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An iterative stochastic procedure with a general step to a linear regular inverse problem

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Abstract. In this paper we consider a linear operator equation in a Hilbert space. Using Hoeffding inequalities, an exponential bound to the solution obtained by a stochastic procedure is established and the values of the step a_k for which the procedure converges almost completely (a.co) are discussed.

An illustrative application was treated to the solution of a Fredholm integral equation of the second kind.

Introduction

Stochastic algorithms are part of modern techniques for numerical solution of many practical problems and are the basis of various advanced industrial applications: signal processing [5–7], adaptive control [20], recursive estimation [12], inverse problems [22, 23]. Stochastic recursive procedures are also used in various fields of considerable technological importance such as communication sciences [21, 28, 29], system identification [8–10] and in the search for solutions of matrix equations [3].

Let (Ω, F, P) be a probability space and H a separable Hilbert space.

Consider the following operator equation

$$Ax = u, \tag{1}$$

where A is a linear operator of H.

We will assume in this work that $\inf\{re\ \lambda;\ \lambda \in \Delta(A)\} > 0$ with $\Delta(A)$ is a spectrum of A.

The solution of the equation (1) is obtained using a stochastic Robbins-Monro type procedure [25]

$$X_{k+1} = X_k - a_k \left[AX_k - u_k\right], \tag{2}$$

with a_k is a sequence of positive numbers decreasing towards zero, called the descent step.

In practice, the second member u is the result of measurements and is only known approximately.

Let's put

$$u_k = u_{ex} + \xi_k,$$

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where u_{ex} represents the exact and unknown value of the second member and $(\xi_k)_{k\in\mathbb{N}^*}$ is a sequence of independent and identically distributed (i.i.d.) random variables with values in H.

We will assume that $(\xi_k)_{k\in\mathbb{N}^*}$ are bounded:

$$\|\xi_i\| < b, \; b \in \mathbb{R}. \tag{3}$$

In a recent work [22] and when the step $a_k = \frac{1}{k}$, the Stochastic Algorithm (2) was applied to solve a linear operator equation in a Hilbert space. The Bernstein-type exponential inequality for the found solution was constructed, which allowed us to determine the almost complete convergence (a.co) of the said solution.

The aim of this work is to establish Hoeffding inequalities to find an exponential bound for the iterative solution obtained by the procedure (2) and to discuss the values of the step a_k for which the procedure converges almost completely (a.co).

Our approach is to apply deterministic iterations in that it uses the full forward model when the noise on the right hand side is stochastic. Other approaches using methods of the stochastic gradient descent, have been recently investigated in the context of linear inverse problems by B. Jin et al [18] and T. Jahn et al [17].

The approach of using multiple independent and identically distributed measurements for linear inverse problems was treated recently by B. Harrach et al [14] and T. Jahn [16].

The study of regular inverse problems appears in works dealing with the fields of linear filtration [1, 2, 15] and linear regression [11, 13].

The efficiency of the Robbins-Monro recursive procedure relies essentially on the choice of the sequence $a_k, k \in \mathbb{N}^*$, this choice remains until now a big open question that is at the heart of the efficiency of stochastic recursive algorithms [19]. Even though experience shows that the choice of the sequence $a_k = \frac{1}{k}$; under some regularity conditions; generates better convergence properties for solving linear equations. Nevertheless, the general case $a_k = \frac{a}{k^\theta}, 0 < a \leq 1, \frac{1}{2} < \theta \leq 1, k \in \mathbb{N}^*$is not sufficiently explored theoretically in the literature.

1. Preliminary results

Using the fact that: $u_k = u_{ex} + \xi_k$ then, the procedure (2) will become

$$X_{k+1} = X_k - a_k \left[AX_k - u_{ex} - \xi_k \right]. \tag{4}$$

where $(X_k)_{k\in\mathbb{N}^*}$, $(\xi_k)_{k\in\mathbb{N}^*}$ are elements of H and $(a_k)_{k\in\mathbb{N}^*}$ is a real sequence satisfying

$$\begin{aligned} \sum_{k=1}^{\infty} a_k &= +\infty, \\ \sum_{k=1}^{\infty} a_k^2 &< +\infty. \end{aligned} \tag{5}$$

The natural choice of steps a_k verifying the hypothesis (5) is $a_k = \frac{a}{k^\theta}, 0 < a \leq 1, \frac{1}{2} < \theta \leq 1, k \in \mathbb{N}^*$.

From an abstract point of view, we are interested in the iterative solution given by the recursive procedure (4) to solve the linear operator equation (1) in a Hilbert space H.

According to [22, lemma 1] and after successive iterations, the following relation is obtained

$$X_{k+1} - X_{ex} = \prod_{i=1}^{k} (I - a_i A)(X_1 - X_{ex}) + \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - a_j A) a_i \xi_i. \tag{6}$$

Where, $1 \leq i, j \leq k$, with the convention $\prod\limits_{j=k+1}^{k} (I - a_j A) = I$, with I is the unit operator in H.

X_{ex} represents the exact solution verifying $AX_{ex} = u_{ex}$.

Lemma 1.1. *Let* $a_i = \frac{a}{i^\theta}$, $0 < a \leq 1$, $\frac{1}{2} < \theta \leq 1$. *Suppose that* $\inf\{re\,\lambda;\ \lambda \in \Delta(A)\} > 0$. *The following expressions hold.*

1) If $\theta = 1$, *then*

$$\exists \gamma > 0,\ \exists p > 0,\ \forall\, 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - a_j A) \right\| \leq \gamma \frac{(i+1)^{ap}}{(k+1)^{ap}}. \tag{7}$$

2) If $\frac{1}{2} < \theta < 1$, *then*

$$\exists\, \gamma > 0,\ \exists\, p > 0,\ \forall\, 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - a_j A) \right\| \leq \gamma \exp\left(\frac{ap}{1-\theta} \left((i+1)^{1-\theta} - (k+1)^{1-\theta} \right) \right). \tag{8}$$

Proof. Under the condition $\inf\{re\,\lambda;\ \lambda \in \Delta(A)\} > 0$, H. Walk obtained the following result [27, Lemma 3.b]

$$\exists \gamma > 0,\ \exists p > 0,\ \forall\, 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - a_j A) \right\| \leq \gamma \left(\prod_{j=i+1}^{k} \left(1 - a_j\right) \right)^p. \tag{9}$$

Then, taking into account that: $\ln(1+x) \leq x$, for $x > -1$, we have

$$p \ln\left(1 - \frac{a}{j^\theta}\right) \leq -\frac{ap}{j^\theta}$$

$$\sum_{j=i+1}^{k} \ln\left(1 - \frac{a}{j^\theta}\right)^p \leq \sum_{j=i+1}^{k} -\frac{ap}{j^\theta} = -ap \sum_{j=i+1}^{k} \frac{1}{j^\theta} = -ap \int_{i+1}^{k+1} \frac{1}{x^\theta} dx.$$

If $\theta = 1$, then

$$\sum_{j=i+1}^{k} \ln\left(1 - \frac{a}{j^\theta}\right)^p \leq ap \ln\left(\frac{i+1}{k+1}\right).$$

This implies that

$$\exp\left(\sum_{j=i+1}^{k} \ln\left(1 - \frac{a}{j^\theta}\right)^p \right) \leq \exp\left(\ln\left(\frac{i+1}{k+1}\right)^{ap} \right).$$

So,

$$\gamma \left(\prod_{j=i+1}^{k} \left(1 - \frac{a}{j^\theta}\right) \right)^p \leq \gamma \frac{(i+1)^{ap}}{(k+1)^{ap}}. \tag{10}$$

If $\frac{1}{2} < \theta < 1$, then

$$\sum_{j=i+1}^{k} \ln\left(1 - \frac{a}{j^\theta}\right)^p \leq \frac{ap}{1-\theta} \left((i+1)^{1-\theta} - (k+1)^{1-\theta} \right).$$

This implies that

$$\exp\left(\sum_{j=i+1}^{k}\ln\left(1-\frac{a}{j^{\theta}}\right)^{p}\right)\leq\exp\left(\frac{ap}{1-\theta}\left((i+1)^{1-\theta}-(k+1)^{1-\theta}\right)\right).$$

The latter is equivalent to the following relation

$$\gamma\left(\prod_{j=i+1}^{k}\left(1-\frac{a}{j^{\theta}}\right)\right)^{p}\leq\gamma\exp\left(\frac{ap}{1-\theta}\left((i+1)^{1-\theta}-(k+1)^{1-\theta}\right)\right). \tag{11}$$

From (10) and (11), we deduce that

$$\lim_{k\to\infty}\left\|\prod_{j=1}^{k}(I-a_jA)\right\|\leq\lim_{k\to\infty}\gamma\left(\prod_{j=1}^{k}\left(1-\frac{a}{j^{\theta}}\right)\right)^{p}=0. \tag{12}$$

□

Lemma 1.2. *Let $a_i=\frac{a}{i^{\theta}}$, $0<a\leq 1$, $\frac{1}{2}<\theta\leq 1$. Under assumptions of Lemma 1, the following expressions hold.*
1) If $\theta=1$, then

$$\exists\gamma>0,\ \exists p>0,\ \forall\ 1\leq i\leq k:\sum_{i=1}^{k}\left\|\prod_{j=i+1}^{k}(I-a_jA)a_i\right\|^{2}\leq C\frac{(\gamma a)^{2}}{(k+1)^{2ap}}, \tag{13}$$

with C is a constant.
2) If $\frac{1}{2}<\theta<1$, then

$$\exists\gamma>0,\ \exists p>0,\ \forall\ 1\leq i\leq k:\sum_{i=1}^{k}\left\|\prod_{j=i+1}^{k}(I-a_jA)a_i\right\|^{2}\leq\frac{2\,(\gamma a)^{2}\,D_{\theta}}{(ap)^{\frac{1}{1-\theta}}}\frac{1}{k^{\theta}}, \tag{14}$$

with $D_{\theta}=4+\frac{2}{2\theta-1}\left(\frac{\theta}{e(2-2^{\theta})}\right)^{\frac{\theta}{1-\theta}}$.

Proof. 1) Let $\theta=1$, by virtue of the relation (7) one has

$$\exists\gamma>0,\ \exists p>0,\ \forall\ 1\leq i\leq k:\left\|\prod_{j=i+1}^{k}(I-a_jA)a_i\right\|^{2}\leq(\gamma a)^{2}\,\frac{(i+1)^{2ap}}{(k+1)^{2ap}i^{2}}.$$

Then

$$\exists\gamma>0,\ \exists p>0,\ \forall\ 1\leq i\leq k:\sum_{i=1}^{k}\left\|\prod_{j=i+1}^{k}(I-a_jA)a_i\right\|^{2}\leq\frac{(\gamma a)^{2}}{(k+1)^{2ap}}\sum_{i=1}^{k}\frac{(i+1)^{2ap}}{i^{2}}. \tag{15}$$

By Kronecker's lemma, $\frac{(\gamma a)^{2}}{(k+1)^{2ap}}\sum_{i=1}^{k}\frac{(i+1)^{2ap}}{i^{2}}$ tends to 0 when k tends to infinity.
In fact, $\left(\frac{1}{i^{2}}\right)_{i\in\mathbb{N}^{*}}$ is a convergent sequence and $\lim\limits_{i\to+\infty}(i+1)^{2ap}=+\infty$.
So,

$$\lim_{i\to+\infty}\frac{1}{(k+1)^{2ap}}\sum_{i=1}^{k}\frac{(i+1)^{2ap}}{i^{2}}=0. \tag{16}$$

From the relation (16) one deduces that: $\exists(N \in \mathbb{N}^*)$ such that

$$\lim_{k\to\infty} \frac{1}{(k+1)^{2ap}} \sum_{i=N+1}^{k} \frac{(i+1)^{2ap}}{i^2} = 0. \tag{17}$$

We have the following relationship

$$\frac{1}{(k+1)^{2ap}} \sum_{i=1}^{k} \frac{(i+1)^{2ap}}{i^2} = \frac{1}{(k+1)^{2ap}} \sum_{i=1}^{N} \frac{(i+1)^{2ap}}{i^2} + \frac{1}{(k+1)^{2c}} \sum_{i=N+1}^{k} \frac{(i+1)^{2ap}}{i^2}$$

$$\leq \frac{1}{(k+1)^{2ap}} \sum_{i=1}^{N} \frac{(i+1)^{2ap}}{i^2} = \frac{C}{(k+1)^{2ap}},$$

with, $\sum_{i=1}^{N} \frac{(i+1)^{2ap}}{i^2} = C$.
Replacing in (15) we find (13).
2) Let $\frac{1}{2} < \theta < 1$, by virtue of the relation (8) we have

$$\sum_{i=1}^{k} \left\| \prod_{j=i+1}^{k} (I - a_j A) a_i \right\|^2 \leq (\gamma a)^2 \sum_{i=1}^{k} \frac{1}{i^{2\theta}} \exp\left(\frac{2ap}{1-\theta} \left((i+1)^{1-\theta} - (k+1)^{1-\theta} \right) \right). \tag{18}$$

The second member of the inequality (18) is estimated in [26, Lemma A1] using estimates based on the Gamma function.
Therefore,

$$(\gamma a)^2 \sum_{i=1}^{k} \frac{1}{i^{2\theta}} \exp\left(\frac{2ap}{1-\theta} \left((i+1)^{1-\theta} - (k+1)^{1-\theta} \right) \right) \leq \frac{2(\gamma a)^2 D_\theta}{(ap)^{\frac{\theta}{1-\theta}}} \frac{1}{k^\theta}, \tag{19}$$

where $D_\theta = 4 + \frac{2}{2\theta - 1} \left(\frac{\theta}{e(2-2^\theta)} \right)^{\frac{\theta}{1-\theta}}$.
Then, the relation (14) is obtained from (18) and (19). □

2. Exponential inequalities and convergence results

In this section, exponential inequalities of the Hoeffding type are established. These allow us to establish the values of the steps a_k for which the iterative procedure (4) converges almost completely to the exact solution.

Definition 2.1. *The sequence of random variables $(X_k)_{k\in\mathbb{N}^*}$ converges almost completely (a.co) to a random variable X, when k tends to infinity, if and only if: $\forall \varepsilon > 0, \sum_{k=1}^{+\infty} \mathbb{P}(\|X_{k+1} - X_{ex}\| > \varepsilon) < +\infty$.*

Theorem 2.2. *Let $a_i = \frac{a}{i^\theta}$, $0 < a \leq 1$, $\frac{1}{2} < \theta \leq 1$, $i \in \mathbb{N}^*$, $A \in L(H)$: the set of linear applications in H. Under the condition $\inf\{re\ \lambda;\ \lambda \in \Delta(A)\} > 0$, the following exponential inequality holds.*

$$\mathbb{P}(\|X_{k+1} - X_{ex}\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{8b^2 \sum_{i=1}^{k} \left\| \prod_{j=i+1}^{k} (I - a_j A) \right\|^2 a_i^2} \right). \tag{20}$$

Proof. By virtue of the relation (6), one has the following expression

$$\mathbb{P}\left(\|X_{k+1}-X_{ex}\|>\varepsilon\right)=P\left(\left\|\prod_{i=1}^{k}(I-a_iA)(X_1-X_{ex})+\sum_{i=1}^{k}\prod_{j=i+1}^{k}(I-a_jA)a_i\xi_i\right\|>\varepsilon\right)$$

$$\leq \mathbb{P}\left(\left\|\sum_{i=1}^{k}\prod_{j=i+1}^{k}(I-a_jA)a_i\xi_i\right\|>\varepsilon-\left\|\prod_{i=1}^{k}(I-a_iA)(X_1-X_{ex})\right\|\right).$$

The relation (12) proves that:

$$\exists\varepsilon>0,\left\|\prod_{i=1}^{k}(I-a_iA)(X_1-X_{ex})\right\|\leq\frac{\varepsilon}{2}. \tag{21}$$

Then,

$$\mathbb{P}\left(\|X_{k+1}-X_{ex}\|>\varepsilon\right)\leq \mathbb{P}\left(\left\|\sum_{i=1}^{k}\prod_{j=i+1}^{k}(I-a_jA)a_i\xi_i\right\|>\varepsilon-\frac{\varepsilon}{2}\right)$$

$$\leq \mathbb{P}\left(\left\|\sum_{i=1}^{k}\prod_{j=i+1}^{k}(I-a_jA)a_i\xi_i\right\|>\frac{\varepsilon}{2}\right).$$

We pose

$$\eta_i=\prod_{j=i+1}^{k}(I-a_jA)a_i\xi_i. \tag{22}$$

$(\eta_i)_{i\in\mathbb{N}^*}$ is a sequence of bounded and i.i.d random variables in a Hilbert space such that

$$\left\|\eta_i\right\|<\left\|\prod_{j=i+1}^{k}(I-a_jA)\right\|a_ib=d_i. \tag{23}$$

Thus

$$\mathbb{P}\left(\|X_{k+1}-X_{ex}\|>\varepsilon\right)\leq \mathbb{P}\left(\left\|\sum_{i=1}^{k}\eta_i\right\|>\frac{\varepsilon}{2}\right). \tag{24}$$

We give the Pinelis-Hoeffding inequality for the sequence $(\eta_i)_{i\in\mathbb{N}^*}$ such that $\left\|\eta_i\right\|<d_i$ in a Hilbert space H ([24]).

$$\mathbb{P}\left(\left\|\sum_{i=1}^{k}\eta_i\right\|>\varepsilon\right)\leq 2\exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^{k}d_i^2}\right). \tag{25}$$

Then, we deduce from (23) and (25) the following relation.

$$\mathbb{P}\left(\left\|\sum_{i=1}^{k}\eta_i\right\|>\frac{\varepsilon}{2}\right)\leq 2\exp\left(-\frac{\varepsilon^2}{8b^2\sum_{i=1}^{k}\left\|\prod_{j=i+1}^{k}(I-a_jA)\right\|^2a_i^2}\right). \tag{26}$$

Finally, by virtue of the relation (24) we get (20). $\square$

Corollary 2.3. *Let $a_i = \frac{a}{i^\theta}$, $0 < a \leq 1$, $\frac{1}{2} < \theta \leq 1$, $A \in L(H)$.Under the condition* $\inf\{re\ \lambda;\ \lambda \in \Delta(A)\} > 0$, *the following exponential inequalities hold.*

1) If $a_i = \frac{a}{i}$, then

$$\mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) \leq 2\exp\left(-\frac{(k+1)^{2ap}\varepsilon^2}{\alpha}\right). \tag{27}$$

2) If $a_i = \frac{a}{i^\theta}$, $\frac{1}{2} < \theta < 1$, then

$$\mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) \leq 2\exp\left(-\frac{k^\theta\varepsilon^2}{\beta}\right). \tag{28}$$

Proof. 1) By virtue of the relation (13) and from the relation (20), we obtain

$$\mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) \leq 2\exp\left(-\frac{\varepsilon^2}{\frac{8C(\gamma ab)^2}{(k+1)^{2ap}}}\right).$$

By putting $\alpha = 8C(\gamma ab)^2$ we will find (27).
2) By virtue of the relation (14) and from the relation (20), we obtain

$$\mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) \leq 2\exp\left(-\frac{\varepsilon^2}{\frac{16(\gamma ab)^2 D_\theta}{(ap)^{\frac{\theta}{1-\theta}}}\frac{1}{k^\theta}}\right).$$

By putting $\beta = \frac{16(\gamma ab)^2 D_\theta}{(ap)^{\frac{\theta}{1-\theta}}}$ we will find (28). □

In the next corollary, we give the value of a_i so that the iterative procedure (4) converges almost completely to the solution of the equation (1).

Corollary 2.4. *Under the conditions of Theorem 1:*
The recursive procedure (4) *converges almost completely (a.co) to the solution of the equation* (1) *if $a_i = \frac{a}{i}$, $i \in \mathbb{N}^*$:*

$$\forall\varepsilon > 0,\ \sum_{k=1}^{+\infty}\mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) < +\infty. \tag{29}$$

Additionally,

$$\|X_{k+1} - X_{ex}\| = O(k^{-2ap}),\ ap > \frac{1}{2}. \tag{30}$$

Proof. 1) Let us pose

$$v_k = 2\exp\left(-\frac{(k+1)^{2ap}\varepsilon^2}{\alpha}\right) \leq 2\exp\left(-(k+1)^{2ap}\,\varepsilon^2\right),$$

and,

$$u_k = 2\exp\left(-\frac{k^\theta\varepsilon^2}{\beta}\right) \leq 2\exp\left((-k)^\theta\varepsilon^2\right) \tag{31}$$

Applying Cauchy's rule to the positive term series v_k and u_k it follows that:
When $ap > \frac{1}{2}$, $\sum_{k=1}^{+\infty} v_k$ is a convergent series.
This implies that

$$\forall \varepsilon > 0, \sum_{k=1}^{+\infty} \mathbb{P}\left(\|X_{k+1} - X_{ex}\| > \varepsilon\right) \leq \sum_{k=1}^{+\infty} 2\exp\left(-(k+1)^{2ap}\,\varepsilon^2\right) < +\infty. \tag{32}$$

$\sum_{k=1}^{+\infty} u_k$ is a divergent sequence (because: $\frac{1}{2} < \theta < 1$).
Therefore, almost complete convergence is assured when $a_i = \frac{a}{i}$.
2) To obtain (30), it is sufficient to choose $A = \varepsilon k^{2ap}$ in (32) to have

$$\sum_{k=1}^{+\infty} P\left(\|X_{k+1} - X_{ex}\| > A k^{-2ap}\right) < +\infty.$$

□

3. Numerical application

Consider the Fredholm integral equation of the second kind [4]

$$x(s) = y(s) + h\int_a^b K(s,t)\,x(t)\,dt,\ h \in \mathbb{R}, \tag{33}$$

on the interval $[a,b] = [0, \frac{\pi}{2}]$, with the kernel K and the function y given by $K(s,t) = \cos(s-t)$ and $y(s) = \frac{-2}{\pi}\cos(s)$.

By the method of quadratures, the integral $\int_a^b K(s,t)\,x(t)\,dt$ is approximated by a sum.

$$\int_a^b K(s,t)\,x(t)\,dt \approx \sum_{j=1}^{n} w_j K(s_i,t_j)\,x(t_j). \tag{34}$$

Particularly, for the method of midpoints (rectangles), we use $w_j = \frac{b-a}{n}$ at $t_j = \frac{(j-\frac{1}{2})(b-a)}{n}$, $j = 1,...,n$, $s_i = \frac{(i-\frac{1}{2})(b-a)}{n}$, $i = 1,...,n$.

Then, the following relation will be checked

$$x(s_i) = y(s_i) + h\sum_{j=1}^{n} w_j K(s_i,t_j)\,x(t_j). \tag{35}$$

So, we will obtain a system of linear equations

$$x = u + hKx, \tag{36}$$

where, the matrix K is of finite dimension given by the elements

$$k_{ij} = w_j K(s_i,t_j) = \frac{\pi}{2n}\cos\left(\frac{\pi(i-j)}{2n}\right),\ i,j = 1,...,n. \tag{37}$$

And, the vectors x, u are of elements

$$x_i = x(s_i) = x(t_j),\ i,j = 1,...,n, \tag{38}$$

$$u_i = y(s_i) = \frac{-2}{\pi}\cos\left(\frac{\pi(i-\frac{1}{2})}{2n}\right),\ i = 1,...,n \tag{39}$$

The system of matrix equations (36) can be rewritten as follows

$$(I - hK)\,x = u, \tag{40}$$

where I is the unit matrix.

The matrix $(I - hK)$ is regular and its inverse $(I - hK)^{-1}$ exists, so the equation (40) admits a unique solution x.

We apply the iterative procedure

$$X_{k+1} = X_k - a_k\,[(I - hK)\,X_k - u_{ex} - \xi_k\,] \tag{41}$$

for solving the equation (40), with X_1 is an arbitrary vector, ξ_k are random vectors of elements $\xi_k\,(s)$.

We choose for this simulation functional random errors: $\xi_k\,(s) = \theta_k \cos 2\pi s$, $0 \le s \le \frac{\pi}{2}$, $i \ge 1$. With, θ_1, $\theta_2, ...$ is a family of real random variables such that $E\theta_i^2 < +\infty$, $i \ge 1$ and $\|\xi_k\,(s)\| \le \|\theta_k\| < M$, $M \in \mathbb{R}$.

We considered steps $a_k = \frac{a}{k^\theta}$, $0 < a \le 1$, $\frac{1}{2} < \theta \le 1$, $k \in \mathbb{N}^*$, which are the natural choices for verifying the hypothesis (5).

The table below, shows the influence of the choice of the step on the numerical convergence of X_k to the exact solution. In this way, we have chosen three types of steps verifying the hypothesis (5) and we have repatriated the error committed for each choice of step, after n iterations.

	n=100	n=500	n=1000
$a_k = \frac{1}{k}$	0.239	0.109	0.025
$a_k = \frac{1/2}{k}$	0.210	0.101	0.018
$a_k = \frac{1}{k^{0.6}}$	1.912	1.810	1.719

We can clearly see that the procedure (41) is more accurate when $a_k = \frac{1}{k}$ and $a_k = \frac{1/2}{k}$. On the other hand, the error made when $a_k = \frac{1}{k^{0.6}}$ is very significant.

The graph below, gives the look of the solution obtained by the procedure (41) when $a_k = \frac{1/2}{k}$ and $a_k = \frac{1}{k^{0.6}}$ compared to the exact solution: $x(t) = \sin(t)$.

4. Conclusion

In this paper, a stochastic recursive method for solving regular inverse problems, defined by linear operator equations in a Hilbert space has been applied. Through the theoretical results demonstrated, it has been deduced that the step size of stochastic algorithms has a real influence on their convergence. Indeed, the almost complete convergence is obtained when the step takes the value $a_k = \frac{a}{k}$, with $0 < a \le 1$. Numerical simulation results for the solution of the Fredholm equation of the second kind support the theoretical results obtained.

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of Interests None.

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