Filomat 37:6 (2023), 1815–1824
https://doi.org/10.2298/FIL2306815Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some studies of *SEP* elements in a ring with involution

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Abstract. In this paper, we give some new characterizations of *SEP* elements and partial isometries in rings with involution. Especially, we discuss these characterizations from the perspectives of the existence of solutions to certain equations, and the form of the general solutions to some equations.

1. Introduction

Let R be a ring with 1. $a \in R$ is called group invertible if there exists $b \in R$ such that

$$aba = a, \qquad bab = b, \qquad ab = ba.$$

In this case, b is uniquely determined by the above equations [2]. We call it the group inverse of a and denote it by $a^{\#}$. The set of all group invertible elements of R is denoted by $R^{\#}$.

R is called a $*-$ring if there exists an anti-isomorphism $*$ of degree 2 in R, which satisfies

$$(a^*)^* = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^*,$$

for all $a, b \in R$.

$a \in R$ is said to be Moore-Penrose invertible (or MP-invertible) if there exists $b \in R$ such that the following Penrose equations hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one b such that the above conditions hold, see [4, 5, 7–9, 17]. We call it the Moore-Penrose inverse (or Moore-inverse) of a and denote it by $a^{\dagger}$. Denote by $R^{\dagger}$ the set of all MP-invertible elements of R.

a is said to be *EP* [6] if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. We denote the set of all *EP* elements of R by R^{EP}. On *EP* elements, the readers can refer to [14, 15, 18–20].

If $a^* = a^{\dagger}$, then the element $a \in R^{\dagger}$ is called a partial isometry. The set of all partial isometries of R is denoted by R^{PI}. In recent years, the study of partial isometry elements are discussed by many authors such as [3, 11–13, 15].

If $a \in R^{EP}$ and $a^* = a^{\dagger}$, then the element a is called strongly *EP* element [21, 22]. We denote by R^{SEP} the set of all strongly *EP* elements of R.

2020 *Mathematics Subject Classification*. 15A09; 16U99; 16W10
Keywords. *EP* element, *SEP* element, partial isometry, solutions of equation, general solutions.
Received: 7 December 2021; Accepted: 5 January 2023
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (11471282) and SuQian Sci&Tech Program (Grant No. K202225)

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In [10, 12, 16, 21, 22], many characterizations of strongly EP are discussed. Motivated by these results, this paper is intended to give a number of new characterizations of strongly EP elements from some different angles. We characterize it by considering the existence of solutions to certain equations in a definite set, the general solutions of certain equations, and invertible elements in rings, which are all new approaches to study generalized inverses in rings.

2. Some characterizations of *SEP* elements

Let $a \in R^{\#} \cap R^{+}$. Then, by [12, Theorem 1.5.3], $a \in R^{SEP}$ if and only if $a^*a^+ = a^{\#}a^+$. We can generalize this result as follows.

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^*a^+(a^{\#})^* = a^{\#}a^+(a^+)^*$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then $a^*a^+ = a^{\#}a^+$ by [12, Theorem 1.5.3] and $a \in R^{EP}$. Hence $a^+ = a^{\#}$, which implies $a^*a^+(a^{\#})^* = a^{\#}a^+(a^+)^*$.
" $\Leftarrow$ " From the condition $a^*a^+(a^{\#})^* = a^{\#}a^+(a^+)^*$, we obtain $a^*a^+ = a^*a^+(a^{\#})^*a^* = a^{\#}a^+(a^+)^*a^* = a^{\#}a^+$. Hence $a \in R^{SEP}$ by [12, Theorem 1.5.3]. $\square$

Lemma 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then*
1) $a^*a^+(a^{\#})^* \in R^+$ and $(a^*a^+(a^{\#})^*)^+ = aa^+a^*a(a^{\#})^*a^+a$;
2) $a^*a^+(a^{\#})^* \in R^{\#}$ with $(a^*a^+(a^{\#})^*)^{\#} = a^*a(a^{\#})^*$;
3) $a^{\#}a^+(a^+)^* \in R^+$ with $(a^{\#}a^+(a^+)^*)^+ = a^*a^3a^+$;
4) $a^{\#}a^+(a^+)^* \in R^{\#}$ with $(a^{\#}a^+(a^+)^*)^{\#} = aa^{\#}a^*a^2$.

Proof. 1) Since

$$(a^*a^+(a^{\#})^*)(aa^+a^*a(a^{\#})^*a^+a) = a^*a^+(a^{\#})^*a^*a(a^{\#})^*a^+a = a^*a^+a(a^{\#})^*a^+a = a^*(a^{\#})^*a^+a = a^+a,$$

$$(a^*a^+(a^{\#})^*)(aa^+a^*a(a^{\#})^*a^+a)(a^*a^+(a^{\#})^*) = a^+a(a^*a^+(a^{\#})^*) = a^*a^+(a^{\#})^*,$$

and

$$((a^*a^+(a^{\#})^*)(aa^+a^*a(a^{\#})^*a^+a))^* = (a^+a)^* = a^+a = (a^*a^+(a^{\#})^*)(aa^+a^*a(a^{\#})^*a^+a).$$

Owing to

$$(aa^+a^*a(a^{\#})^*a^+a)(a^*a^+(a^{\#})^*) = aa^+a^*a(a^{\#})^*a^*a^+(a^{\#})^* = aa^+a^*aa^+(a^{\#})^* = aa^+a^*(a^{\#})^* = aa^+,$$

$$(aa^+a^*a(a^{\#})^*a^+a)(a^*a^+(a^{\#})^*)(aa^+a^*a(a^{\#})^*a^+a) = aa^+(aa^+a^*a(a^{\#})^*a^+a) = aa^+a^*a(a^{\#})^*a^+a,$$

and

$$((aa^+a^*a(a^{\#})^*a^+a)(a^*a^+(a^{\#})^*))^* = (aa^+)^* = aa^+ = (aa^+a^*a(a^{\#})^*a^+a)(a^*a^+(a^{\#})^*).$$

Hence $a^*a^+(a^{\#})^* \in R^+$ and $(a^*a^+(a^{\#})^*)^+ = aa^+a^*a(a^{\#})^*a^+a$.
2) Since

$$(a^*a^+(a^{\#})^*)(a^*a(a^{\#})^*) = a^*a^+a(a^{\#})^* = a^*(a^{\#})^* = (aa^{\#})^*,$$
$$(a^*a^+(a^{\#})^*)(a^*a(a^{\#})^*)(a^*a^+(a^{\#})^*) = (aa^{\#})^*a^*a^+(a^{\#})^* = a^*a^+(a^{\#})^*.$$

Owing to

$$(a^*a(a^{\#})^*)(a^*a^+(a^{\#})^*) = a^*aa^+(a^{\#})^* = a^*(a^{\#})^* = (aa^{\#})^*,$$
$$(a^*a(a^{\#})^*)(a^*a^+(a^{\#})^*)(a^*a(a^{\#})^*) = (aa^{\#})^*a^*a(a^{\#})^* = a^*a(a^{\#})^*,$$

and

$$(a^*a^+(a^\#)^*)(a^*a(a^\#)^*) = (aa^\#)^* = (a^*a(a^\#)^*)(a^*a^+(a^\#)^*).$$

Hence $a^*a^+(a^\#)^* \in R^\#$ with $(a^*a^+(a^\#)^*)^\# = a^*a(a^\#)^*$.
3) and 4) can be shown similarly. □

Lemma 2.1 and Lemma 2.2 imply the following Theorem.

Theorem 2.3. *Let $a \in R^\# \cap R^+$. Then*
1) $a \in R^{SEP}$ if and only if $aa^+a^*a(a^\#)^*a^+a = a^*a^3a^+$;
2) The following conditions are equivalent:
(1) $a \in R^{EP}$;
(2) $a^*a^+(a^\#)^* \in R^{EP}$;
(3) $a^+a^\#(a^\#)^* \in R^{EP}$;
(4) $aa^+a^*a(a^\#)^*a^+a = a^*a(a^\#)^*$;
(5) $a^*a^3a^+ = aa^\#a^*a^2$.

Proof. 2) (1)″ ⇒ ″(2) By Lemma 2.2, we know that $(a^*a^+(a^\#)^*)^+ = aa^+a^*a(a^\#)^*a^+a$ and $(a^*a^+(a^\#)^*)^\# = a^*a(a^\#)^*$. Since $a \in R^{EP}$, $aa^+a^*a(a^\#)^*a^+a = a^+aa^*a(a^\#)^*aa^+ = a^*a(a^\#)^*$. This gives $(a^*a^+(a^\#)^*)^+ = (a^*a^+(a^\#)^*)^\#$. Hence $a^*a^+(a^\#)^* \in R^{EP}$.
(2)″ ⇒ ″(3) Since $a^*a^+(a^\#)^* \in R^{EP}$, by Lemma 2.2, we get $aa^+a^*a(a^\#)^*a^+a = a^*a(a^\#)^*$. Multiplying the equality on the left by $aa^\#$, one yields $aa^\#a^*a(a^\#)^* = a^*a(a^\#)^*$. Again multiply the equality on the right by $a^*a^+(a^+)^*$, one gets $aa^\# = a^+a$. Hence $a \in R^{EP}$. It follows that $(a^+a^\#(a^\#)^*)^\# = (a^\#a^+(a^\#)^*)^\# = aa^\#a^*a^2 = a^+aa^*a^2 = a^*a^2 = a^*a^3a^+ = (a^\#a^+(a^\#)^*)^+ = (a^+a^\#(a^\#)^*)^+$. Hence $a^+a^\#(a^\#)^* \in R^{EP}$ by Lemma 2.2.
(3)″ ⇒ ″(1) Since $a^+a^\#(a^\#)^* \in R^{EP}$, $(a^+a^\#(a^\#)^*)^\# = (a^+a^\#(a^\#)^*)^+$, that is,

$$a^*a^+a^3(aa^\#)^* = aa^+a^*a^+a^3.$$

Multiplying the equality on the right by aa^+, one gets

$$aa^+a^*a^+a^3 = aa^+a^*a^+a^4a^+.$$

Multiplying the last equality on the left by $a^\#(a^\#)^*$, one yields $a = a^2a^+$. Hence $a \in R^{EP}$. □

The following corollary is an immediate result of Theorem 2.3.

Corollary 2.4. *Let $a \in R^\# \cap R^+$. Then the following conditions are equivalent:*
1) $a \in R^{SEP}$;
2) $aa^+a^*a(a^\#)^* = a^*a^3a^+$;
3) $aa^+a^*a(a^\#)^* = a^*a^2$.

Proof. (1)″ ⇒ ″(2) Since $a \in R^{SEP}$, $a^+ = a^\# = a^*$. It follows that

$$aa^+a^*a(a^\#)^* = aa^\#a^*a(a^*)^* = a^\#aa^*aa = a^+aa^*a^2 = a^*a^2 = a^*a^3a^\# = a^*a^3a^+.$$

(2)″ ⇒ ″(3) From the assumption, we have

$$a^*a^3a^+ = aa^+a^*a(a^\#)^* = aa^+(aa^+a^*a(a^\#)^*) = aa^+a^*a^3a^+.$$

Multiplying the equality on the right by $a^\#a^+$, one gets $a^* = aa^+a^*$. Hence $a \in R^{EP}$. This gives

$$a^*a(a^\#)^* = aa^+a^*a(a^\#)^* = a^*a^3a^+ = a^*a^2.$$

(3)″ $\Longrightarrow$ ″ (1) Assume that $aa^+a^*a(a^\#)^* = a^*a^2$. Multiplying the last equality on the right by aa^+, one has

$$a^*a^2 = a^*a^3a^+,$$

and

$$a^2 = (a^+)^*a^*a^2 = (a^+)^*a^*a^3a^+ = a^3a^+.$$

Hence $a \in R^{EP}$ by [**?**], it follows that

$$a^*a(a^\#)^* = aa^+a^*a(a^\#)^* = a^*a^2$$

. Multiplying the last equality on the left by $a^+(a^+)^*$, one obtains $(a^\#)^* = a$. Hence $a \in R^{SEP}$.

□

Observing Lemma 2.1, we have the following corollary.

Corollary 2.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^\#(a^\#)^* = a^\#a^+(a^+)^*$.*

Proof. ″ ⇒ ″ Since $a \in R^{SEP}$, $a^+ = a^\# = a^*$. It follows that

$$a^*a^\#(a^\#)^* = a^\#a^+(a^+)^*.$$

″ ⇐ ″ Assume that $a^*a^\#(a^\#)^* = a^\#a^+(a^+)^*$. Multiplying the equality on the left by a^+a, one gets

$$a^+aa^\#a^+(a^+)^* = a^\#a^+(a^+)^*.$$

Again multiply the last equality on the right by a^*a, one has $a^+aa^\# = a^\#$. Hence $a \in R^{EP}$. It gives that $a^*a^+(a^\#)^* = a^\#a^+(a^+)^*$. Thus $a \in R^{SEP}$ by Lemma 2.1. □

Lemma 2.6. *Let $a \in R^\# \cap R^+$. Then $a^*a^\#(a^\#)^* \in R^+$ with $(a^*a^\#(a^\#)^*)^+ = aa^+a^*a^+a^2(a^+)^*$.*

Hence, Corollary 2.5 and Lemma 2.6 lead to the following Corollary.

Corollary 2.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^+a^*a^+a^2(a^+)^* = a^*a^3a^+$.*

3. Consistency of certain equations

Observing Lemma 2.1, we construct the following equation

$$a^*x(a^\#)^* = a^\#a^+(a^+)^*. \tag{3.1}$$

Theorem 3.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(3.1) is consistent and the general solution is given by*

$$x = a^+ + u - aa^+ua^+a, \text{where } u \in R. \tag{3.2}$$

Proof. ″ ⇒ ″ Assume that $a \in R^{SEP}$. Then $a^*a^+(a^\#)^* = a^\#a^+(a^+)^*$ by Lemma 2.1. Hence

$$a^*(a^+ + u - aa^+ua^+a)(a^\#)^* = a^*a^+(a^\#)^* = a^\#a^+(a^+)^*,$$

which implies formula (3.2) is exact the solution of Eq.(3.1).

Now, let $x = x_0$ be any solution of Eq.(3.1). Then $a^*x_0(a^\#)^* = a^\#a^+(a^+)^*$. It follows that

$$aa^+x_0a^+a = (a^+)^*a^*x_0(a^\#)^*a^*a^+a = (a^+)^*(a^\#a^+(a^+)^*)a^*a^+a = (a^+)^*a^\#a^+aa^+a^+a = (a^+)^*a^\#a^+a^+a.$$

Noting that $a \in R^{SEP}$. Then $(a^+)^* = a; a^+ = a^\#$. Hence $aa^+x_0a^+a = a^\# = a^+$, this gives $x_0 = a^+ + x_0 - aa^+x_0a^+a$. Thus the general solution of Eq.(3.1) is given by (3.2).
" $\Leftarrow$ " If the general solution of Eq.(3.1) is given by (3.2), then

$$a^*(a^+ + u - aa^+ua^+a)(a^\#)^* = a^\#a^+(a^+)^*,$$

e.g. $a^*a^+(a^\#)^* = a^\#a^+(a^+)^*$. Hence $a \in R^{SEP}$ by Lemma 2.1. □

Remark 3.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{EP}$ if and only if Eq.(3.1) is consistent. In this case, the general solution of Eq.(3.1) is given by*

$$x = (a^\#)^*a^+a^+ + u - aa^+ua^+a, \text{where } u \in R. \tag{3.3}$$

Proof. " $\Rightarrow$ " Assume that $a \in R^{EP}$. Then $a^+ = a^\#$, it follows that

$$a^*((a^\#)^*a^+a^+)(a^\#)^* = a^+a^+(a^\#)^* = a^\#a^+(a^+)^*.$$

Hence Eq.(3.1) is consistent. Clearly, the formula (3.3) is exact the solution of Eq.(3.1). Now, let $x = x_0$ be any solution of Eq.(3.1). Then $a^*x_0(a^\#)^* = a^\#a^+(a^+)^*$. So

$$aa^+x_0a^+a = (a^+)^*a^\#a^+a^+a = (a^\#)^*a^+a^\#a^\#a = (a^\#)^*a^+a^\# = (a^\#)^*a^+a^+$$

because $a \in R^{EP}$. Hence $x_0 = (a^\#)^*a^+a^+ + x_0 - aa^+x_0a^+a$, one obtains the general solution of Eq.(3.1) is given by the formula (3.3).
" $\Leftarrow$ " From the assumption, we have

$$a^*((a^\#)^*a^+a^+ + u - aa^+ua^+a)(a^\#)^* = a^\#a^+(a^+)^*,$$

that is, $a^+a^+(a^\#)^* = a^\#a^+(a^+)^*$. Post-multiplying the equality by a^*, one gets $a^+a^+ = a^\#a^+$. Hence $a \in R^{EP}$. □

Now, we construct equation as follows

$$a^*x(a^\#)^* = a^\# \tag{3.4}$$

Theorem 3.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(3.4) is consistent and the general solution is given by*

$$x = a^* + u - aa^+ua^+a, \text{where } u \in R. \tag{3.5}$$

Proof. " $\Rightarrow$ " If $a \in R^{SEP}$, then $(a^+)^* = a$, it follows $a^\#a^+(a^+)^* = a^\#a^+a = a^\#$. Hence Eq.(3.1) is the same as Eq.(3.4). Noting that $a^* = a^+$. Then, by Theorem 3.1, we have the general solution of Eq.(3.4) is given by (3.5).
" $\Leftarrow$ " From the assumption, we have $a^*(a^* + u - aa^+ua^+a)(a^\#)^* = a^\#$, e.g. $a^* = a^\#$. Thus $a \in R^{SEP}$ by [12, Theorem 1.5.3]. □

Consider

$$a^*x(a^\#)^* = a^+ \tag{3.6}$$

It is easy to show that Eq.(3.6) is consistent and the general solution is given by

$$x = (a^+)^*a^+a^* + u - aa^+ua^+a, \text{where } u \in R. \tag{3.7}$$

In fact, by a simple computation, we get the formula (3.7) is indeed the solution to Eq.(3.6). On the other hand, for any solution $x = x_0$ to Eq.(3.6), we have

$$a^*x_0(a^\#)^* = a^+.$$

Choose $u = x_0 - (a^+)^*a^+a^*$. Then

$$aa^+ua^+a = aa^+(x_0-(a^+)^*a^+a^*)a^+a = aa^+x_0a^+a-aa^+(a^+)^*a^+a^*a^+a = aa^+x_0a^+a-(a^+)^*(a^*x_0(a^\#)^*)a^*a^+a = aa^+x_0a^+a-aa^+x_0a^+a = .$$

Thus $x_0 = (a^+)^*a^+a^* + u - aa^+ua^+a$ has the form of the formula (3.7).

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(3.1) and Eq.(3.6) have the same solutions.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then

$$a^{\#}a^{+}(a^{+})^{*} = a^{\#}a^{+}a = a^{\#} = a^{+}.$$

Hence Eq.(3.1) is the same as Eq.(3.6), certainly, they have the same solutions.
" $\Leftarrow$ " If Eq.(3.1) and Eq.(3.6) have the same solutions, then by Remark 3.2, $a \in R^{EP}$. Noting that the general solution of Eq.(3.6) is given by the formula (3.7). Then the formula (3.7) is the general solutions of Eq.(3.1), this gives

$$a^{*}((a^{+})^{*}a^{+}a^{*} + u - aa^{+}ua^{+}a)(a^{\#})^{*} = a^{\#}a^{+}(a^{+})^{*},$$

e.g. $a^{+} = a^{\#}a^{+}(a^{+})^{*}$, it follows that

$$a = a^{2}a^{+} = a^{2}a^{\#}a^{+}(a^{+})^{*} = (a^{+})^{*}.$$

Hence $a \in R^{SEP}$. □

Remark 3.5. *Clearly, $x = (a^{\#})^{*}a^{+}a^{*}$ is a solution of Eq.(3.6). How to express the solution in the form of the formula (3.7)?*

4. Univariate equation

Lemma 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then*
1) $a^{+} \in R^{\#}$ with $(a^{+})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}$;
2) $a^{\#} \in R^{+}$ with $(a^{\#})^{+} = a^{+}a^{3}a^{+}$.

Proof. It is routine. □

Lemma 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{*}a(a^{+})^{*} = (a^{+})^{*}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then $a^{*}a(a^{+})^{*} = a^{\#}a^{2} = a = (a^{+})^{*}$.
" $\Leftarrow$ " If $a^{*}a(a^{+})^{*} = (a^{+})^{*}$, then

$$aa^{+} = (a^{+})^{*}a^{*} = a^{*}a(a^{+})^{*}a^{*} = a^{*}a^{2}a^{+},$$

this gives $a = a^{*}a^{2}$. Hence $a \in R^{SEP}$ by [11, Theorem 2.3]. □

Observing Lemma 2.1, we consider the following equation

$$a^{*}x(a^{\#})^{*} = a^{\#}x(a^{+})^{*}. \tag{4.1}$$

Theorem 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(4.1) has at least one solution in $\rho_a=\{a, a^{\#}, a^{+}, a^{*}, (a^{+})^{*}, (a^{\#})^{*}, (a^{+})^{\#}, (a^{\#})^{+}\}$.*

Proof. " $\Rightarrow$ " If $a \in R^{SEP}$, then $x = a^{+}$ is a solution by Lemma 2.1.
" $\Leftarrow$ " 1) If $x = a$ is a solution, then $a^{*}a(a^{\#})^{*} = a^{\#}a(a^{+})^{*} = (a^{+})^{*}$. Post-multiplying the equality by $a^{*}a^{+}$, one yields $a^{*} = aa^{+}a^{+}$, it follows that $a^{*}a^{*} = a^{*}(aa^{+}a^{+}) = a^{*}a^{+}$. Hence $a \in R^{PI}$ by [21, Corollary 2.10], this gives

$$a = (a^{+})^{*} = a^{*}a(a^{\#})^{*} = a^{+}a(a^{\#})^{*} = (a^{\#})^{*}.$$

Thus $a \in R^{SEP}$ by [12, Theorem 1.5.3];
2) If $x = a^{\#}$, then $a^{*}a^{\#}(a^{\#})^{*} = a^{\#}a^{\#}(a^{+})^{*}$, this gives

$$(1 - aa^{+})a^{*}a^{\#}(a^{\#})^{*} = (1 - aa^{+})a^{\#}a^{\#}(a^{+})^{*} = 0,$$

and

$$(1 - aa^{+})a^{*}a^{\#} = (1 - aa^{+})a^{*}a^{\#}(a^{\#})^{*}a^{*}a^{+}a = 0,$$

one has $(1-aa^+)a^+ = (1-aa^+)a^*a^\# a(a^+)^* a^+ = 0$. Hence $a \in R^{EP}$, which implies

$$a^*a^\# = a^*a^\#(a^\#)^* a^*a^+a = a^\# a^\#(a^+)^* a^*a^+a = a^\# a^+a^+a = a^+a^\#.$$

Thus $a \in R^{PI}$ by [12, Theorem 1.5.2]. Therefore $a \in R^{SEP}$;
3) If $x = a^+$, then $a^*a^+(a^\#)^* = a^\# a^+(a^+)^*$, which implies $a \in R^{SEP}$ by Lemma 2.1;
4) If $x = a^*$, then $a^*a^*(a^\#)^* = a^\# a^*(a^+)^*$, e.g. $a^* = a^\#$. Hence $a \in R^{SEP}$ by [12, Theorem 1.5.3];
5) If $x = (a^+)^*$, then $a^*(a^+)^*(a^\#)^* = a^\#(a^+)^*(a^+)^*$, e.g. $(a^\#)^* = a^\#(a^+)^*(a^+)^*$, one obtains

$$(1-aa^+)(a^\#)^* = (1-aa^+)a^\#(a^+)^*(a^+)^* = 0.$$

Hence $(1-aa^+)a^+ = (1-aa^+)(a^\#)^* a^*a^+ = 0$, which implies $a \in R^{EP}$. Now we have

$$(a^\#)^* = a^*(a^+)^*(a^\#)^* = a^\#(a^+)^*(a^+)^* = a^\#(a^\#)^*(a^\#)^*,$$

and

$$a^* = (a^\#)^* a^*a^* = a^\#(a^\#)^*(a^\#)^* a^*a^* = a^\#(a^\#)^* a^* = a^+(a^\#)^* a^* = a^+.$$

Thus $a \in R^{SEP}$;
6) If $x = (a^\#)^*$, then $a^*(a^\#)^*(a^\#)^* = a^\#(a^\#)^*(a^+)^*$, e.g. $(a^\#)^* = a^\#(a^\#)^*(a^+)^*$, this gives $(1-aa^+)(a^\#)^* = 0$. Hence $a \in R^{EP}$, which implies $x = (a^\#)^* = (a^+)^*$ is a solution. Hence $a \in R^{SEP}$ by 5);
7) If $x = (a^+)^\#$, then $a^*(a^+)^\#(a^\#)^* = a^\#(a^+)^\#(a^+)^*$. By Lemma 4.1, we have

$$a^*(aa^\#)^* a(aa^\#)^*(a^\#)^* = a^\#(aa^\#)^* a(aa^\#)^*(a^+)^*,$$

that is, $a^*a(a^\#)^* = a^\#(aa^\#)^* a(aa^\#)^*(a^+)^*$. Since

$$(1-aa^+)a^*a(a^\#)^* = (1-aa^+)a^\#(aa^\#)^* a(aa^\#)^*(a^+)^* = 0.$$

$$(1-aa^+)a^* = (1-aa^+)a^*a(a^\#)^* a^*a^+ = 0.$$

Hence $a \in R^{EP}$. Now we have

$$a^*a(a^+)^* = a^*a(a^\#)^* = a^\#(aa^\#)^* a(aa^\#)^*(a^+)^* = a^+(aa^\#)^* a(aa^\#)^*(a^\#)^* = a^+a(a^\#)^* = (a^\#)^* = (a^+)^*.$$

Hence $a \in R^{SEP}$ by Lemma 4.2;
8) If $x = (a^\#)^+$, then $a^*(a^\#)^+(a^\#)^* = a^\#(a^\#)^+(a^+)^*$. By Lemma 4.1, we have

$$a^*a^+a^3a^+(a^\#)^* = a^\#a^+a^3a^+(a^+)^* = (a^+)^*,$$

this gives

$$(1-a^+a)(a^+)^* = (1-a^+a)a^*a^+a^3a^+(a^\#)^* = 0,$$

which implies $a \in R^{EP}$. Now we get $(a^+)^* = a^*a^+a^3a^+(a^\#)^* = a^*a(a^+)^*$. Hence $a \in R^{SEP}$ by Lemma 4.2. □

It is known that $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$. Hence a^* instead of a in Eq.(4.1), we have

$$axa^\# = (a^\#)^* xa^+. \tag{4.2}$$

Thus Theorem 4.3 implies the following Theorem:

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.2) has at least one solution in ρ_a.*

Applying the involution on Eq.(4.2), one obtains the following equation.

$$(a^\#)^* xa^* = (a^+)^* xa^\#. \tag{4.3}$$

Theorem 4.4 implies the following Theorem:

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.3) has at least one solution in ρ_a.*

5. Bivariate equations

We modify Eq.(4.1) as follows:

$$a^*x(a^\#)^* = a^\# y(a^+)^*. \tag{5.1}$$

Theorem 5.1. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(5.1) is given by*

$$\begin{cases} x = (a^+)^* a^+ p a^+ a^* a^+ a + u - aa^+ u a^+ a \\ y = a^+ a^2 a^+ p a^+ a^* + v - a^+ a v a a^+ \end{cases}, \text{where } u, v, p \in R \tag{5.2}$$

and satisfying $a^+p = aa^+a^+p$ and $pa^+ = pa^+a^+a$.

Proof. First,

$$a^*((a^+)^* a^+ p a^+ a^* a^+ a + u - aa^+ u a^+ a)(a^\#)^* = a^+ p a^+;$$

and

$$a^\#(a^+ a^2 a^+ p a^+ a^* + v - a^+ a v a a^+)(a^+)^* = a^\# a a^+ p a^+ a^+ a = a^\# a(aa^+ a^+ p a^+ a^+ a) = aa^+ a^+ p a^+ a^+ a = a^+ p a^+.$$

Hence the formula (5.2) is the solution of Eq.(5.1). Next, let

$$\begin{cases} x &= x_0 \\ y &= y_0 \end{cases}$$

is a solution of Eq.(5.1). Then $a^* x_0 (a^\#)^* = a^\# y_0 (a^+)^*$. Choose

$$p = aa^\# y_0 (a^+)^* a, \quad u = x_0, \quad v = y_0.$$

Then

$$a^+ p = a^+ a(a^\# y_0 (a^+)^*) a = a^+ a(a^* x_0 (a^\#)^*) a = a^* x_0 (a^\#)^* a = a^\# y_0 (a^+)^* a,$$

one has

$$aa^+ a^+ p = aa^+(a^\# y_0 (a^+)^* a) = a^\# y_0 (a^+)^* a = a^+ p.$$

Also

$$pa^+ = a(a^\# y_0 (a^+)^*) aa^+ = a(a^* x_0 (a^\#)^*) aa^+ = aa^* x_0 (a^\#)^* = aa^\# y_0 (a^+)^*,$$

hence $pa^+a^+a = aa^\# y_0 (a^+)^* a^+ a = aa^\# y_0 (a^+)^* = pa^+$. Noting that

$$\begin{aligned} & aa^+ u a^+ a = aa^+ x_0 a^+ a = (a^+)^* a^* x_0 ((a^\#)^* a^* a^+) a \\ = \; & (a^+)^* (a^\# y_0 (a^+)^*) a^* a^+ a = ((a^+)^* a^+ a)(a^\# y_0 (a^+)^*) a^* a^+ a \\ = \; & (a^+)^* a^+ (aa^\# y_0 (a^+)^*) a^* a^+ a = (a^+)^* a^+ (aa^* x_0 (a^\#)^*) a^* a^+ a \\ = \; & (a^+)^* a^+ (aa^* x_0 (a^\#)^* aa^+) a^* a^+ a = (a^+)^* a^+ (aa^\# y_0 (a^+)^* a) a^+ a^* a^+ a \\ = \; & (a^+)^* a^+ p a^+ a^* a^+ a. \end{aligned}$$

Then

$$x_0 = (a^+)^* a^+ p a^+ a^* a^+ a + x_0 - aa^+ u a^+ a = (a^+)^* a^+ p a^+ a^* a^+ a + u - aa^+ u a^+ a.$$

Since

$$\begin{aligned} & a^+ a v a a^+ = a^+ a y_0 a a^+ = a^+ a^2 a^+ (aa^\# y_0 (a^+)^* a^*) \\ = \; & a^+ a^2 a^+ (aa^* x_0 (a^\#)^*) a^* = a^+ a^2 a^+ (aa^* x_0 (a^\#)^* aa^+) a^* \\ = \; & a^+ a^2 a^+ (aa^\# y_0 (a^+)^* a)(a^+ a^*) \\ = \; & a^+ a^2 a^+ p a^+ a^*, \end{aligned}$$

$$y_0 = a^+ a^2 a^+ p a^+ a^* + y_0 - a^+ a v a a^+ = a^+ a^2 a^+ p a^+ a^* + v - a^+ a v a a^+.$$

Thus the general solution of Eq.(5.1) is given by (5.2). □

Corollary 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the general solution of Eq.(5.1) is given by*

$$\begin{cases} x = (a^{+})^{*}a^{+}pa^{+}a^{*}a^{+}a + u - aa^{+}ua^{+}a \\ y = a^{+}a^{2}a^{+}pa^{+}a^{*} + v - a^{+}avaa^{+} \end{cases}, where \ \ u, v, p \in R. \tag{5.3}$$

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{EP}$. Then for all $p \in R$,

$$a^{+}p = aa^{+}a^{+}p \quad and \quad pa^{+} = pa^{+}a^{+}a.$$

Hence, by Theorem 5.1, the general solution of Eq.(5.1) is given by (5.3).
$'' \Leftarrow ''$ If the general solution of Eq.(5.1) is given by (5.3), then

$$\begin{aligned} & a^{*}((a^{+})^{*}a^{+}pa^{+}a^{*}a^{+}a + u - aa^{+}ua^{+}a)(a^{\#})^{*} \\ = \ & a^{\#}(a^{+}a^{2}a^{+}pa^{+}a^{*} + v - a^{+}avaa^{+})(a^{+})^{*}, \end{aligned}$$

e.g. $a^{+}pa^{+} = a^{\#}aa^{+}pa^{+}a^{+}a$ for each $p \in R$. Choose $p = a$, one has $a^{+} = a^{\#}aa^{+}a^{+}a$, this gives

$$aa^{+}a^{+} = aa^{+}(a^{\#}aa^{+}a^{+}a) = a^{\#}aa^{+}a^{+}a = a^{+}.$$

Thus $a \in R^{EP}$. □

Lemma 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{+} = a^{+}a^{+}(a^{+})^{*}$.*

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{SEP}$. Then $a^{+} = a^{\#}$ and $(a^{+})^{*} = a$. It follows that $a^{+}a^{+}(a^{+})^{*} = a^{\#}a^{+}a = a^{\#} = a^{+}$.
$'' \Leftarrow ''$ If $a^{+} = a^{+}a^{+}(a^{+})^{*}$, then $a^{+}a^{+} = a^{+}a^{+}(a^{+})^{*}a^{+}$, this gives $a^{+} = a^{+}(a^{+})^{*}a^{+}$ by [21, Lemma 2.11], so

$$a = aa^{+}a = aa^{+}(a^{+})^{*}a^{+}a = (a^{+})^{*}.$$

This induces $a \in R^{PI}$ and $a^{+} = a^{+}a^{+}(a^{+})^{*} = a^{+}a^{+}a$. Hence $a \in R^{SEP}$. □

Theorem 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution of Eq.(5.1) is given by*

$$\begin{cases} x = (a^{+})^{*}a^{+}pa^{+}a^{*}a^{+}a + u - aa^{+}ua^{+}a \\ y = a^{+}a^{2}a^{+}pa^{+}a^{+} + v - a^{+}avaa^{+} \end{cases}, where \ \ u, v, p \in R. \tag{5.4}$$

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{SEP}$. Then $a^{*} = a^{+}$ and $a \in R^{EP}$, this infers the formula (5.3) is the same as the formula (5.4). Hence, by Corollary 5.2, we know that the general solution of Eq.(5.1) is given by (5.4).
$'' \Leftarrow ''$ If the general solution of Eq.(5.1) is given by (5.4), then

$$\begin{aligned} & a^{*}((a^{+})^{*}a^{+}pa^{+}a^{*}a^{+}a + u - aa^{+}ua^{+}a)(a^{\#})^{*} \\ = \ & a^{\#}(a^{+}a^{2}a^{+}pa^{+}a^{+} + v - a^{+}avaa^{+})(a^{+})^{*}, \end{aligned}$$

e.g. $a^{+}pa^{+} = a^{\#}aa^{+}pa^{+}a^{+}(a^{+})^{*}$ for all $p \in R$. Choose $p = a$. Then $a^{+} = a^{\#}aa^{+}a^{+}(a^{+})^{*}$, one obtains

$$aa^{+}a^{+} = aa^{+}(a^{\#}aa^{+}a^{+}(a^{+})^{*}) = a^{\#}aa^{+}a^{+}(a^{+})^{*} = a^{+}.$$

Hence $a \in R^{EP}$,which implies

$$a^{+} = a^{\#}aa^{+}a^{+}(a^{+})^{*} = a^{+}aa^{+}a^{+}(a^{+})^{*} = a^{+}a^{+}(a^{+})^{*}.$$

Hence $a \in R^{SEP}$ by Lemma 5.3. □

Inspired by Lemma 2.2, we can guess the following results, however, we do not know how to prove it. Let $a \in R^{\#} \cap R^{+}$. Then
(1) $a^{*}x(a^{\#})^{*} \in R^{+}$ with $(a^{*}x(a^{\#})^{*})^{+} = aa^{+}a^{*}a^{+}ax^{\#}(a^{+})^{*}$ for $x \in \chi_{a} = \{a, a^{\#}, a^{+}, a^{*}, (a^{+})^{*}, (a^{\#})^{*}\}$.
(2) $a^{*}x(a^{\#})^{*} \in R^{\#}$ with $(a^{*}x(a^{\#})^{*})^{\#} = a^{*}x^{+}(a^{\#})^{*}$ for $x \in \chi_{a}$.

Acknowledgments

We would like to express our heartfelt thanks to referees, for their instructive advice and useful suggestions on our thesis.

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