Filomat 37:6 (2023), 1777–1792
https://doi.org/10.2298/FIL2306777K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Anholonomic surfaces via directional motion of curves

Talat Korpinar[a], Rıdvan Cem Demirkol[a], Zeliha Korpinar[b]

[a]*Department of Mathematics, Mus Alparslan University*
[b]*Department of Administration, Mus Alparslan University*

Abstract. While there are many scientific manuscripts devoted to various details of the evolution of curves and their models, there are still many perspectives of the research subject that required comprehensive investigation. In particular, this manuscript is devoted to the search of a new class of time evolution equations of space curves written in the anholonomic coordinates. In this way, we have direct observation and access to form a new class of anholonomic surfaces induced by the given evolution systems. Thus, the proper representation of the evolution equations allows us to obtain useful characteristics of the corresponding evolution dynamics.

1. Introduction

In the traditional differential geometry, the Frenet-Serret frame possesses a vital role to describe space curves in given spaces whether they are characterized by definite or semi-definite metric structures. In these spaces, there exist three well-known classes of planes that are spanned by the Frenet-Serret vectors at each point of the curve i.e. osculating, rectifying, and normal planes. It also exists three corresponding classes of space curves i.e. osculating, rectifying, and normal curves associated with each plane. To be more specific, the position vector of osculating curves lies in its osculating space. Similarly, the position vector of normal curves (rectifying curves) lies in its normal space (rectifying space). Chen [1] investigated astonishing features of rectifying curves including their geometric invariants to construct rectifying curves from Darboux vectors and spherical curves. Izumiya and Takeuchi [2] described rectifying curves and slant helices to prove that rectifying developable of the conical geodesic is a cone.

The two basic concepts of curvature for a surface in a given space are the Gaussian and mean curvatures. If the mean curvature of the surface vanishes then it is called minimal surface. It has been extensively studied for both its significant prominence in applications and its unique mathematical features. If the Gaussian curvature of the surface vanishes then the surface is called developable. It is also of special attention since it can be flattened smoothly onto a plane without contraction or stretching. The geometric literature dealing with the developable surface is very large and has led to comprehensive generalization. Accordingly, a developable surface can also be constructed as the union of generalized cylinders, generalized cones, tangent developable, planar developable types. This surface is also formed by conserving the measure along with any curve between its points.

2020 *Mathematics Subject Classification*. Primary 53A10; Secondary 53C44, 14J25, 53Z05
Keywords. Osculating motion, Rectifying motion, Normal motion, Directional derivatives, Anholonomic surfaces
Received: 28 November 2021; Accepted: 6 July 2022
Communicated by Ljubica Velimirović
Email addresses: talatkorpinar@gmail.com (Talat Korpinar), rcdemirkol@gmail.com (Rıdvan Cem Demirkol), z.korpinar@alparslan.edu.tr (Zeliha Korpinar)

A large class of physical processes and engineering mechanisms is formed in terms of the flow and motion of curves, including the increase of dendritic crystals, the dynamics of vortex filaments, and the planar motion of interfaces. Moreover, if this motion is governed by the integrable systems then it preserves its global invariants containing both its enclosed area and the total length of the curve. For this reason, there are many papers attempting to find the answer to the question: What is the connection between the motion curves and integrable evolution equations either in 3D space or in the plane? A pioneering study of Hasimoto [3] demonstrated that the evolution of motion of thin filament considered as a curve represented by the well-known Da-Rios equation is equivalent to the famous nonlinear integrable Schrodinger equation. Considering the Hasimoto transformation relating complex curvature functions and space curves, Lamb [4] gave a more generalized form of the soliton-bearing equations.

The manuscript is structured in the following manner. Section 2 is devoted to some of the fundamental knowledge and facts of the differential geometry of curves and anholonomic coordinates in the three-dimensional ordinary space. Section 3 deals with defining transformations of the space curve flows to special surfaces through the effective use of anholonomic coordinates. Section 4 is concerned with the investigation of some pure geometric characteristics of these surfaces. Section 5 is dedicated to details of the physical dynamics of the numerical and analytical solution of the transformation of the space curve flows. Section 6 includes additional comments and results regarding the evolution of the special transformations and new classes of surfaces.

2. The Differential Geometry of Curves and Anholonomic Coordinates in $\mathbb{E}^3$

Let α be a space curve in $\mathbb{E}^3$ such that three-dimensional coordinate location (x, y, z) indicates a point on α. Furthermore, let $\mathcal{R}$ be a position vector placed at the same reference frame pointing to the location on α.

The Frenet-Serret frame of space curve α is described by the ordered triad of unit orthonormal vectors such that they are mutually perpendicular to each other. It is also called the moving trihedron or moving triple. It includes tangential axis $\mathbf{T}$, principal normal axis $\mathbf{N}$, and binormal axis $\mathbf{B}$. $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ triad satisfies the following cross product or vector product rule due to cyclic permutations, i.e.

$$\mathbf{T} = \mathbf{N} \times \mathbf{B}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}, \quad \mathbf{N} = \mathbf{B} \times \mathbf{T}. \tag{1}$$

Let (s, n, b) denotes arc distance along with Frenet-Serret vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$, respectively. (s, n, b) establishes a suitable curvilinear coordinate frame if arc distances are restricted appropriately around the origin, for instance $(s = s_0, n = 0, b = 0)$.

In the schematic trihedron, Frenet-Serret vectors of $(\mathbf{T}, \mathbf{B})$ span the rectifying plane; Frenet-Serret vectors of $(\mathbf{T}, \mathbf{N})$ span the osculating plane; Frenet-Serret vectors of $(\mathbf{N}, \mathbf{B})$ span the normal plane.

Vector calculus components on vector or scalar fields are typically expressed by the fundamental three vector operators. For instance, the divergent operator acts on an arbitrary vector field $\mathcal{Z}$ in the following manner

$$div\mathcal{Z} = \mathbf{T} \cdot \frac{\delta}{\delta s}\mathcal{Z} + \mathbf{N} \cdot \frac{\delta}{\delta s}\mathcal{Z} + \mathbf{B} \cdot \frac{\delta}{\delta s}\mathcal{Z}. \tag{2}$$

The curl operator acts on an arbitrary vector field $\mathcal{Z}$ in the following manner

$$curl\mathcal{Z} = \mathbf{T} \times \frac{\delta}{\delta s}\mathcal{Z} + \mathbf{N} \times \frac{\delta}{\delta s}\mathcal{Z} + \mathbf{B} \times \frac{\delta}{\delta s}\mathcal{Z}. \tag{3}$$

Finally, the gradient operator acts on an arbitrary scalar field $\mathcal{Y}$ in the following manner

$$grad\mathcal{Y} = \frac{\delta \mathcal{Y}}{\delta s}\mathbf{T} + \frac{\delta \mathcal{Y}}{\delta s}\mathbf{N} + \frac{\delta \mathcal{Y}}{\delta s}\mathbf{B}. \tag{4}$$

The Frenet-Serret formulas describe the motion of the ordered triad of unit orthonormal vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ along with the $s - line$ coordinate curve (vector line of s). In this case, the motion of the triad frame follows a space curve α parametrized by the arc-length s. By definition, the directional derivative of α with respect to the arc-length s is equal to $\mathbf{T}$ i.e.

$$\frac{\delta}{\delta s}\alpha = \mathbf{T}. \tag{5}$$

It is denoted by the following identity

$$\frac{\delta}{\delta s}\alpha = \frac{\partial}{\partial s}\alpha \frac{1}{\sqrt{\frac{\partial}{\partial s}\alpha \cdot \frac{\partial}{\partial s}\alpha}}. \tag{6}$$

In this paper, we always consider the special case in which the curve is supposed to be a unit speed curve i.e.

$$\frac{\delta}{\delta s}\alpha = \frac{\partial}{\partial s}\alpha, \ \left|\frac{\partial}{\partial s}\alpha\right| = 1. \tag{7}$$

The Frenet-Serret formulas are characterized by taking the directional derivative of the unit orthonormal vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ with respect to the vector line of s in the following manner

$$\begin{aligned} \frac{\delta}{\delta s}\mathbf{T} &= \kappa\mathbf{N}, \\ \frac{\delta}{\delta s}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \frac{\delta}{\delta s}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{8}$$

where κ measures the bending or rate of change of the tangent vector in the $(\mathbf{T}, \mathbf{N})$ plane along with the vector line of s. Torsion τ measures the twisting or amount of rotation of the Frenet-Serret triad frame about the $\mathbf{T}$ along with the vector line of s.

The directional derivative of α in the normal and binormal directions are expressed by considering the parametrization with respect to arc-lentgh n and b, respectively. In the normal direction, the directional derivative of α with respect to the arc-length n is equal to $\mathbf{N}$ i.e.

$$\frac{\delta}{\delta n}\alpha = \mathbf{N}, \tag{9}$$

where the unit speed curve parametrization is guaranteed by the following assumption

$$\frac{\delta}{\delta n}\alpha = \frac{\partial}{\partial n}\alpha, \ \left|\frac{\partial}{\partial n}\alpha\right| = 1. \tag{10}$$

Eqs. (9, 10) imply that the tangent vector of the $n - line$ coordinate curve (vector line of n) is $\mathbf{N}$ in the normal direction.

Similarly, in the binormal direction, the directional derivative of α with respect to the arc-length b is equal to $\mathbf{B}$ i.e.

$$\frac{\delta}{\delta b}\alpha = \mathbf{B}, \tag{11}$$

where the unit speed curve parametrization is guarenteed by the following assumption

$$\frac{\delta}{\delta b}\alpha = \frac{\partial}{\partial b}\alpha, \ \left|\frac{\partial}{\partial b}\alpha\right| = 1. \tag{12}$$

Eqs. (11, 12) imply that the tangent vector of the $b-line$ coordinate curve (vector line of b) is $\mathbf{B}$ in the binormal direction. These implications lead to define a new type of variations of the motion of the Frenet-Serret frame by taking the directional derivative of the $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ along with the $n-line$ coordinate curve and $b-line$ coordinate curve. Accordingly, these variations satisfy that

$$\begin{aligned} \frac{\delta}{\delta n}\mathbf{T} &= \theta_{ns}\mathbf{N} + (\Omega_b + \tau)\mathbf{B}, \\ \frac{\delta}{\delta n}\mathbf{N} &= -\theta_{ns}\mathbf{T} - (div\mathbf{B})\mathbf{B}, \\ \frac{\delta}{\delta n}\mathbf{B} &= -(\Omega_b + \tau)\mathbf{T} + (div\mathbf{B})\mathbf{N}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \frac{\delta}{\delta b}\mathbf{T} &= -(\Omega_n + \tau)\mathbf{N} + \theta_{bs}\mathbf{B}, \\ \frac{\delta}{\delta b}\mathbf{N} &= (\Omega_n + \tau)\mathbf{T} + (\kappa + div\mathbf{N})\mathbf{B}, \\ \frac{\delta}{\delta b}\mathbf{B} &= -\theta_{bs}\mathbf{T} - (\kappa + div\mathbf{N})\mathbf{N}. \end{aligned} \tag{14}$$

The entire directional differential equation systems of the Frenet-Serret triad vectors given by Eqs. (8, 13, 14) are called Gauss Weingarten equations [5].

The geometric quantities θ_{ns} and θ_{bs} symbolize the normal deformation of the vector tube in the normal and binormal directions, respectively, in the following manner

$$\theta_{ns} = \mathbf{N}\frac{\delta}{\delta n}\mathbf{T},\ \theta_{bs} = \mathbf{B}\frac{\delta}{\delta n}\mathbf{T}. \tag{15}$$

The divergence of the tangent, normal, and binormal vectors are expressed by the following identities

$$\begin{aligned} div\mathbf{T} &= \theta_{ns} + \theta_{bs}, \\ div\mathbf{N} &= \mathbf{B} \cdot \frac{\delta}{\delta b}\mathbf{N} - \kappa, \\ div\mathbf{B} &= -\mathbf{B} \cdot \frac{\delta}{\delta n}\mathbf{N}. \end{aligned} \tag{16}$$

The curl of the tangent, normal, and binormal vectors are expressed by the following notation

$$curl\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \Omega_s & 0 & \kappa \\ -div\mathbf{B} & \Omega_n & \theta_{ns} \\ \kappa + div\mathbf{N} & -\theta_{bs} & \Omega_b \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{17}$$

where $\Omega_s, \Omega_n, \Omega_b$ are called the abnormalities of the $\mathbf{T}-field$, $\mathbf{N}-field$, and $\mathbf{B}-field$ [5]. They are computed by considering the comparison of the two forms for the *curl* operator. There also exists the following relation among these functions

$$2(\Omega_s - \tau) = \Omega_s + \Omega_n + \Omega_b. \tag{18}$$

3. Transformation of Curve Flows to Special Surfaces via Anholonomic Coordinates

To obtain the evolution of space or plane curves in the different geometric or physical spacetime structures inducing the special equation systems attracts the attention of many researchers for a long time. The main motivation comes from the research of geometric characterization and the physical dynamics of moving curves, which have plentiful applications in various branches. From the subject, it is known that many fascinating soliton equations or completely integrable equations relate to evolution of space curves in the two or higher dimensional spaces which are homogenous or inhomogeneous. For example, the well-known Da Rios equation relates the binormal motion of space curves and the completely integrable non-linear Schrödinger (NLS) equation [6]. A couple system of KDV equations, the Regge-Lund equation, the defocusing nonlinear Schrödinger equation arise from hyperbolic evolution of curves [7]. The smoke ring or vortex filament equation relates the motion of space curves having the unchanged type of traveling wave solutions of the NLS equation [3]. In short, understanding the nature of the evolutionary equation systems and models is a very important subject to describe further classes of geometric and physical models and examine their behavior. These models are generally derived from the time evolution equations of the moving space curves satisfying particular conditions. In this section, we choose to apply these ideas in a straightforward and concrete manner within the different cases. First of all, we introduce a new class of time evolution equations, which define the directional motion of curves in the three-dimensional ordinary space $\mathbb{E}^3$. Then, we present the evolved geometric quantities and anholonomic coordinates with respect to that motion. Finally, we form a new class of directional surfaces and give their geometric characterizations and physical dynamics.

3.1. Osculating Motion of Curves and Osculating Surfaces in $\mathbb{E}^3$

Let s and n be arc-length parameters along with the curve α in the tangent direction ($\mathbf{T}$) and normal direction ($\mathbf{N}$) in the three-dimensional ordinary space $\mathbb{E}^3$, respectively. Let us also define the time parameter u and suppose that it is not dependent on neither s nor n. From Eqs. $(5, 9)$, the time evolution of a space curve obeying the osculating motion is given by the following equation

$$\frac{\partial}{\partial u}\alpha = \frac{\partial}{\partial s}\alpha \times \frac{\partial}{\partial n}\alpha = \mathbf{T} \times \mathbf{N} = \mathbf{B}. \tag{19}$$

Here, we choose to define the time evolution of a curve in that order due to the rule of positive cyclic permutations. It is also a trivial fact that we use the following abbreviation for the above equality

$$\frac{\partial}{\partial u}\alpha\left(s, n, u\right) = \frac{\partial}{\partial s}\alpha\left(s, n, u\right) \times \frac{\partial}{\partial n}\alpha\left(s, n, u\right). \tag{20}$$

For the rest of the paper, we rather choose to consider the type of Eq. (19) instead of the type of Eq. (20) for the brevity purpose. We also make the same choice for similar situations.

Case 1. In this case, we investigate the osculating motion of curves and osculating surfaces in the tangent direction in $\mathbb{E}^3$. In the tangent direction, when the time evolution of a space curve α is given by Eq. (19) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial s}\alpha = \frac{\partial}{\partial s}\frac{\partial}{\partial u}\alpha, \tag{21}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{T} = \frac{\partial}{\partial s}\mathbf{B}. \tag{22}$$

The further calculations are processed by the orthonormality condition of the Frenet-Serret vectors together with the compatibility condition between the arc-length parameter s and time parameter u associated with these vectors. Here, we should remind that the compatibility condition is not allowed between the partial differentiation of the s and n parameter of the space curve α due to the nature of the evolution in the tangent direction. From Eqs. $(5 - 8, 19, 21, 22)$, the time evolution equation of Frenet-Serret vectors obeying the osculating motion is expressed in the following manner

$$\frac{\delta}{\delta u}\left(\mathbf{T}, \mathbf{N}, \mathbf{B}\right) = \left(\mathbf{T}, \mathbf{N}, \mathbf{B}\right)\begin{pmatrix} 0 & -\tau & 0 \\ \tau & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix}, \tag{23}$$

where λ is a function of (s, n, u). The matrix is an antisymmetric, as follows from the generation process of the vectors and orthonormality conditions. Comparing the coefficients of $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ in Eq. (23) leads to time evolution equations of scalar geometric quantities along with the osculating motion of the curve α. Thus, we get the following three identities

$$\frac{\partial}{\partial u}\kappa = -\frac{\partial}{\partial s}\tau,\ \frac{\partial}{\partial u}\tau = -\frac{\partial}{\partial s}\frac{\tau^2}{\kappa},\ \lambda = \frac{\tau^2}{\kappa},\ \kappa \neq 0. \tag{24}$$

By the definition of the Frenet-Serret frame equation, it is already supposed that the curvature κ is non-vanishing. Therefore Eq. (24) is well-defined. If one also considers the compatibility condition of the scalar geometric quantities then it is obtained that

$$\frac{\partial^2}{\partial s^2}\frac{\tau^2}{\kappa} + \frac{\partial^2}{\partial u^2}\kappa = 0. \tag{25}$$

Eq. (25) is a second-order non-linear Laplacian-like partial differential equation and its solution family for some special cases helps to analyze physical dynamics of the osculating motion of the space curve. We choose to deal with that problem in the application section not to distract our attention while determining the geometric characterization of the osculating motion.

Thus, we completely describe the osculating motion of the space curve in the tangent direction. The osculating motion of a curve is determined by six equations: three in Eq. (23) and the three in Eq. (24) . The set of these identities is the fundamental result of this subsection. Now, we shall present the formation of the first osculating type of the anholonomic surface, which is called the osculating surface in the tangent direction, due to the osculating motion of the space curve. It is denoted by $O_s^{\mathcal{A}}$. This surface is generated by the corresponding osculating motion throughout the space curve α. In the case of the osculating motion of the curve in the tangent direction, the coefficients of 1^{st} and 2^{nd} fundamental forms of the first osculating type of the anholonomic surface are computed respectively by the following equalities

$$\mathcal{I} = d\alpha \cdot d\alpha = (\frac{\partial}{\partial s}\alpha ds + \frac{\partial}{\partial u}\alpha du) \cdot (\frac{\partial}{\partial s}\alpha ds + \frac{\partial}{\partial u}\alpha du), \tag{26}$$

and

$$\mathcal{II} = -d\alpha \cdot d\mathcal{N} = -(\frac{\partial}{\partial s}\alpha ds + \frac{\partial}{\partial u}\alpha du) \cdot (\frac{\partial}{\partial s}\mathcal{N} ds + \frac{\partial}{\partial u}\mathcal{N} du), \tag{27}$$

where $\mathcal{N}$ is a normal vector field of the surface given by

$$\mathcal{N} = \frac{\frac{\partial}{\partial s}\alpha \times \frac{\partial}{\partial u}\alpha}{\left\| \frac{\partial}{\partial s}\alpha \times \frac{\partial}{\partial u}\alpha \right\|} = \mathbf{T} \times \mathbf{B} = -\mathbf{N}. \tag{28}$$

From Eqs. $(8, 19, 23, 24, 26 - 28)$, we have

$$\mathcal{I} = ds^2 + du^2, \tag{29}$$

and

$$\mathcal{II} = \kappa ds^2 - 2\tau ds du - \lambda du^2, \tag{30}$$

where $\lambda = \frac{\tau^2}{\kappa},\ \kappa \neq 0$. Thus, from Eq. (29) , the coefficients of 1^{st} fundamental forms of the osculating surface in the tangent direction are given by

$$\mathcal{I}_{\mathcal{E}} = 1,\ \mathcal{I}_{\mathcal{F}} = 0,\ \mathcal{I}_{\mathcal{G}} = 1, \tag{31}$$

and similarly from Eq. (30) , the coefficients of 2^{nd} fundamental forms of the osculating surface in the tangent direction are given by

$$\mathcal{II}_{\mathcal{E}} = \kappa,\ \mathcal{II}_{\mathcal{F}} = -2\tau,\ \mathcal{II}_{\mathcal{G}} = -\frac{\tau^2}{\kappa}. \tag{32}$$

As a result, the Gaussian curvature $\mathcal{K}$ and mean cuvature $\mathcal{H}$ of the osculating surface in the tangent direction takes the following form

$$\mathcal{K} = -5\tau^2,\ \mathcal{H} = \frac{\kappa^2 - \tau^2}{2\kappa}. \tag{33}$$

Further geometric characterizations associated with the first osculating type of the anholonomic surface are given in the next section.

Case 2. In this case, we investigate the osculating motion of curves and osculating surfaces in the normal direction in $\mathbb{E}^3$. In the normal direction, when the time evolution of a space curve α is given by Eq. (19) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial n}\alpha = \frac{\partial}{\partial n}\frac{\partial}{\partial u}\alpha, \tag{34}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{N} = \frac{\partial}{\partial n}\mathbf{B}. \tag{35}$$

The further calculations are processed by the orthonormality condition of the Frenet-Serret vectors together with the compatibility condition between the arc-length parameter n and time parameter u associated with these vectors. Here, we should remind that the compatibility condition is not allowed between the partial differentiation of the s and n parameter of the space curve α due to the nature of the evolution in the normal direction. From Eqs. (9, 10, 13, 19, 34, 35) , the time evolution equation of Frenet-Serret vectors obeying the osculating motion is expressed in the following manner

$$\frac{\delta}{\delta u}(\mathbf{T}, \mathbf{N}, \mathbf{B}) = (\mathbf{T}, \mathbf{N}, \mathbf{B}) \begin{pmatrix} 0 & (\Omega_b + \tau) & \mu \\ -(\Omega_b + \tau) & 0 & 0 \\ -\mu & 0 & 0 \end{pmatrix}, \tag{36}$$

where μ is a function of (s, n, u) and $div\mathbf{B} = 0$. The matrix is an antisymmetric, as follows from the generation process of the vectors and orthonormality conditions. Comparing the coefficients of $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ in Eq. (36) leads to time evolution equations of scalar geometric quantities along with the osculating motion of the curve α. Thus, we get the following three identities

$$\frac{\partial}{\partial u}\theta_{ns} = \frac{\partial}{\partial n}(\Omega_b + \tau),\ \frac{\partial}{\partial u}(\Omega_b + \tau) = \frac{\partial}{\partial n}\frac{(\Omega_b + \tau)^2}{\theta_{ns}},\ \mu = \frac{(\Omega_b + \tau)^2}{\theta_{ns}},\ \theta_{ns} \neq 0. \tag{37}$$

Here, we suppose that the geometric quantity of θ_{ns} is non-vanishing. Therefore Eq. (37) is well-defined. If one also considers the compatibility condition of the scalar geometric quantities then it is obtained that

$$\frac{\partial^2}{\partial n^2}\frac{(\Omega_b + \tau)^2}{\theta_{ns}} + \frac{\partial^2}{\partial u^2}\theta_{ns} = 0. \tag{38}$$

Thus, we completely describe the osculating motion of the space curve in the normal direction. The osculating motion of a curve is determined by six equations: three in Eq. (36) and the three in Eq. (37) . The set of these identities is the fundamental result of this subsection. Now, we shall present the formation of the second osculating type of the anholonomic surface, which is called the osculating surface in the normal direction, due to the osculating motion of the space curve. It is denoted by $O_n^{\mathcal{A}}$. This surface is generated by the corresponding osculating motion throughout the space curve α. In the case of the osculating motion of the curve in the normal direction, the coefficients of 1^{st} and 2^{nd} fundamental forms of the second osculating type of the anholonomic surface are computed respectively by the following equalities

$$\mathcal{I} = d\alpha \cdot d\alpha = (\frac{\partial}{\partial n}\alpha dn + \frac{\partial}{\partial u}\alpha du) \cdot (\frac{\partial}{\partial n}\alpha dn + \frac{\partial}{\partial u}\alpha du), \tag{39}$$

and

$$\mathcal{II} = -d\alpha \cdot d\mathcal{N} = -(\frac{\partial}{\partial n}\alpha dn + \frac{\partial}{\partial u}\alpha du) \cdot (\frac{\partial}{\partial n}\mathcal{N} dn + \frac{\partial}{\partial u}\mathcal{N} du), \tag{40}$$

where $\mathcal{N}$ is a normal vector field of the surface given by

$$\mathcal{N} = \frac{\frac{\partial}{\partial n}\alpha \times \frac{\partial}{\partial u}\alpha}{\left\| \frac{\partial}{\partial n}\alpha \times \frac{\partial}{\partial u}\alpha \right\|} = \mathbf{N} \times \mathbf{B} = \mathbf{T}. \tag{41}$$

From Eqs. $(13, 19, 36, 37, 39 - 41)$, we have

$$\mathcal{I} = dn^2 + du^2, \tag{42}$$

and

$$\mathcal{II} = -\theta_{ns} dn^2 + \mu du^2, \tag{43}$$

where $\mu = \frac{(\Omega_b+\tau)^2}{\theta_{ns}}$, $\theta_{ns} \neq 0$. Thus, from Eq. (42), the coefficients of 1^{st} fundamental forms of the osculating surface in the normal direction are given by

$$\mathcal{I}_{\mathcal{E}} = 1, \ \mathcal{I}_{\mathcal{F}} = 0, \ \mathcal{I}_{\mathcal{G}} = 1, \tag{44}$$

and similarly from Eq. (43), the coefficients of 2^{nd} fundamental forms of the osculating surface in the normal direction are given by

$$\mathcal{II}_{\mathcal{E}} = -\theta_{ns}, \ \mathcal{II}_{\mathcal{F}} = 0, \ \mathcal{II}_{\mathcal{G}} = \frac{(\Omega_b + \tau)^2}{\theta_{ns}}. \tag{45}$$

As a result, the Gaussian curvature $\mathcal{K}$ and mean cuvature $\mathcal{H}$ of the osculating surface in the normal direction takes the following form

$$\mathcal{K} = -(\Omega_b + \tau)^2, \ \mathcal{H} = \frac{(\Omega_b + \tau)^2 - \theta_{ns}^2}{2\theta_{ns}}. \tag{46}$$

3.2. Rectifying Motion of Curves and Rectifying Surfaces in $\mathbb{E}^3$

Let b and s be arc-length parameters along with the curve α in the binormal direction $(\mathbf{B})$ and tangent direction $(\mathbf{T})$ in the three-dimensional ordinary space $\mathbb{E}^3$, respectively. Let us also define the time parameter u and suppose that it is not dependent on neither b nor s. From Eqs. $(5, 11)$, the time evolution of a space curve obeying the rectifying motion is given by the following equation

$$\frac{\partial}{\partial u}\alpha = \frac{\partial}{\partial b}\alpha \times \frac{\partial}{\partial s}\alpha = \mathbf{B} \times \mathbf{T} = \mathbf{N}. \tag{47}$$

Case 1. In this case, we define the rectifying motion of curves and rectifying surfaces in the binormal direction in $\mathbb{E}^3$. In the binormal direction, when the time evolution of a space curve α is given by Eq. (47) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial b}\alpha = \frac{\partial}{\partial b}\frac{\partial}{\partial u}\alpha, \tag{48}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{B} = \frac{\partial}{\partial b}\mathbf{N}. \tag{49}$$

The time evolution equation of Frenet-Serret vectors obeying the rectifying motion is expressed in the following manner

$$\frac{\delta}{\delta u}(\mathbf{T},\mathbf{N},\mathbf{B}) = (\mathbf{T},\mathbf{N},\mathbf{B})\begin{pmatrix} 0 & \nu & -(\Omega_n+\tau) \\ -\nu & 0 & 0 \\ (\Omega_n+\tau) & 0 & 0 \end{pmatrix}, \tag{50}$$

where ν is a function of (s,b,u). Further details regarding the rectifying motion of curves and rectifying surfaces in the binormal direction in $\mathbb{E}^3$ can be computed similarly as in the case of the osculating motion of curves and osculating surfaces.

Case 2. In this case, we attempt to define the rectifying motion of curves and rectifying surfaces in the tangent direction in $\mathbb{E}^3$. In the tangent direction, when the time evolution of a space curve α is given by Eq. (47) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial s}\alpha = \frac{\partial}{\partial s}\frac{\partial}{\partial u}\alpha, \tag{51}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{T} = \frac{\partial}{\partial s}\mathbf{N}. \tag{52}$$

It is computed that the curvature vanishes throughout the rectifying motion of the space curve α. This contadicts with the fact that κ has to be a non-vanishing scalar to define the Frenet-Serret equations. Thus, we conclude that there exist no rectifying surface in the tangent direction since the time evolution equation of the rectifying motion is not valid.

3.3. *Normal Motion of Curves and Normal Surfaces in* $\mathbb{E}^3$

Let n and b be arc-length parameters along with the curve α in the normal direction ($\mathbf{N}$) and binormal direction ($\mathbf{B}$) in the three-dimensional ordinary space $\mathbb{E}^3$, respectively. Let us also define the time parameter u and suppose that it is not dependent on neither n nor b. From Eqs. $(9,11)$, the time evolution of a space curve obeying the normal motion is given by the following equation

$$\frac{\partial}{\partial u}\alpha = \frac{\partial}{\partial n}\alpha \times \frac{\partial}{\partial b}\alpha = \mathbf{N}\times\mathbf{B} = \mathbf{T}. \tag{53}$$

Case 1. In this case, we define the normal motion of curves and normal surfaces in the normal direction in $\mathbb{E}^3$. In the normal direction, when the time evolution of a space curve α is given by Eq. (53) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial n}\alpha = \frac{\partial}{\partial n}\frac{\partial}{\partial u}\alpha, \tag{54}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{N} = \frac{\partial}{\partial n}\mathbf{T}. \tag{55}$$

The time evolution equation of Frenet-Serret vectors obeying the normal motion is expressed in the following manner

$$\frac{\delta}{\delta u}(\mathbf{T},\mathbf{N},\mathbf{B}) = (\mathbf{T},\mathbf{N},\mathbf{B})\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & (\Omega_b+\tau) \\ -\beta & -(\Omega_b+\tau) & 0 \end{pmatrix}, \tag{56}$$

where β is a function of (n, b, u) . Further details regarding the normal motion of curves and normal surfaces in the normal direction in $\mathbb{E}^3$ can be computed similarly as in the case of the osculating motion of curves and osculating surfaces.

Case 2. In this case, we define the normal motion of curves and normal surfaces in the binormal direction in $\mathbb{E}^3$. In the binormal direction, when the time evolution of a space curve α is given by Eq. (53) the evolution of the time derivative of Frenet-Serret vectors is computed by using the following compatibility condition

$$\frac{\partial}{\partial u}\frac{\partial}{\partial b}\alpha = \frac{\partial}{\partial b}\frac{\partial}{\partial u}\alpha, \tag{57}$$

which implies that

$$\frac{\partial}{\partial u}\mathbf{B} = \frac{\partial}{\partial b}\mathbf{T}. \tag{58}$$

The time evolution equation of Frenet-Serret vectors obeying the normal motion is expressed in the following manner

$$\frac{\delta}{\delta u}(\mathbf{T}, \mathbf{N}, \mathbf{B}) = (\mathbf{T}, \mathbf{N}, \mathbf{B})\begin{pmatrix} 0 & \varpi & 0 \\ -\varpi & 0 & (\Omega_n + \tau) \\ 0 & -(\Omega_n + \tau) & 0 \end{pmatrix}, \tag{59}$$

where ϖ is a function of $(n, b, u) = 0$. Further details regarding the normal motion of curves and normal surfaces in the binormal direction in $\mathbb{E}^3$ can be computed similarly as in the case of the osculating motion of curves and osculating surfaces.

4. Some Pure Geometric Characteristics of the Anholonomic Surfaces

In the research of pure differential geometry, the theory of surfaces has been of great importance. In the investigation of the characteristics of a surface, the most significant step is to comprehend the nature of the formation and the process of the construction of the surface. There exist many well-known construction methods for the surfaces. For example, they can be constructed provided satisfying particular equations for different parameters or they can be created by continuous motion of a generating curve or a line. They can also be defined by the rigid kinematical motion of uniparametric family of curves. However, they can also be described by the fully abstract motion without referencing to an ambient space such as the Klein bottle. Recently, different parametric basis functions and operators have been improved to define special surfaces. As a result of this extensive research effort, it has been developed various surfaces such as ruled surfaces, developable surfaces, minimal surfaces, Roman surfaces, Boy's surfaces, Steiner surfaces, etc. together with their topological and or differential classifications [8 – 17]. In the present section, we investigate further geometric characterizations including the local behavior of the anholonomic surfaces generated by the directional motion of space curves.

Theorem 1. Let $O_s^{\mathcal{A}}$ be the osculating surface in the tangent direction. Then the followings hold.

i. If α is a plane curve then a point of a surface $O_s^{\mathcal{A}}$ is parabolic. In this case, $O_s^{\mathcal{A}}$ is also a developable surface.

ii. If α is not a plane curve then a point of a surface $O_s^{\mathcal{A}}$ is hyperbolic.

Proof. It is known that the torsion τ of the plane curve vanishes. Thus, if one considers Eqs. (33) then the proof comes from the fact that a point of a surface is parabolic (hyperbolic) when the Gaussian curvature is zero (negative), respectively. Here it is also obvious that one of the principal curvatures is non-vanishing.

Theorem 2. Let $O_n^{\mathcal{A}}$ be the osculating surface in the normal direction. Then the followings hold.

i. If $\tau = -\Omega_b$ then a point of a surface $O_n^{\mathcal{A}}$ is parabolic. In this case, $O_s^{\mathcal{A}}$ is also a developable surface.

ii. If $\tau \neq -\Omega_b$ then a point of a surface $O_n^{\mathcal{A}}$ is hyperbolic.

Proof. It is obvious from Eq. (46).

Theorem 3. Let $O_s^{\mathcal{A}}$ be the osculating surface in the tangent direction. Then the followings hold.
i. s-parameter curves of the osculating surface $O_s^{\mathcal{A}}$ are geodesics.
ii. u-parameter curves of the osculating surface $O_s^{\mathcal{A}}$ are geodesics.

Proof. i. From Eq. (8), we know that

$$\frac{\partial^2}{\partial s^2}\alpha = \kappa \mathbf{N}, \tag{60}$$

and from Eq. (28), the normal of the osculating surface is given by

$$\mathcal{N} = -\mathbf{N}. \tag{61}$$

From Eqs. (60, 61), it is seen that $\frac{\partial^2}{\partial s^2}\alpha$ is parallel to the osculating surface normal in the tangent direction, which proves the first part of the theorem.
ii. From Eq. (23), we compute that

$$\frac{\partial^2}{\partial u^2}\alpha = -\lambda \mathbf{N}. \tag{62}$$

Thus, it is seen from Eq. (62) that the binormal component of $\frac{\partial^2}{\partial u^2}\alpha$ vanishes. As a result, the proof is completed.

Theorem 4. Let $O_n^{\mathcal{A}}$ be the osculating surface in the normal direction. Then the followings hold.
i. n-parameter curves of the osculating surface $O_n^{\mathcal{A}}$ are geodesics.
ii. u-parameter curves of the osculating surface $O_n^{\mathcal{A}}$ are geodesics.

Proof. i. From Eq. (13), we know that

$$\frac{\partial^2}{\partial n^2}\alpha = -\theta_{ns}\mathbf{T} - (div\mathbf{B})\mathbf{B}, \tag{63}$$

and from Eq. (36), it is computed that $div\mathbf{B} = 0$. Thus, we get the following equality

$$\frac{\partial^2}{\partial n^2}\alpha = -\theta_{ns}\mathbf{T}. \tag{64}$$

Moreover, from Eq. (41), the normal of the osculating surface is given by

$$\mathcal{N} = \mathbf{T}. \tag{65}$$

From Eqs. (64, 65), it is seen that $\frac{\partial^2}{\partial n^2}\alpha$ is parallel to the osculating surface normal in the normal direction, which proves the first part of the theorem.
ii. From Eq. (36), we compute that

$$\frac{\partial^2}{\partial u^2}\alpha = -\mu \mathbf{T}. \tag{66}$$

Thus, it is seen from Eq. (66) that the binormal component of $\frac{\partial^2}{\partial u^2}\alpha$ vanishes. As a result, the proof is completed.

Theorem 5. Let $O_s^{\mathcal{A}}$ be the osculating surface in the tangent direction. Then the followings hold.
i. s-parameter curves of the osculating surface $O_s^{\mathcal{A}}$ cannot be asymptotics.
ii. u-parameter curves of the osculating surface $O_s^{\mathcal{A}}$ are asymptotics if and only if

$$\tau = 0.$$

Proof. **i**. From Eq. (8), we know that

$$\frac{\partial^2}{\partial s^2}\alpha = \kappa \mathbf{N}, \tag{67}$$

and from Eq. (28), the normal of the osculating surface is given by

$$\mathcal{N} = -\mathbf{N}. \tag{68}$$

From Eqs. (67, 68), it is seen that the normal component of $\frac{\partial^2}{\partial s^2}\alpha$ vanishes in the tangent direction if and only if $\kappa = 0$. However, this contradicts with our choice of the non-vanishing curvature due to Eq. (24). Thus, it proves the first part of the theorem.

ii. From Eq. (23), we compute that

$$\frac{\partial^2}{\partial u^2}\alpha = -\lambda \mathbf{N},\ \lambda = \frac{\tau^2}{\kappa},\ \kappa \neq 0. \tag{69}$$

Thus, it is seen from Eq. (69) that the normal component of $\frac{\partial^2}{\partial u^2}\alpha$ vanishes in the tangent direction if and only if $\tau = 0$, i.e. α is a plane curve. As a result, the proof is completed.

Theorem 6. Let $O_n^{\mathcal{A}}$ be the osculating surface in the normal direction. Then the followings hold.
i. n-parameter curves of the osculating surface $O_n^{\mathcal{A}}$ are asymptotics.
ii. u-parameter curves of the osculating surface $O_n^{\mathcal{A}}$ are asymptotics.

Proof. **i**. From Eq. (13), we know that

$$\frac{\partial^2}{\partial n^2}\alpha = -\theta_{ns}\mathbf{T} - (div\mathbf{B})\mathbf{B}, \tag{70}$$

and from Eq. (36), it is computed that $div\mathbf{B} = 0$. Thus, we get the following equality

$$\frac{\partial^2}{\partial n^2}\alpha = -\theta_{ns}\mathbf{T}. \tag{71}$$

Moreover, from Eq. (41), the normal of the osculating surface is given by

$$\mathcal{N} = \mathbf{T}. \tag{72}$$

From Eqs. (71, 72), it is seen that the normal component of $\frac{\partial^2}{\partial n^2}\alpha$ vanishes in the normal direction. Thus, it proves the first part of the theorem.

ii. From Eq. (36), we compute that

$$\frac{\partial^2}{\partial u^2}\alpha = -\mu \mathbf{T}. \tag{73}$$

Thus, it is seen from Eq. (73) that the normal component of $\frac{\partial^2}{\partial u^2}\alpha$ vanishes in the normal direction. As a result, the proof is completed.

Corollary 7. The parameter curves of the first osculating type of the anholonomic surface $O_s^{\mathcal{A}}$ are lines of curvature if and only if α is a plane curve i.e. $\tau = 0$.

Proof. For the first osculating type of the anholonomic surface $O_s^{\mathcal{A}}$, we know from Eqs. (31, 32) that $\mathcal{I}_{\mathcal{F}} = \mathcal{II}_{\mathcal{F}} = 0$ if and only if $\tau = 0$. This completes the proof.

Corollary 8. The parameter curves of the second osculating type of the anholonomic surface $O_n^{\mathcal{A}}$ are lines of curvature.

Proof. For the second osculating type of the anholonomic surface $O_n^{\mathcal{A}}$, we know from Eqs. (44, 45) that $\mathcal{I}_{\mathcal{F}} = \mathcal{II}_{\mathcal{F}} = 0$. This completes the proof.

Similar results regarding the rectifying (normal) motion of curves and rectifying (normal) surfaces in the tangent, normal or binormal direction in $\mathbb{E}^3$ can be obtained similarly as in the case of the osculating motion of curves and osculating surfaces.

5. Application: A Fractional Solution of the Directional Motion of the Helical Model

In this section, we apply a relatively effective approach known as Adomian decomposition technique for solving the second-order non-linear Laplacian-like partial differential equation, which are obtained to characterize physical dynamics of the directional motion of the space curve. Accordingly, we first transform the second-order non-linear Laplacian-like partial differential equation into a time fractional boundary value problems of wave equation. Then, we apply the fractional derivative in the sense of the Caputo derivative. Moreover, the Adomian decomposition technique is considered to establish approximate analytical solutions of time dependent fractional wave equation with the certain constraints of boundary conditions.

Now, from Eqs. (24, 25), we have the following identity

$$\frac{\partial^2}{\partial s^2}\frac{\tau^2}{\kappa} + \frac{\partial^2}{\partial u^2}\kappa = 0, \tag{74}$$

which explains physical dynamics of the osculating motion of the space curve in the tangent direction in $\mathbb{E}^3$. Then, if we assume that the space curve is a helix then the ratio of the torsion and curvature of the curve is a constant. This assumption provides to analyze more specific and attractive case. Finally, Eq. (74) is written in the following form

$$\tau_{uu} + k^2\tau_{ss} = 0. \tag{75}$$

From the fractional differential approach and Eq. (75) we obtain that

$$D_{*u}^{\alpha}\tau(s,u) = -k^2\frac{\partial^2}{\partial s^2}\tau(s,u), \quad 0 < u < 1, \quad 1 < \alpha \leq 2. \tag{76}$$

Now, let us consider the following boundary value problem

$$D_{*u}^{\alpha}\tau(s,u) = -k^2\frac{\partial^2}{\partial s^2}\tau(s,u), \quad 0 < u < 1, \quad 1 < \alpha \leq 2, \tag{77}$$

$$\tau(s,0) = \sin s, \quad \tau(s,1) = \sin s, \tag{78}$$

where D_{*u}^{α} is the Caputo fractional derivative of order α, $1 < \alpha \leq 2$ with respect to time variable u. If the inverse operator I_u^{α} is applied to the both sides of Eq. (77), then we compute that

$$\tau(s,u) = \tau(s,0) + u\ \tau_u(s,0) + I_u^{\alpha}(-k^2\frac{\partial^2}{\partial s^2}\tau(s,u)). \tag{79}$$

Then, the Adomian decomposition method [18, 19] implies that

$$\sum_{i=0}^{\infty}\tau_i(s,u) = \tau(s,0) + u\,\tau_u(s,0) + I_u^{\alpha}(-k^2\frac{\partial^2}{\partial s^2}(\sum_{i=0}^{\infty}\tau_i(s,u))), \tag{80}$$

$$\begin{aligned}\tau_0(s,u) &= \tau(s,0) + u\,\tau_u(s,0),\\ \tau_{i+1}(s,u) &= I_u^{\alpha}(-k^2\frac{\partial^2}{\partial s^2}\tau_i(s,u)), \quad i \geq 0.\end{aligned}$$

Furthermore, from the Taylor series expansion of $\sin s$, we calculate the following equalities

$$\begin{aligned}\tau_0(s,u) &= (1-u)\sin s + us,\\ \tau_1(s,u) &= -k^2\frac{u^{\alpha}}{\Gamma(\alpha+1)}(1-u)\sin s - u^{\alpha+1}\frac{s^3}{3!},\\ &\vdots\\ \tau(s,u) &= \sum_{i=0}^{\infty}\frac{(-1)^i u^{i\alpha}}{\Gamma(i\alpha+1)}(k^2)^i(1-u)\sin s + \sum_{i=0}^{\infty}\frac{(-1)^i u^{i\alpha+1}s^{2i+1}}{(2i+1)!}.\end{aligned}$$

$$\tau(s,u) = (1-u)E_{\alpha}(k^2u^{\alpha})\sin s + u^{1-\frac{\alpha}{2}}\sin(u^{\frac{\alpha}{2}}s), \tag{81}$$

where $E_{\alpha}(u)$ is the Mittag–Leffler function defined by

$$E_{\alpha}(u) = \sum_{i=0}^{\infty}\frac{u^i}{\Gamma(i\alpha+1)}, \quad |u| < \infty.$$

When $\alpha = 2$, the solution of the wave equation Eq. (77) is given by the following equality

$$\tau(s,u) = (1-u)\sum_{i=0}^{\infty}(k^2)^i\cos u\sin s + \sin(us).$$

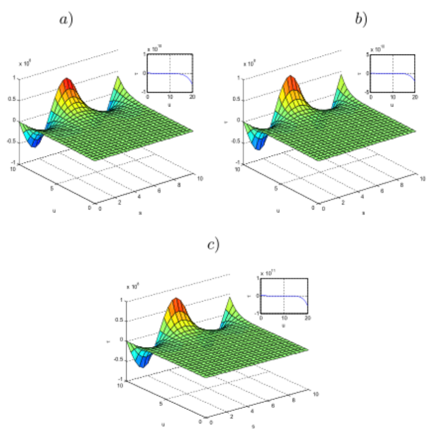

Figure 1: The 3D graphics of the fractional differential equation Eq. (77) for (k=0.5) (a) $\alpha = 1.25$, (b) $\alpha = 1.5$, (c) $\alpha = 1.75$.

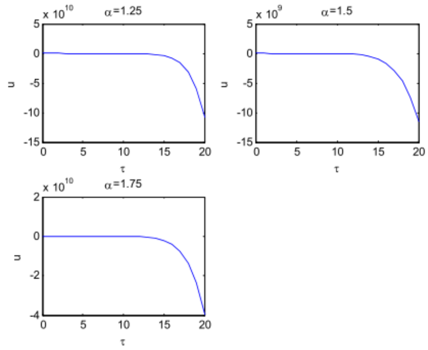

Figure 2: The 2D graphics of the fractional differential equation Eq. (77) for different values of α. (k=0.5, s=0.4)

$s \setminus u$	0.5	1	1.5	2	2.5
0.5	0.278376	0.603928	0.508998	9.09058	96.0454
1	0.427139	0.841471	0.366079	16.9675	170.203
1.5	0.531204	0.960396	0.146163	20.8174	202.874
2	0.599487	0.999591	0.001564	19.5354	185.758
2.5	0.638366	0.979623	0.0258319	13.4167	123.008

Table 1. ADM solutions of the fractional differential equation Eq. (77) for different values of s and u.($\alpha = 1.25, k = 0.5$)

In Figure 1, Figure 2 and Table1, six-dependent terms of the decomposition series are obtained by the ADM solutions.

We can also solve similar type equations to analyze the physical dynamics of the other directional motions of the space curve. However, we left that exercise to the reader to keep the manuscript in the concise and compact form.

6. Conclusion

The manuscript focuses on the directional motion of curves in the three-dimensional ordinary space $\mathbb{E}^3$. We report evolved geometric quantities and anholonomic coordinates with respect to those motions. At the same time, it has been defined a new class of directional surfaces. Moreover, some geometric characterization and local behavior of these surfaces are determined.

In differential geometry, there exists quite a number of fascinating special cases (minimal surfaces, ruled surfaces, parallel surfaces, surfaces of revolution), which become a beautiful example of differentiable methods. In future studies, we consider these special cases to focus more on defining new directional surfaces. For example, based on the definition of the ruled surface, it is quite natural to describe the osculating ruled surface in the tangent direction due to the osculating motion of the space curve. Similar definitions may also be obtained for other cases by using the same argument. Thus, we can further discuss whether a new family of surfaces developable or not. Moreover, the inextensibility, bi-inextensibility, and parallelism conditions of the surface evolution and its flow can also be characterized.

In determining the characterization of many dynamical and physical systems, one encounters the mathematical ideas of geometric quantities and variables such as curvatures, abnormality functions, etc. Examples of such dynamics mostly contain connections between the integrable systems and the differential geometry of curve motion. One example is that the sine-Gordon equation describes the dynamics of more generalized types of evolution, while the nonlinear Schrödinger equation deals with the special cases of evolution systems that are connected to solitons and other integrable systems. These systems are generally represented by time-dependent equations. In the future, we plan to investigate purely local properties of anholonomic surface dynamics preserving global restrictions. Concentrating on the fundamental classes of directional motions, we attempt to obtain special partial differential equations associated with these motions and finally compute their particular and explicit solutions.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] B. Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, The American mathematical monthly 110 (2003) 147-152.
[2] S. Izumiya, N. Takeuchi, New special curves and developable surfaces, Turkish Journal of Mathematics 28 (2004) 153-164.
[3] H. Hasimoto, A soliton on a vortex filament, Journal of Fluid Mechanics 51 (1972) 477-485.
[4] G.L. Lamb, Solitons on moving space curves, J. Mathematical Phys. 18 (1977) 1654-1661.
[5] W.K. Schief, C. Rogers, Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 455 (1999) 3163-3188.
[6] L.S. Da Rios, Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque, Rendiconti del Circolo Matematico di Palermo 22 (1906) 117-135.
[7] K. Nakayama, Motion of curves in hyperboloid in the Minkowski space, Journal of the Physical Society of Japan 67 (1998) 3031-3037.
[8] M. Erdoğdu, M. Özdemir, Geometry of Hasimoto surfaces in Minkowski 3-space, Mathematical Physics, Analysis and Geometry 17 (2014) 169-181.
[9] A. Kelleci, M. Bektaş, M. Ergüt, The Hasimoto surface according to bishop frame, Adıyaman Üniversitesi Fen Bilimleri Dergisi 9 (2019) 13-22.
[10] A. Sym, Soliton surfaces, Lettere al Nuovo Cimento 36 (1983) 307-312.
[11] S.C. Anco, R. Myrzakulov, Integrable generalizations of Schrödinger maps and Heisenberg spin models from Hamiltonian flows of curves and surfaces, Journal of Geometry and Physics 60 (2010) 1576-1603.
[12] N. Gürbüz, The motion of timelike surfaces in timelike geodesic coordinates, Int. J. Math. Anal. 4 (2010) 349-356.
[13] Ö. Ceyhan, A.S. Fokas, M. Gürses, Deformations of surfaces associated with integrable Gauss-Mainardi–Codazzi equations, Journal of Mathematical Physics 41 (2000) 2251-2270.
[14] M. Grbović, E. Nešović, On Bäcklund transformation and vortex filament equation for pseudo null curves in Minkowski 3-space, International Journal of Geometric Methods in Modern Physics 13 (2016) 1650077.
[15] M. Melko, I. Sterling, Application of soliton theory to the construction of pseudospherical surfaces in R^3, Annals of Global Analysis and Geometry 11 (1993) 65-107.
[16] A. Bobenko, U. Pinkall, Discrete surfaces with constant negative Gaussian curvature and the Hirota equation, Journal of Differential Geometry 43 (1996) 527-611.
[17] N. Gürbüz, D.W. Yoon, Hasimoto surfaces for two classes of curve evolution in Minkowski 3-space, Demonstratio Mathematica 53 (2020) 277-284.
[18] Z.M. Odibat, S. Momani, Approximate solutions for boundary value problems of time-fractional wave equation, Applied Mathematics and Computation 181 (2006) 767-774.
[19] L.V.C. Hoan, Z. Korpinar, M. Inç, Y.M. Chu, B. Almohsen, On convergence analysis and numerical solutions of local fractional Helmholtz equation, Alexandria Engineering Journal 59 4335-4341.