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Bounds for the Berezin number of reproducing kernel Hilbert space operators

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Abstract. In this paper, we find new upper bounds for the Berezin number of the product of bounded linear operators defined on reproducing kernel Hilbert spaces. We also obtain some interesting upper bounds concerning one operator, the upper bounds obtained here refine the existing ones. Further, we develop new lower bounds for the Berezin number concerning one operator by using their Cartesian decomposition. In particular, we prove that $\mathbf{ber}(A) \geq \frac{1}{\sqrt{2}}\mathbf{ber}\big(\Re(A) \pm \Im(A)\big)$, where $\mathbf{ber}(A)$ is the Berezin number of the bounded linear operator A.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the usual inner product $\langle .,. \rangle$ and $\|\cdot\|$ be the corresponding norm. Throughout this article I stands for the identity operator on $\mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A|$ denotes the positive operator $(A^*A)^{1/2}$. The Cartesian decomposition of A is given by $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. Let $A \in \mathcal{B}(\mathcal{H})$. The operator norm, the numerical range and the numerical radius of A are, respectively, denoted by $\|A\|$, $W(A)$ and $w(A)$ and are respectively given by $\|A\| = \sup\{\|Ax\| \ : \ x \in \mathcal{H}, \ \|x\| = 1\}$, $W(A) = \{\langle Ax, x\rangle \ : \ x \in \mathcal{H}, \ \|x\| = 1\}$ and $w(A) = \sup\{|\langle Ax, x\rangle| \ : \ x \in \mathcal{H}, \ \|x\| = 1\}$. It is easy to verify that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and is equivalent to the operator norm $\|\cdot\|$. In particular, $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$ holds. For refinements of this inequality we refer to [5–8] and references therein.
A reproducing kernel Hilbert space (RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of all complex valued functions on a set Ω such that for every $\lambda \in \Omega$, the linear evaluation functional $E_\lambda : \mathcal{H} \to \mathbb{C}$, defined by $E_\lambda(f) = f(\lambda)$, is continuous (see [14]). Throughout this paper, we denote by $\mathcal{H}$ the reproducing kernel Hilbert space on the set Ω. The Riesz representation theorem ensure that for each $\lambda \in \Omega$, there exists unique $k_\lambda \in \mathcal{H}$ such that for every $f \in \mathcal{H}, f(\lambda) = E_\lambda(f) = \langle f, k_\lambda\rangle$. The function k_λ is called the reproducing kernel associated with λ and the family $\{k_\lambda : \lambda \in \Omega\}$ is called reproducing kernel of the Hilbert space $\mathcal{H}$. For each λ in Ω, the normalized reproducing kernel is defined by $\hat{k}_\lambda = k_\lambda/\|k_\lambda\|$. For a bounded linear operator A on $\mathcal{H}$, the

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function $\widetilde{A}$ defined on Ω by $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle$ is called the Berezin symbol of A, which was introduced by Berezin (see [3, 4]). The Berezin set of A, denoted by $\mathbf{Ber}(A)$, is defined as $\mathbf{Ber}(A) := \{\widetilde{A}(\lambda) : \lambda \in \Omega\}$. In [13] Karaev introduced Berezin number of A, which is denoted by $\mathbf{ber}(A)$ and is defined as

$$\mathbf{ber}(A) := \sup_{\lambda\in\Omega} |\widetilde{A}(\lambda)|.$$

The least Berezin number of A, to be denoted by $c(A)$, is defined as $c(A) := \inf_{\lambda\in\Omega} |\widetilde{A}(\lambda)|$ (see [19]). It is clear from the definition that for any $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subseteq W(A) \text{ and } \mathbf{ber}(A) \le w(A).$$

Moreover, For any $A, B \in \mathcal{B}(\mathcal{H})$, we infer that

$$\mathbf{ber}(A + B) \le \mathbf{ber}(A) + \mathbf{ber}(B),$$
$$\mathbf{ber}(\alpha A) = |\alpha|\mathbf{ber}(A) \text{ for all } \alpha \in \mathbb{C},$$
$$c(\alpha A) = |\alpha|c(A) \text{ for all } \alpha \in \mathbb{C}.$$

In [18], by using the classical Jensen and Young inequalities, Yamancı and Garayev obtained upper bounds for the Berezin number of $A^\alpha XB^\alpha$ and $A^\alpha XB^{1-\alpha}$ for $0 \le \alpha \le 1$ and $A, B, X \in \mathcal{B}(\mathcal{H})$ where A, B are positive operators. In [1], Bakherad and Yamancı developed Berezin number inequalities involving the operator geometric mean, i.e., $A\sharp B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2} A^{1/2}$ for positive operators $A, B \in \mathcal{B}(\mathcal{H})$. In [20], Yamancı and Gürdal obtained inverse power inequalities for the Berezin number of bounded linear operators acting on $\mathcal{H}$. Many other mathematicians have also studied different Berezin numbers inequalities, we refer the readers to [2, 11, 15, 17, 19] and references therein.

This article is demarcated into three sections, including the introductory one. In section 2, we obtain some new upper bounds for the Berezin number of the product operators on reproducing kernel Hilbert spaces. Also, we find some upper bounds concerning one operator. In section 3, we obtain lower bounds for the Berezin number of bounded linear operators via Cartesian decomposition.

2. On upper bounds for the Berezin number

Our first theorem in this section provides upper bounds for the Berezin numbers of A^*XB, in particular, we find bounds for $\mathbf{ber}^r(A^*XB)$ and $\mathbf{ber}^2(A^*XB)$, respectively, where $A, B, X \in \mathcal{B}(\mathcal{H})$. To do so we need the following four lemmas, first two of which deal with bounded linear operator $A \in \mathcal{B}(\mathcal{H})$. Third lemma is on scalars and remaining one is on vectors.

Lemma 2.1. *([16]) Let $A \in \mathcal{B}(\mathcal{H})$ be positive and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$(a)\ \langle Ax, x\rangle^r \le \langle A^r x, x\rangle, \quad \text{for } r \ge 1.$$
$$(b)\ \langle A^r x, x\rangle \le \langle Ax, x\rangle^r, \quad \text{for } 0 < r \le 1.$$

Lemma 2.2. *([10]) Let $A \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then*

$$|\langle Ax, y\rangle|^2 \le \langle |A|^{2\alpha}x, x\rangle\ \langle |A^*|^{2(1-\alpha)}y, y\rangle, \quad \text{for all } \alpha \in [0, 1].$$

Lemma 2.3. *([12]) If $a, b \ge 0$ and $0 \le \alpha \le 1$, then*

$$(a^\alpha b^{1-\alpha})^2 \le (\alpha a + (1-\alpha)b)^2 - r_0^2 (a-b)^2, \text{ where } r_0 = min\{\alpha, 1-\alpha\}.$$

Lemma 2.4. *([9]) If $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, then*

$$|\langle x, e\rangle\langle e, y\rangle| \le \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

We now prove the first result of this section.

Theorem 2.5. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$. Then*

$$(i)\ \boldsymbol{ber}^r(A^*XB) \leq \frac{\|X\|^r}{2}\sqrt{\boldsymbol{ber}^2(|A|^{2r}+|B|^{2r}) - c^2(|A|^{2r}-|B|^{2r})}, \quad \text{for all } r \geq 1.$$

$$(ii)\ \boldsymbol{ber}^2(A^*XB) \leq \frac{1}{4}\Big(\boldsymbol{ber}^2(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A) - c^2(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\Big),$$

for every $\alpha \in [0,1]$.

Proof. (i) Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
& |\langle (A^*XB)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \\
= \ & |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle|^{2r} \\
\leq \ & \|X\|^{2r}\|A\hat{k}_\lambda\|^{2r}\|B\hat{k}_\lambda\|^{2r} \quad \Big(\text{using Cauchy-Schwarz inequality}\Big) \\
= \ & \|X\|^{2r}\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \\
\leq \ & \|X\|^{2r}\langle |A|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle |B|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \ \Big(\text{using Lemma 2.1(a)}\Big) \\
= \ & \frac{\|X\|^{2r}}{4}\Big(\langle (|A|^{2r}+|B|^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - \langle (|A|^{2r}-|B|^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\Big) \\
\leq \ & \frac{\|X\|^{2r}}{4}\Big(\mathbf{ber}^2(|A|^{2r}+|B|^{2r}) - c^2(|A|^{2r}-|B|^{2r})\Big).
\end{aligned}$$

Therefore, by taking supremum over all $\lambda \in \Omega$, we get the result as desired.
(ii) Again,

$$\begin{aligned}
& |\langle (A^*XB)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2} \\
= \ & |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle|^{2} \\
\leq \ & \langle |X|^{2\alpha}B\hat{k}_\lambda, B\hat{k}_\lambda\rangle \langle |X^*|^{2(1-\alpha)}A\hat{k}_\lambda, A\hat{k}_\lambda\rangle \ \Big(\text{by Lemma 2.2}\Big) \\
= \ & \langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
= \ & \frac{1}{4}\Big(\langle (B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - \langle (B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\Big) \\
\leq \ & \frac{1}{4}\Big(\mathbf{ber}^2(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A) - c^2(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\Big).
\end{aligned}$$

Therefore, by considering supremum over all $\lambda \in \Omega$, we get the desired inequality. □

Remark 2.6. *Following [11, Theorem 2.5 (i)], for the case $p = q = 2$, we get,*

$$\boldsymbol{ber}^r(A^*XB) \leq \frac{\|X\|^r}{2}\boldsymbol{ber}(|A|^{2r}+|B|^{2r}). \tag{1}$$

It is clear that the inequality obtained in Theorem 2.5 (i) is better than that in (1).
We also remark that the inequality obtained in Theorem 2.5 (ii) improves on the inequality [11, Theorem 2.5 (ii)], namely,

$$\boldsymbol{ber}(A^*XB) \leq \frac{1}{2}\boldsymbol{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A), \ \forall \alpha \in [0,1].$$

In particular, considering $X = I$ in Theorem 2.5 (i) we get the following corollary.

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2}\sqrt{\mathbf{ber}^2(|A|^{2r} + |B|^{2r}) - c^2(|A|^{2r} - |B|^{2r})}.$$

Remark 2.8. *Following [17, Corollary 3.7 (ii)], for the case $\nu = \frac{1}{2}$, we get,*

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2}\mathbf{ber}(|A|^{2r} + |B|^{2r}). \tag{2}$$

Clearly, Corollary 2.7 is sharper than that in (2).

Theorem 2.9. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ where A, B are positive. Then for all $r \geq 1$ and for all $\alpha \in [0, 1]$,*

$$\mathbf{ber}^{2r}(A^\alpha X B^{1-\alpha}) \leq \|X\|^{2r}\sqrt{\mathbf{ber}^2(\alpha A^{2r} + (1-\alpha)B^{2r}) - r_0^2 c^2(A^{2r} - B^{2r})},$$

where $r_0 = min\{\alpha, 1-\alpha\}$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
& |\langle (A^\alpha X B^{1-\alpha})\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \\
= \; & |\langle X B^{1-\alpha}\hat{k}_\lambda, A^\alpha \hat{k}_\lambda\rangle|^{2r} \\
\leq \; & \|X\|^{2r}\|B^{1-\alpha}\hat{k}_\lambda\|^{2r}\|A^\alpha\hat{k}_\lambda\|^{2r} \quad \big(\text{by Cauchy-Schwarz inequality}\big) \\
= \; & \|X\|^{2r}\langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \langle A^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \\
\leq \; & \|X\|^{2r}\langle A^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle^\alpha \langle B^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-\alpha)} \quad \big(\text{by Lemma 2.1}\big) \\
\leq \; & \|X\|^{2r}\sqrt{\langle (\alpha A^{2r} + (1-\alpha)B^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - r_0^2\langle (A^{2r} - B^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2} \quad \big(\text{by Lemma 2.3}\big) \\
\leq \; & \|X\|^{2r}\sqrt{\mathbf{ber}^2(\alpha A^{2r} + (1-\alpha)B^{2r}) - r_0^2 c^2(A^{2r} - B^{2r})}.
\end{aligned}$$

Hence, by taking supremum over all $\lambda \in \Omega$, we get the desired result. □

In particular, considering $X = I$ in Theorem 2.9 we get the following corollary.

Corollary 2.10. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then for all $r \geq 1$ and $\alpha \in [0, 1]$,*

$$\mathbf{ber}^{2r}(A^\alpha B^{1-\alpha}) \leq \sqrt{\mathbf{ber}^2(\alpha A^{2r} + (1-\alpha)B^{2r}) - r_0^2 c^2(A^{2r} - B^{2r})},$$

where $r_0 = min\{\alpha, 1-\alpha\}$. In particular, if $AB = BA$, then

$$\mathbf{ber}\left(\sqrt{AB}\right) \leq \frac{1}{\sqrt{2}}\left(\mathbf{ber}^2(A^2 + B^2) - c^2(A^2 - B^2)\right)^{1/4}.$$

Theorem 2.11. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^2(A) \leq \frac{1}{4}\mathbf{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\mathbf{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha}),$$

for all $\alpha \in [0, 1]$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
&|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2\\
\leq\ & \langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle \ \big(\text{by Lemma 2.2}\big)\\
=\ & \langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, |A^*|^{2(1-\alpha)}\hat{k}_\lambda\rangle\\
\leq\ & \frac{1}{2}\| \,|A|^{2\alpha}\hat{k}_\lambda\| \,\| \,|A^*|^{2(1-\alpha)}\hat{k}_\lambda\|| + \frac{1}{2}|\langle |A|^{2\alpha}\hat{k}_\lambda, |A^*|^{2(1-\alpha)}\hat{k}_\lambda\rangle| \ \big(\text{by Lemma 2.4}\big)\\
=\ & \frac{1}{2}\langle |A|^{4\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |A^*|^{4(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} + \frac{1}{2}|\langle |A^*|^{2(1-\alpha)}|A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle|\\
\leq\ & \frac{1}{4}\langle (|A|^{4\alpha} + |A^*|^{4(1-\alpha)})\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}|\langle |A^*|^{2(1-\alpha)}|A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle|\\
\leq\ & \frac{1}{4}\mathbf{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\mathbf{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha}).
\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we get the desired inequality. □

Based on the above theorem we have the following upper bound for $\mathbf{ber}^2(A)$.

Corollary 2.12. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\boldsymbol{ber}^2(A) &\leq \min_{\alpha\in[0,1]}\left\{\frac{1}{4}\boldsymbol{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\boldsymbol{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha})\right\}\\
&\leq \frac{1}{4}\boldsymbol{ber}(|A|^2 + |A^*|^2) + \frac{1}{2}\boldsymbol{ber}(|A^*||A|).
\end{aligned}$$

Our next inequality reads as follows.

Theorem 2.13. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^4(A) \leq \frac{1}{4}\Big(\boldsymbol{ber}^2(A^2) + \frac{1}{4}\boldsymbol{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\boldsymbol{ber}(A^*A^2A^*) + \boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2 + |A^*|^2)\Big).$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
&\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
=\ & \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, AA^*\hat{k}_\lambda\rangle\\
\leq\ & \frac{1}{2}\|A^*A\hat{k}_\lambda\|\|AA^*\hat{k}_\lambda\| + \frac{1}{2}|\langle AA^*\hat{k}_\lambda, A^*A\hat{k}_\lambda\rangle| \ \big(\text{by Lemma 2.4}\big)\\
\leq\ & \frac{1}{4}(\|A^*A\hat{k}_\lambda\|^2 + \|AA^*\hat{k}_\lambda\|^2) + \frac{1}{2}|\langle A^*A^2A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle|\\
=\ & \frac{1}{4}\langle (|A|^4 + |A^*|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}|\langle A^*A^2A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle|\\
\leq\ & \frac{1}{4}\mathbf{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\mathbf{ber}(A^*A^2A^*).
\end{aligned}$$

Now,

$$\begin{aligned}
& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^4 \\
= \;& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 \\
\leq \;& \frac{1}{4}\Big(|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle| + \|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\Big)^2 \quad \Big(\text{by Lemma 2.4}\Big) \\
= \;& \frac{1}{4}\Big(|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 + \|A\hat{k}_\lambda\|^2\|A^*\hat{k}_\lambda\|^2 + 2|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\Big) \\
\leq \;& \frac{1}{4}\Big(|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle)\Big) \\
= \;& \frac{1}{4}\Big(|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle(|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\leq \;& \frac{1}{4}\Big(\mathbf{ber}^2(A^2) + \frac{1}{4}\mathbf{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\mathbf{ber}(A^*A^2A^*) + \mathbf{ber}(A^2)\mathbf{ber}(|A|^2 + |A^*|^2)\Big).
\end{aligned}$$

Therefore, by taking supremum over all $\lambda \in \Omega$, we get the desired result. □

Remark 2.14. *Following [17, Corollary 3.5 (i)], for the case $r = 4$, we get*

$$\boldsymbol{ber}^4(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^4 + |A^*|^4). \tag{3}$$

If $A^2 = 0$, then from Theorem 2.13 it follows that

$$\boldsymbol{ber}^4(A) \leq \frac{1}{16}\boldsymbol{ber}(|A|^4 + |A^*|^4). \tag{4}$$

Therefore, we note that for the case $A^2 = 0$, Theorem 2.13 gives stronger bound than that in (3).

Theorem 2.15. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
4\,\boldsymbol{ber}^4(A) \;\leq\;& \boldsymbol{ber}(t|A|^2 + (1-t)|A^*|^2)\boldsymbol{ber}((1-t)|A|^2 + t|A^*|^2) + \boldsymbol{ber}^2(A^2) \\
& + \boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2 + |A^*|^2),
\end{aligned}$$

for all $t \in [0, 1]$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^4 \\
= \;& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 \\
\leq \;& \frac{1}{4}\Big(\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\| + |\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\Big)^2 \quad \Big(\text{by Lemma 2.4}\Big) \\
= \;& \frac{1}{4}\Big(\|A\hat{k}_\lambda\|^2\|A^*\hat{k}_\lambda\|^2 + |\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 + 2|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\Big) \\
\leq \;& \frac{1}{4}\Big(\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle(A^*A + AA^*)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
= \;& \frac{1}{4}\Big(\langle|A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^t\langle|A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-t}\langle|A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-t}\langle|A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^t \\
& \qquad + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle(|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\leq \;& \frac{1}{4}\Big(\langle(t|A|^2 + (1-t)|A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle((1-t)|A|^2 + t|A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
& \qquad + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle(|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\leq \;& \frac{1}{4}\Big(\mathbf{ber}(t|A|^2 + (1-t)|A^*|^2)\mathbf{ber}((1-t)|A|^2 + t|A^*|^2) \\
& \qquad + \mathbf{ber}^2(A^2) + \mathbf{ber}(A^2)\mathbf{ber}(|A|^2 + |A^*|^2)\Big).
\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we have the desired inequality of the theorem. □

Based on the above theorem we have the following corollary.

Corollary 2.16. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \boldsymbol{ber}^4(A) &\leq \frac{1}{4}\Big(\min_{t\in[0,1]}\Big\{\boldsymbol{ber}(t|A|^2+(1-t)|A^*|^2)\boldsymbol{ber}((1-t)|A|^2+t|A^*|^2)\Big\} \\ &\qquad +\boldsymbol{ber}^2(A^2)+\boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2+|A^*|^2)\Big) \\ &\leq \frac{1}{4}\left(\frac{1}{4}\boldsymbol{ber}^2(|A|^2+|A^*|^2)+\boldsymbol{ber}^2(A^2)+\boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2+|A^*|^2)\right). \end{aligned}$$

Remark 2.17. *Following [17, Corollary 3.5 (i)], for the case $r = 2$, we get*

$$\boldsymbol{ber}^2(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^2+|A^*|^2). \tag{5}$$

If $A^2 = 0$, then it follows from Corollary 2.16 that

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\boldsymbol{ber}(|A|^2+|A^*|^2). \tag{6}$$

Therefore, we remark that for the case $A^2 = 0$, Corollary 2.16 gives stronger bound than that in (5).

3. On lower bounds for the Berezin number

We begin this section with the elementary identity that $\max\{\beta,\gamma\} = \frac{1}{2}(\beta+\gamma)+\frac{1}{2}|\beta-\gamma|$ for all real numbers β and γ. Based on the above identity we obtain our first lower bound for the Berezin number of bounded linear operators on reproducing kernel Hilbert space $\mathcal{H}$.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}(A) \geq \frac{1}{2}\boldsymbol{ber}\Big(\Re(A) \pm \Im(A)\Big) + \frac{1}{2}\Big|\boldsymbol{ber}(\Re(A)) - \boldsymbol{ber}(\Im(A))\Big|.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then using the Cartesian decomposition of A we get,

$$|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 = |\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2.$$

From this we infer that

$$\mathbf{ber}(A) \geq \mathbf{ber}(\Re(A)) \tag{7}$$

and

$$\mathbf{ber}(A) \geq \mathbf{ber}(\Im(A)). \tag{8}$$

Therefore, combining (7) together with (8) we have,

$$\begin{aligned} \mathbf{ber}(A) &\geq \max\{\mathbf{ber}(\Re(A)), \mathbf{ber}(\Im(A))\} \\ &= \frac{\mathbf{ber}(\Re(A)) + \mathbf{ber}(\Im(A))}{2} + \frac{|\mathbf{ber}(\Re(A)) - \mathbf{ber}(\Im(A))|}{2} \\ &\geq \frac{\mathbf{ber}(\Re(A) \pm \Im(A))}{2} + \frac{|\mathbf{ber}(\Re(A)) - \mathbf{ber}(\Im(A))|}{2}, \end{aligned}$$

as required. □

Next inequality reads as:

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\boldsymbol{ber}^2(A) &\geq \frac{1}{4}\boldsymbol{ber}^2\Big(\Re(A) \pm \Im(A)\Big) + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}\\ &\quad + \left|\frac{\boldsymbol{ber}^2(\Re(A)) - \boldsymbol{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|.\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then using the Cartesian decomposition of A we get,

$$|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 = |\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2.$$

Therefore, we infer that

$$\mathbf{ber}^2(A) \geq \mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) \tag{9}$$

and

$$\mathbf{ber}^2(A) \geq \mathbf{ber}^2(\Im(A)) + c^2(\Re(A)). \tag{10}$$

Now we have by combining (9) together with (10) that

$$\begin{aligned}\mathbf{ber}^2(A) &\geq \max\{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)), \mathbf{ber}^2(\Im(A)) + c^2(\Re(A))\}\\ &= \frac{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) + \mathbf{ber}^2(\Im(A)) + c^2(\Re(A))}{2}\\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) - \mathbf{ber}^2(\Im(A)) - c^2(\Re(A))}{2}\right|\\ &= \frac{\mathbf{ber}^2(\Re(A)) + \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) + c^2(\Re(A))}{2}\\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) - \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|\\ &\geq \frac{1}{4}\Big(\mathbf{ber}(\Re(A)) + \mathbf{ber}(\Im(A))\Big)^2 + \frac{c^2(\Im(A)) + c^2(\Re(A))}{2}\\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) - \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|\\ &\geq \frac{1}{4}\mathbf{ber}^2\Big(\Re(A) \pm \Im(A)\Big) + \frac{c^2(\Im(A)) + c^2(\Re(A))}{2}\\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) - \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|.\end{aligned}$$

This completes the proof. □

Finally, we prove the following lower bound.

Theorem 3.3. *If $A \in \mathcal{B}(\mathcal{H})$, then $\boldsymbol{ber}^2(A) \geq \max\{\beta, \gamma\}$, where*

$$\beta = \frac{\boldsymbol{ber}^2\Big(\Re(A) + \Im(A)\Big)}{2} + \frac{c^2\Big(\Re(A) - \Im(A)\Big)}{2}$$

and

$$\gamma = \frac{\boldsymbol{ber}^2\Big(\Re(A) - \Im(A)\Big)}{2} + \frac{c^2\Big(\Re(A) + \Im(A)\Big)}{2}.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then from the Cartesian decomposition of A we get,

$$\begin{aligned}|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 &= \langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 \\ &= \frac{\langle (\Re(A)+\Im(A))\hat{k}_\lambda, \hat{k}_\lambda\rangle^2}{2} + \frac{\langle (\Re(A)-\Im(A))\hat{k}_\lambda, \hat{k}_\lambda\rangle^2}{2}.\end{aligned}$$

Therefore, we have the following two inequalities:

$$\mathbf{ber}^2(A) \geq \frac{\mathbf{ber}^2(\Re(A)+\Im(A))}{2} + \frac{c^2(\Re(A)-\Im(A))}{2} \tag{11}$$

and

$$\mathbf{ber}^2(A) \geq \frac{\mathbf{ber}^2(\Re(A)-\Im(A))}{2} + \frac{c^2(\Re(A)+\Im(A))}{2}. \tag{12}$$

By combining (11) together with (12) we get the desired inequality. □

As an easy consequence of the above theorem we infer the following inequality.

Corollary 3.4. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$\boldsymbol{ber}(A) \geq \frac{1}{\sqrt{2}}\boldsymbol{ber}\Big(\Re(A) \pm \Im(A)\Big).$$

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