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A new combinatorial identity for Bernoulli numbers and its application in Ramanujan's expansion of harmonic numbers

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Abstract. We establish a new combinatorial identity related to the well-known Bernoulli numbers, which generalizes the result due to Feng and Wang. By means of the identity, we find a recursive formula for successively determining the coefficients of Ramanujan's asymptotic expansion for the generalized harmonic numbers.

1. Introduction

Harmonic numbers and generalized harmonic numbers have been widely studied in many fields of mathematics, such as in mathematical analysis, number theory, special functions, combinatorics and so on. Recently, more attention on harmonic numbers and generalized harmonic numbers has been paid to their generating functions and related properties [9, 10], summation formulas [6, 18], the identities involving other special numbers [17]. Specifically, Dattoli and Srivastava [10] proposed several generating functions involving harmonic numbers by making use of an interesting approach based on the umbral calculus. Subsequently, Cvijović[8] showed the truth of the conjectured relations in [10] by using some simple analytical arguments. By the similar method, Giuseppe Dattoli et al. [9] introduced higher-order harmonic numbers and derived their relevant properties and generating functions. Their work shows that the combinations of umbral and other techniques yield a very efficient tool to explore the properties of these numbers. Different from [9, 10], J. Choi and H.M. srivastava[6] proposed to present further identities for series associated with harmonic numbers and generalized harmonic numbers by making use of the unique series expansion of classical hypergeometric summary formulas. In [17], A. Sofo and H.M. Srivastava extended some results of Euler related sums. Integral and closed-form representations of sums with products of harmonic numbers and binomial coefficients were developed in terms of Polygamma functions. In their paper[18], A. Sofo and H.M. Srivastava further developed a set of identities for Euler-type sums and investigate products of the shifted harmonic numbers and the reciprocal binomial coefficients.

The asymptotic expansion of harmonic numbers is another subject of great concern [4, 5, 11–15, 20, 21, 23]. In his lost notebook, Ramanujan [2, 19] proposed the following asymptotic expansion for the nth harmonic

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number:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots \tag{1}$$

as $n \to \infty$, where $m = \frac{1}{2}n(n+1)$ is the nth triangular number and γ is the Euler-Mascheroni constant. Berndt [2] described Eq.(1) as "somewhat enigmatic" and mentioned that "we cannot find a 'natural' method to produce such asymptotic series". He converted it into powers of $1/n$ and found it agrees with Euler's asymptotic expansion

$$H_n \sim \ln n + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{kn^k}, \quad n \to \infty,$$

where B_n are the well-known Bernoulli numbers.

In 2008, Villarino [21] gave the general formula and error estimate for the Ramanujan asymptotic expansion, and proved that for every integer $r \geq 1$, there exists a Θ_r with $0 < \Theta_r < 1$, for which the following representation is true:

$$H_n = \frac{1}{2}\ln(2m) + \gamma + \sum_{i=1}^{r} \frac{R_i}{m^i} + \Theta_r \cdot \frac{R_{r+1}}{m^{r+1}}. \tag{2}$$

where

$$R_i = \frac{(-1)^{i-1}}{2i \cdot 8^i} \left\{ 1 + \sum_{j=1}^{i} \binom{i}{j} (-4)^j B_{2j}\left(\frac{1}{2}\right) \right\} \tag{3}$$

and $B_i(x)$ are the Bernoulli polynomials defined by the following generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{i=0}^{\infty} B_i(x) \frac{t^i}{i!}.$$

Recently, Chen and Cheng [5] provided a recurrence relation for successively determining the coefficients R_i:

$$R_1 = \frac{1}{12}, \quad R_i = \frac{1}{2^i} \left\{ \frac{1}{4i} - \frac{B_{2i}}{2i} - \sum_{j=1}^{i-1} 2^j R_j P_{2(i-j)}(j) \right\}, \ i \geq 2, \tag{4}$$

where

$$P_j(k) = (-1)^j \binom{k+j-1}{j}.$$

The coefficient sequence $(R_i)_{i\geq1}$ in Eq.(2) or Eq.(3) satisfies the recurrence and initial condition Eq.(4), which is an open problem proposed by Chen and Cheng [5]. For more works related to the Ramanujan harmonic number expansion, one is referred to [3, 4, 14, 15, 20, 23].

It is remarkable that in order to give an affirmative answer to the open problem, Feng and Wang [11] established two key binomial coefficients identities by using the Riordan array method, and then they proved the following interesting identity involving Bernoulli numbers holds true:

$$\sum_{p=1}^{l} \sum_{q=0}^{p} \binom{2l-p-1}{p-1} \binom{p}{q} \frac{(-1)^{p+q-1} B_{2q}\left(\frac{1}{2}\right)}{2p \cdot 4^{p-q}} = \frac{1}{4l} - \frac{B_{2l}}{2l}. \tag{5}$$

Equivalently, we rewrite Eq.(5) as

$$\sum_{p=1}^{l}\sum_{q=0}^{p}\binom{2l-p-1}{p-1}\binom{p}{q}\frac{(-1)^{p+q}(4^q-2)B_{2q}}{2p\cdot 4^p}=\frac{1}{4l}-\frac{B_{2l}}{2l} \tag{6}$$

because of the following relation (see for example [1])

$$B_i\left(\frac{1}{2}\right)=-(1-2^{1-i})B_i. \tag{7}$$

For more identities on the Bernoulli numbers, one is referred to [16].

Let $(x)_k=\Gamma(x+1)/\Gamma(k+1)$. In fact, for $l\geq 1$, if we write

$$A(l,p)=\begin{cases}\frac{1}{p}\binom{2l-p-1}{2l-2p}, & p\geq 1,\\ \lim_{p\to 0}\frac{(2l-p-1)_{2l-2p-1}}{(2l-2p)!}=\frac{1}{2l}, & p=0,\end{cases}$$

then Eq.(6) can be written in a more compact form

$$\sum_{p=0}^{l}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^{p+q}(4^q-2)A(l,p)B_{2q}}{2^{2p+1}}=-\frac{B_{2l}}{2l}. \tag{8}$$

It looks very nice that the combinatorial identity is actually related to the well-known Bernoulli numbers with even indexes.

In this paper, we extend the definition of $A(l,p)$ to

$$A(l,p;r)=\begin{cases}\frac{1}{p+r}\binom{r+2l-p-1}{2l-2p}, & p\geq 1 \text{ or } r\geq 1,\\ \lim_{p\to 0}\lim_{r\to 0}\frac{(r+2l-p-1)_{2l-2p-1}}{(2l-2p)!}=\frac{1}{2l}, & r=p=0,\end{cases}$$

for any integer $r\geq 0$. Obviously, $A(l,p;0)=A(l,p)$. Hence we obtain an extended version of Identity Eq.(8), namely,

Theorem 1.1. *Let l and r be two integers such that $l\geq 1$ and $r\geq 0$. Then there holds*

$$\sum_{p=0}^{l}\sum_{q=0}^{p}\binom{p+r}{q+r}\binom{2r+2q}{2q}\frac{(-1)^{p+q}(4^q-2)A(l,p;r)B_{2q}}{2^{2p+1}}=-\frac{B_{2l}}{(2l)!}(2r-1+2l)_{2l-1}. \tag{9}$$

By means of the above combinatorial identity involving Bernoulli numbers, we find a recursive formula for successively determining the coefficients of Ramanujan's asymptotic expansion for the generalized harmonic numbers (see Theorem 3.1).

2. The Proof of Theorem 1.1

Let $[t^n]$ be the operator which gives the nth coefficient in the series development of a generating function. As usual, the coefficient of t^n in $f(t)$ may be denoted by $[t^n]f(t)$. In this section, we give a proof for the main theorem by using the method of generating function [7, 22]. First of all, we present two identities as lemmas.

Lemma 2.1. *Let s and n be non-negative integers. The following identity holds true:*

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{2n+s-k}{2n-2k}\binom{s+k}{k}=\frac{1}{4^n}\binom{2n+2s+1}{2n}. \tag{10}$$

Proof. It is clear that the generating function of the sequence $\left\{\left(-\frac{1}{4}\right)^k \binom{s+k}{k}\right\}_{k\geq 0}$ is

$$h(t) = \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \binom{s+k}{k} t^k = \frac{1}{(1+\frac{1}{4}t)^{s+1}}. \tag{11}$$

It implies that

$$[t^k]\left\{\frac{1}{(1+\frac{1}{4}t)^{s+1}}\right\} = \left(-\frac{1}{4}\right)^k \binom{s+k}{k}. \tag{12}$$

Similarly, we have

$$\binom{2n+s-k}{2n-2k} = [u^{2n-2k}]\left\{\frac{1}{(1-u)^{s+1+k}}\right\} = [u^{2n}]\left\{\frac{1}{(1-u)^{s+1}}\left(\frac{u^2}{1-u}\right)^k\right\}. \tag{13}$$

Then, by Eqs.(11),(12) and Eq.(13), we have

$$\begin{aligned}
&\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k \binom{2n+s-k}{2n-2k}\binom{s+k}{k} \\
&= \sum_{k=0}^{n} [t^k]\left\{\frac{1}{(1+\frac{1}{4}t)^{s+1}}\right\} \cdot [u^{2n}]\left\{\frac{1}{(1-u)^{s+1}}\left(\frac{u^2}{1-u}\right)^k\right\} \\
&= [u^{2n}]\left\{\frac{1}{(1-u)^{s+1}} h\left(\frac{u^2}{1-u}\right)\right\} \\
&= [u^{2n}]\left\{\frac{1}{(1-\frac{u}{2})^{2s+2}}\right\} = \frac{1}{4^n}\binom{2n+2s+1}{2n},
\end{aligned}$$

which gives the desired result. □

It is well known that the Bernoulli polynomials satisfy the following identity

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}.$$

Taking $x = 1/2$ and $y = -1/2$, and using $B_n(0) = B_n$ and $B_{2n+1}\left(\frac{1}{2}\right) = 0$ for $n \geq 0$, we get a fact as follows.

Lemma 2.2 ([1, 11]). *For $n \geq 0$, we have*

$$B_{2n} = \sum_{k=0}^{n} \binom{2n}{2k} \frac{1}{4^{n-k}} B_{2k}\left(\frac{1}{2}\right). \tag{14}$$

Now, we are in a position to prove the main theorem. It is obvious that Eq.(9) reduces to Eq.(8) when $r = 0$. So let us consider the case for $r \geq 1$. It suffices to prove that

$$\begin{aligned}
&\sum_{p=0}^{l} \left(-\frac{1}{4}\right)^p \binom{r+2l-1-p}{2l-2p} \sum_{q=0}^{p} (-1)^q \binom{p+r-1}{p-q}\binom{2r+2q}{2q} \frac{4^q-2}{2(r+q)} B_{2q} \\
&= -\frac{B_{2l}}{(2l)!}(2r-1+2l)_{2l-1}.
\end{aligned} \tag{15}$$

Changing the order of summation in the left hand side of Eq.(15) we arrive at

$$\sum_{q=0}^{l}(-1)^q\binom{2r+2q}{2q}\frac{4^q-2}{2(r+q)}B_{2q}\sum_{p=q}^{l}\left(-\frac{1}{4}\right)^p\binom{r+2l-1-p}{2l-2p}\binom{p+r-1}{p-q}$$
$$=-\frac{B_{2l}}{(2l)!}(2r-1+2l)_{2l-1}. \tag{16}$$

Replacing p by $k+q$ gives

$$\sum_{p=q}^{l}\left(-\frac{1}{4}\right)^p\binom{r+2l-1-p}{2l-2p}\binom{p+r-1}{p-q}$$
$$=\left(-\frac{1}{4}\right)^q\sum_{k=0}^{l-q}\left(-\frac{1}{4}\right)^k\binom{r+2l-q-1-k}{2l-2q-2k}\binom{q+r-1+k}{k}.$$

Let $n=l-q$ and $s=q+r-1$. By Lemma 2.1 we have

$$\sum_{p=q}^{l}\left(-\frac{1}{4}\right)^p\binom{r+2l-1-p}{2l-2p}\binom{p+r-1}{p-q}$$
$$=\frac{(-1)^q}{4^l}\binom{2r+2l-1}{2l-2q}.$$

Thus, combining with Eq.(16) we need to prove

$$\frac{1}{4^l}\sum_{q=0}^{l}\binom{2r+2q}{2q}\binom{2r+2l-1}{2l-2q}\frac{4^q-2}{2(r+q)}B_{2q}=-\frac{B_{2l}}{(2l)!}(2r-1+2l)_{2l-1}. \tag{17}$$

From Eq.(7) we have

$$B_{2q}=-\frac{1}{1-2^{1-2q}}B_{2q}\left(\frac{1}{2}\right). \tag{18}$$

Substituting Eq.(18) into the left hand side of Eq.(17) gives

$$\frac{1}{4^l}\sum_{q=0}^{l}\binom{2r+2q}{2q}\binom{2r+2l-1}{2l-2q}\frac{4^q-2}{2(r+q)}B_{2q}$$
$$=-\sum_{q=0}^{l}\binom{2r+2q}{2q}\binom{2r+2l-1}{2l-2q}\frac{1}{2(r+q)}\frac{1}{4^{l-q}}B_{2q}\left(\frac{1}{2}\right)$$
$$=-\frac{(2r+2l-1)_{2l-1}}{(2l)!}\sum_{q=0}^{l}\binom{2l}{2q}\frac{1}{4^{l-q}}B_{2q}\left(\frac{1}{2}\right),$$

since

$$\frac{1}{2(r+q)}\binom{2r+2q}{2q}\binom{2r+2l-1}{2l-2q}=\frac{(2r+2l-1)_{2l-1}}{(2l)!}\binom{2l}{2q}.$$

By Lemma 2.2 we immediately obtain Eq.(17), which implies Eq.(9) is true.

3. Application in Ramanujan's Expansion of Harmonic Numbers

Based on a natural derivation for the Ramanujan asymptotic expansion, Hirschhorn [12] obtained the following results for the odd powers, that is $2i+1$ for $1 \le i \le 11$ of the tail of the Riemann zeta function in terms of $1/m$.

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{4m} - \frac{1}{16m^2} + \frac{1}{48m^3} - \frac{1}{96m^4} + \frac{1}{120m^5} - \frac{1}{96m^6} + \cdots, \\
&\sum_{k=n+1}^{\infty} \frac{1}{k^5} \sim \frac{1}{16m^2} - \frac{1}{24m^3} + \frac{11}{384m^3} - \frac{5}{192m^5} + \frac{13}{384m^6} - \frac{1}{16m^7} + \cdots, \\
&\qquad \vdots \\
&\sum_{k=n+1}^{\infty} \frac{1}{k^{23}} \sim \frac{1}{45056m^{11}} - \cdots,
\end{aligned}$$

as $n \to \infty$.

Recently, Issaka [13] provided a rigorous proof of those expansions for the generalized harmonic numbers with odd powers mentioned in Hirschhorn [12]. He obtained

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{2i+1}} = -\frac{1}{(2m)^i} \left\{ \sum_{p=0}^{r} \frac{A_p^i}{m^p} + \kappa_r(n,i) \frac{A_{r+1}^i}{m^{r+1}} \right\}, \tag{19}$$

where $0 < \kappa_r(n,i) < 1$ and

$$A_p^i = \frac{(-1)^p}{8^p} \sum_{q=0}^{p} (-1)^q \binom{p+i-1}{p-q} \binom{2i+2q}{2q} \frac{(2^{2q}-2)B_{2q}}{2i+2q}. \tag{20}$$

Let $H_n^{(2i+1)} = \sum_{k=1}^{n} \frac{1}{k^{2i+1}}$ and $C_p^i = \frac{1}{2^i} A_p^i$ for $i \ge 1$. We rewrite (3.1) as

$$H_n^{(2i+1)} = \zeta(2i+1) + \sum_{p=0}^{r} \frac{C_p^i}{m^{i+p}} + E_{r+1}^i, \tag{21}$$

where $\zeta(z)$ is the Riemman zeta function and the error term $E_{r+1}^i = \frac{A_{r+1}^i}{2^i m^{i+r+1}} \kappa_r(n,i)$.

In this section, by using the combinatorial identity derived in Theorem 1, we find a recursive formula for determining the coefficients C_p^i for $p \ge 0$, which is similar to Eq.(4). Specifically, we have

Theorem 3.1. *For $i \ge 1$, the coefficients C_p^i in Eq.(21) can be recursively determined by*

$$\begin{aligned}
C_0^i &= -\frac{1}{i \cdot 2^{i+1}}, \\
C_p^i &= \frac{1}{2^{i+p}} \left\{ -\frac{B_{2p}}{(2p)!} (2i-1+2p)_{2p-1} - \sum_{k=0}^{p-1} 2^{i+k} \binom{i+2p-1-k}{2p-2k} C_k^i \right\}, \ p \ge 1.
\end{aligned} \tag{22}$$

Proof. Taking $r = i \ge 1$ in Eq.(9) yields

$$\begin{aligned}
&\sum_{p=0}^{l} \sum_{q=0}^{p} \binom{p+i}{q+i} \binom{2i+2q}{2q} \binom{i+2l-p-1}{2l-2p} \frac{(-1)^{p+q}(4^q-2)B_{2q}}{(p+i)2^{2p+1}} \\
&= -\frac{B_{2l}}{(2l)!} (2i-1+2l)_{2l-1}.
\end{aligned}$$

It is clear that

$$\binom{p+i}{q+i} = \frac{p+i}{q+i}\binom{p+i-1}{p-q},$$

which leads to

$$\sum_{p=0}^{l}\sum_{q=0}^{p}\binom{p+i-1}{p-q}\binom{2i+2q}{2q}\binom{i+2l-p-1}{2l-2p}\frac{(-1)^{p+q}(4^q-2)B_{2q}}{2(q+i)4^p}$$
$$= -\frac{B_{2l}}{(2l)!}(2i-1+2l)_{2l-1}.$$

Replacing l by p and p by k, we obtain

$$\sum_{k=0}^{p} 2^{i+k}\binom{i+2p-1-k}{2p-2k}C_k^i = -\frac{B_{2p}}{(2p)!}(2i-1+2p)_{2p-1}, \quad p \geq 1,$$

which is equivalent to Eq.(22). □

By the above theorem, several first few coefficients are calculated as follows.

$$\begin{aligned}
&C_0^1 = -\frac{1}{4}, \quad C_1^1 = \frac{1}{16}, \quad C_2^1 = -\frac{1}{48}, \quad C_3^1 = \frac{1}{96}, \quad C_4^1 = -\frac{1}{120}, \quad C_5^1 = \frac{1}{96},\\
&C_0^2 = -\frac{1}{16}, \quad C_1^2 = \frac{1}{24}, \quad C_2^2 = -\frac{11}{384}, \quad C_3^2 = \frac{5}{192}, \quad C_4^2 = -\frac{13}{384}, \quad C_5^2 = \frac{1}{16},\\
&C_0^3 = -\frac{1}{48}, \quad C_1^3 = \frac{5}{192}, \quad C_2^3 = -\frac{29}{960}, \quad C_3^3 = \frac{11}{256}, \quad C_4^3 = -\frac{313}{384}, \quad C_5^3 = \frac{1589}{7680},\\
&C_0^4 = -\frac{1}{128}, \quad C_1^4 = \frac{1}{64}, \quad C_2^4 = -\frac{7}{256}, \quad C_3^4 = \frac{25}{448}, \quad C_4^4 = -\frac{592}{4096}, \quad C_5^4 = \frac{1481}{3092},\\
&C_0^5 = -\frac{1}{320}, \quad C_1^5 = \frac{7}{768}, \quad C_2^5 = -\frac{43}{1920}, \quad C_3^5 = \frac{317}{5120}, \quad C_4^5 = -\frac{3227}{15360}, \quad C_5^5 = \frac{54499}{61440}.
\end{aligned}$$

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