)$. Given $0<\varepsilon<1$. We can choose $n_0 \in \mathbb{N}^+$ such that $\lambda^{n_0} < \frac{(1+k)\varepsilon}{4k}$, where $k \ge 1$. We define the sequence in the following way: $x_0 = x$, and $x_{n+1} = Fx_n = F^{n+1}x_0$ for all $n \in \mathbb{N}^+$, $x_0 \in X$, where $F = T^{n_0}$. Then $p_k^s(Fx,Fy) = p_k^s(T^{n_0}x, T^{n_0}y) \le \lambda^{n_0} p_k^s(x,y)$, which implies that $p_k^s(x_n,x_{n+1}) = p_k^s(Fx_{n-1},Fx_n) \le \lambda^{n_0} p_k^s(x_{n-1},x_n)$. By repetition of this process, we have $p_k^s(x_n,x_{n+1}) \le (\lambda^{n_0})^n p_k^s(x_0,x_1)$, which implies $\lim_{n\to+\infty} p_k^s(x_n,x_{n+1}) = 0$. Then for any $0<\varepsilon<\frac{2}{1+k}$, there exists $n_1 \in \mathbb{N}^+$ such that $p_k^s(x_n,x_{n+1}) < \frac{(1+k)\varepsilon}{4k}$, where $n > n_1 \ge n_0$. To prove the existence and uniqueness of the fixed point, we consider the following steps:

Step 1: We denote $\mathbb{B}_{p_k^s}(x,\delta) = \{y \in X : p_k^s(x,y) \le p_k^s(x,x) + \delta\}$. It is obvious that $x_r \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$ for all $k \ge 1$ and $0<\varepsilon<\frac{2}{1+k}$, where $r \in \mathbb{N}^+$. Then $\mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2}) \ne \emptyset$. For each $z \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$, we have

$$p_k^s(Fx_r,Fz) \le \lambda^{n_0} p_k^s(x_r,z) \le \lambda^{n_0}[\frac{(1+k)\varepsilon}{2} + p_k^s(x_r,x_r)].$$

In addition, $p_k^s(x_r,Fx_r) = p_k^s(x_r,x_{r+1})$ and

$$\begin{aligned} & p_k^s(x_r,Fz) \\ \le\ & k[p_k^s(x_r,Fx_r) + p_k^s(Fx_r,Fz)] - \frac{k-1}{2}[p_k^s(x_r,x_r) + p_k^s(Fz,Fz)] - kp_k^s(Fx_r,Fx_r) \end{aligned}$$

by (SPK3). Then we have

$$\begin{aligned} & p_k^s(x_r,Fz) \\ \le\ & k[p_k^s(x_r,Fx_r) + p_k^s(Fx_r,Fz)] - \frac{k-1}{2} p_k^s(x_r,x_r) \\ =\ & k[p_k^s(x_r,x_{r+1}) + p_k^s(Fx_r,Fz)] - \frac{k-1}{2} p_k^s(x_r,x_r) \\ \le\ & k[\frac{(1+k)\varepsilon}{4k} + \frac{(1+k)\varepsilon}{4k}(\frac{(1+k)\varepsilon}{2} + p_k^s(x_r,x_r))] - \frac{k-1}{2} p_k^s(x_r,x_r) \\ =\ & \frac{(1+k)\varepsilon}{4}(1 + \frac{(1+k)\varepsilon}{2}) + (1 - \frac{(1+k)(2-\varepsilon)}{4}) p_k^s(x_r,x_r) \\ <\ & \frac{(1+k)\varepsilon}{2} + p_k^s(x_r,x_r). \end{aligned}$$

Thus $Fz \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$.

Step 2: From step 1, we have $Fx_r \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$ for all $n \in \mathbb{N}^+$. Then $F^m x_r \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$ for all $m \in \mathbb{N}^+$. Namely, $x_n \in \mathbb{B}_{p_k^s}(x_r, \frac{(1+k)\varepsilon}{2})$ for all $n \ge r$. Thus, for any $n, m \ge r$, we obtain that

$$\begin{aligned} & p_k^s(x_n,x_m) \\ <\ & \frac{(1+k)\varepsilon}{2} + p_k^s(x_r,x_r) \\ \le\ & \frac{(1+k)\varepsilon}{2} + p_k^s(x_r,x_{r+1}) \\ \le\ & \frac{(1+k)(2k+1)\varepsilon}{4k}. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence and $\lim_{n,m\to+\infty} p_k^s(x_n,x_m) = 0$. Since (X, p_k^s) is complete, then there exists $x^* \in X$, such that

$$\lim_{n\to+\infty} p_k^s(x_n,x^*) = \lim_{n\to+\infty} p_k^s(x^*,x_n) = p_k^s(x^*,x^*) = 0.$$

Furthermore, we have

$$
\begin{aligned}
& p_k^s(x^*, Tx^*) \\
\leq\ & k[p_k^s(x^*, x_{n+1}) + p_k^s(x_{n+1}, Tx^*)] - \frac{k-1}{2}[p_k^s(x^*, x^*) + p_k^s(Tx^*, Tx^*)] - kp_k^s(x_{n+1}, x_{n+1}) \\
\leq\ & k[p_k^s(x^*, x_{n+1}) + p_k^s(x_{n+1}, Tx^*)] \\
=\ & kp_k^s(x^*, x_{n+1}) + kp_k^s(x_{n+1}, Tx^*) \\
\leq\ & kp_k^s(x^*, x_{n+1}) + \lambda kp_k^s(x_n, x^*).
\end{aligned}
$$

This implies $p_k^s(x^*, Tx^*) = 0$. Thus $Tx^* = x^*$ by Proposition 2.6(1).

Step 3: Suppose $x^* \neq y^*$, where $Ty^* = y^*$. We have

$$p_k^s(x^*, y^*) = p_k^s(Tx^*, Ty^*) \leq \lambda p_k^s(x^*, y^*) < p_k^s(x^*, y^*),$$

which is a contradiction. Hence $x^* = y^*$. □

Corollary 3.5. *Let (X, p_k^s) be a complete strongly partial-quasi k-metric space with coefficient $k \geq 1$, and let $T : X \to X$ be a function satisfying $p_k^s(Tx, Ty) \leq \varphi(t)p_k^s(x, y)$ for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that* $\lim_{n\to+\infty} \varphi^n(r) = 0$ *for some $r > 0$. Then T has a unique fixed point $x^* \in X$, and $p_k^s(x^*, x^*) = 0$.*

Proof. It is similar to Theorem 3.4. □

Conclusions

The purpose of this paper is to introduce a new notion of (strongly) partial-quasi k-metrics by omitting the condition $p(x, y) = p(y, x)$, whenever $x, y \in X$, which is another variant concept of partial-(quasi)-k metrics given by Künzi et al. and Mustafa et al., respectively. In Section 2, we show the relationships among partial-quasi k-metrics, partial quasi-metrics and k-metrics via several examples. Finally, we illustrate some fixed point theorems on strongly partial-quasi k-metric spaces.

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