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On the set of all $\mathcal{I}$-convergent sequences over different spaces

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Abstract. In this article we elaborately study certain characteristics of the set of all $\mathcal{I}$-convergent sequences over various topological spaces. Earlier results of different authors were concerned regarding the closeness property of the sets: set of all bounded statistically convergent sequences, set of all bounded statistically convergent sequences of order α, set of all bounded $\mathcal{I}$-convergent sequences over the space ℓ^∞ (ℓ^∞- endowed with the sup-norm) only. On this context apart from this observation other properties (like connected and dense) of all three above mentioned sets have not yet been discussed over any other spaces. Our approach is to examine different behaviors of the set of all $\mathcal{I}$-convergent sequences over different spaces. Finally we are able to exhibit a condition over sequence spaces for which the set of all $\mathcal{I}$-convergent sequences form a closed set.

1. Introduction

In the year 1951, a subject was commenced as a generalization of usual convergence which is known as statistical convergence. Interestingly Fast [8] and Steinhaus [19] (see also [17]) explored the same conception independently in their own way. So many years later in 1980 and 1985 authors Šalát [15] and Fridy [9] respectively portrayed such remarkable works in this aspect and since then, the topic has become the other genre of domain of research. The definition of statistical convergence is expressed below as: If $\mathbb{N}$ denote the set of all natural numbers and $K \subseteq \mathbb{N}$ then $K(m, n)$ (where $m, n \in \mathbb{N}$) denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural (or, asymptotic) densities of the set K are defined by

$$\overline{d}(K) = \limsup_{n\to\infty} \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \liminf_{n\to\infty} \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists and it is denoted by $d(K)$ and clearly $d(K) = \lim_{n\to\infty} \frac{K(1, n)}{n}$.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number c if for any $\varepsilon > 0$, the set $K_x(\varepsilon) = \{n \in \mathbb{N} : |x_n - c| \geq \varepsilon\}$ has natural density zero. According to the notion of Šalát we denote by m_0 the set of all bounded statistically convergent sequences (see [1, 12, 18, 20] where other

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references can found).

We state here some necessary results which play a relevant role in our work.

Theorem 1.1. *[15, Theorem 2.1] The set m_0 is a closed linear subspace of the linear normed space ℓ^∞ (ℓ^∞- endowed with the sup-norm).*

Theorem 1.2. *[15, Theorem 2.2] The set m_0 is a nowhere dense set in ℓ^∞ (ℓ^∞- endowed with the sup-norm).*

There after a more generalized version of statistical convergence which is known as statistical convergence of order α was improved by Çolak [5] and is defined as follows: Let $0 < \alpha \leq 1$ be a real number. The upper and lower natural (or, asymptotic) densities of order α of the set $K(\subseteq \mathbb{N})$ are defined by $\overline{d}^\alpha(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n^\alpha}$ and $\underline{d}^\alpha(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n^\alpha}$. If $\overline{d}^\alpha(K) = \underline{d}^\alpha(K)$, then we say that the natural density of order α of K exists and it is denoted by $d^\alpha(K)$ and clearly $d^\alpha(K) = \lim_{n\to\infty} \frac{K(1,n)}{n^\alpha}$.

Contextually, a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent of order α to a real number ℓ if for any $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$. We follow the notation m_0^α to denote the set of all statistically convergent sequences of order α [2, 4].

Immediate after, following the line of work of Çolak [5], Bhunia et al. [4] established the following theorem:

Theorem 1.3. *[4, Theorem 3] For a fixed α, $0 < \alpha \leq 1$, the set $m_0^\alpha \cap \ell^\infty$ is a closed linear subspace of the linear normed space ℓ^∞ (ℓ^∞- endowed with the sup-norm).*

Further to this, in the beginning of the 21$^{\text{st}}$ century the creation of $\mathcal{I}$-convergence marked an epoch in this field. We remember Kostyrko et al. [11] for this significant direction of research. The concept of $\mathcal{I}$-convergence was formed on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers which substantially grown up as a further generalization of statistical convergence and statistical convergence of order α. Several works have been done on $\mathcal{I}$-convergence in the last twenty years (see [3, 6, 10, 16]). We now recall some relevant definitions and results.

Definition 1.4. *[10, 11]. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if*
(i) $\varnothing \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.
The ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\varnothing\}$ and $\mathbb{N} \notin \mathcal{I}$.

Definition 1.5. *[10, 11]. A non-empty family $\mathfrak{F} \subset 2^{\mathbb{N}}$ is called a filter if*
(i) $\varnothing \notin \mathfrak{F}$,
(ii) $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$,
(iii) $A \in \mathfrak{F}, A \subset B$ implies $B \in \mathfrak{F}$.
Clearly $\mathcal{I} \subset 2^{\mathbb{N}}$ is a non-trivial ideal of $\mathbb{N}$ iff $\mathfrak{F} = \mathfrak{F}(\mathcal{I}) = \{K \subset \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I}\}$ is a filter on $\mathbb{N}$, called the filter associated with $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all the singleton sets. Throughout in this paper we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.

Definition 1.6. *[10, 11]. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $c \in \mathbb{R}$, if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{n \in \mathbb{N} : |x_n - c| \geq \varepsilon\} \in \mathcal{I}$.*

If $x = \{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to c then we write $\mathcal{I} - \lim x = c$ or $\mathcal{I} - \lim x_n = c$. We introduce the set $F(\mathcal{I})$ as follows:

$$F(\mathcal{I}) = \{x = \{x_n\}_{n\in\mathbb{N}} \text{ is a sequence of real numbers } : \mathcal{I} - \lim x \in \mathbb{R}\}.$$

Definition 1.7. *[7, 12]. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-bounded if there exists a positive real number G such that the set $\{n \in \mathbb{N} : |x_n| \geq G\} \in \mathcal{I}$.*

Theorem 1.8. *[10, Theorem 2.3] Suppose that $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. Then $F(\mathcal{I}) \cap \ell^\infty$ is a closed linear subspace of the linear normed space ℓ^∞ (ℓ^∞- endowed with the sup-norm).*

Theorem 1.9. *[6, Theorem 8] Let m_2 denote the norm linear space of all bounded double sequences of real numbers (with norm, $\|x\| = \sup\limits_{m,n\in\mathbb{N}} |x_{mn}|$ where $\{x_{mn}\}_{m,n\in\mathbb{N}}$). The set of all bounded $\mathcal{I}_2$-convergent double sequence of real numbers form a closed linear subspace of the linear normed space m_2, when $\mathcal{I}_2$ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}$.*

The demonstration of literature survey assures that all the three sets m_0, $m_0^\alpha \cap \ell^\infty$ and $F(\mathcal{I}) \cap \ell^\infty$ had have been studied only over the space ℓ^∞-endowed with the sup-norm. Our focus is to furnish the nature of the set $F(\mathcal{I})$ (irrespective of boundedness) over topological spaces which are may or may not be metrizable. In this short note we improve the characteristic of the set of all $\mathcal{I}$-convergent sequences on sort of topological aspect. Further more, on a conclusion we could able to possess a condition over sequence spaces for which the set of all $\mathcal{I}$-convergent sequences form a closed set.

2. Main Results

In this section our first context of observation is product topological space.

Let $\mathbb{R}^\mathbb{N}$ denote the set of all real sequences and $\pi_m : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ be a projection mapping such that $\pi_m(x_1, x_2, x_3, ...x_{m-1}, x_m, x_{m+1}, ...) = x_m$ for all $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$. Let us define the collection $\mathcal{S}_m = \{\pi_m^{-1}(\mathcal{U}) : \mathcal{U}$ is open in $\mathbb{R}\}$ and $\mathcal{S} = \bigcup\limits_{m\in\mathbb{N}} \mathcal{S}_m$. The topology generated by the subbasis $\mathcal{S}$ is called the product topology over $\mathbb{R}^\mathbb{N}$ [14].

Example 2.1. *The set $F(\mathcal{I})$ is not a closed set in $\mathbb{R}^\mathbb{N}$ with product topology.*

Proof. Let us consider $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Therefore $\mathcal{I}_d$ forms an admissible ideal in $\mathbb{N}$. Our intension is to show, there exists a sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in $F(\mathcal{I}_d)$ such that $x^{(n)} \to x$ as $n \to \infty$ where $x \in \mathbb{R}^\mathbb{N}$ with product topology, but x does not belong to the set $F(\mathcal{I}_d)$. Let $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$,

$$x_k^{(n)} = \begin{cases} 1, & k \in \{1, 3, 5, ..., 2n-1\}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly each $\mathcal{I}_d - \lim x^{(n)} = 0$ where $n \in \mathbb{N}$. Setting $x = \{x_k\}_{k\in\mathbb{N}}$ where

$$x_k = \begin{cases} 1, & k \in \{2m-1 : m \in \mathbb{N}\}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $U = \prod\limits_{i\in\mathbb{N}} U_i$ be a basis element for the product topology that contains $x = \{x_k\}_{k\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$. Then there exists an integer p such that $U_i = \mathbb{R}$ for all $i \geq p$. Hence $x^{(n)} \in U$ for all $n \geq p$. This implies $\lim\limits_{n\to\infty} x^{(n)} = x$.

For each ε (where $0 < \varepsilon < 1$), neither the set $\{k \in \mathbb{N} : |x_k - 0| \geq \varepsilon\}$ nor even the set $\{k \in \mathbb{N} : |x_k - 1| \geq \varepsilon\}$ belong to $\mathcal{I}_d$. Hence we conclude that x does not belong to the set $F(\mathcal{I}_d)$. □

From the construction of Example 2.1 it is clear that $F(\mathcal{I}) \cap \ell^\infty$ is not a closed set over $\mathbb{R}^\mathbb{N}$ with product topology.

Theorem 2.2. *The set $F(\mathcal{I})$ is dense in $\mathbb{R}^\mathbb{N}$ with product topology.*

Proof. Let $a = \{a_1, a_2, ...\}$ be a point of $\mathbb{R}^\mathbb{N}$ and $U = \prod_{i\in\mathbb{N}} U_i$ be a basis element for the product topology that contains a. Then there exists a positive integer m such that $U_i = \mathbb{R}$ for all $i \geq m$. Setting the point $x = \{x_k\}_{k\in\mathbb{N}}$ where

$$x_k = \begin{cases} a_k, & k \in \{1, 2, 3, ..., m-1\}, \\ a_1, & \text{otherwise.} \end{cases}$$

Thus the point x of $F(\mathcal{I})$ belongs to U since $a_i \in U_i$ for all $i < m$ and $a_1 \in U_i$ for all $i \geq m$. □

Next we move to establish another result.

Theorem 2.3. *The subset $F(\mathcal{I})$ of $\mathbb{R}^\mathbb{N}$ forms a connected set with product topology.*

Proof. Let $\widetilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\mathbb{N}$ consisting of all sequences $x = \{x_k\}_{k\in\mathbb{N}}$ such that $x_k = 0$ for $i > n$. The space $\widetilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$ and so it is connected. Let $\mathbb{R}^\infty = \bigcup_{i\in\mathbb{N}} \widetilde{\mathbb{R}}^n$. Since we know that $\mathbb{R}^\infty$ is connected and closure of $\mathbb{R}^\infty$ is equal to $\mathbb{R}^\mathbb{N}$, so from the above Theorem 2.1 it is clear that $\mathbb{R}^\infty \subset F(\mathcal{I}) \subset \overline{\mathbb{R}^\infty}$. Hence $F(\mathcal{I})$ is connected in $\mathbb{R}^\mathbb{N}$ with respect to product topology. □

If we replace $F(\mathcal{I})$ by $F(\mathcal{I}) \cap \ell^\infty$ in Theorem 2.2 and Theorem 2.3 then the results are remain unaltered.

We proceed further to discuss classification of the set $F(\mathcal{I})$ over box topology.

Let us take a basis for a topology on the product space $\mathbb{R}^\mathbb{N}$ which is of the form $\prod_{m\in\mathbb{N}} \mathcal{U}_m$ where each $\mathcal{U}_m$ is an open set in $\mathbb{R}$. The topology generated by this basis is called the box topology over $\mathbb{R}^\mathbb{N}$ [14].

Now we figure out some more results over box topology.

Theorem 2.4. *The set $F(\mathcal{I})$ is closed in $\mathbb{R}^\mathbb{N}$ with box topology.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}} \in \overline{F(\mathcal{I})}$. Consider the basis element $\mathcal{U} = \prod_{k\in\mathbb{N}} (x_k - \frac{1}{2k}, x_k + \frac{1}{2k})$ which contains x. Then there exists an element $y = \{y_k\}_{k\in\mathbb{N}} \in F(\mathcal{I})$ such that $y \in \mathcal{U} \cap F(\mathcal{I})$ and $\mathcal{I} - \lim y = \xi$. Let ε be a positive real number, then there exists a natural number k_0 such that $\frac{1}{k_0} < \frac{\varepsilon}{2}$. Therefore $\{k \in \mathbb{N} : |y_k - \xi| < \frac{\varepsilon}{2}\} \cap \{k_0, k_0+1, k_0+2, ...\} \subseteq \{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\} \in \mathfrak{F}(\mathcal{I})$. Hence the result. □

Example 2.5. *The set $F(\mathcal{I})$ is neither dense nor connected in $\mathbb{R}^\mathbb{N}$ with box topology.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ be defined as

$$x_k = \begin{cases} 1, & k \in \{2m-1 : m \in \mathbb{N}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the open set $\mathcal{U} = (1-1, 1+1) \times (0 - \frac{1}{2}, 0 + \frac{1}{2}) \times (1 - \frac{1}{3}, 1 + \frac{1}{3}) \times (0 - \frac{1}{4}, 0 + \frac{1}{4}) \times ...$ contains the point x but $F(\mathcal{I}) \cap \mathcal{U} = \emptyset$. So $F(\mathcal{I})$ is not dense in $\mathbb{R}^\mathbb{N}$.

Now we proceed to the second part of the example. We can express $\mathbb{R}^{\mathbb{N}}$ as the union of the set A consisting of all bounded sequences of real numbers and the set B of all unbounded sequences of real numbers. Both the sets are nonempty, disjoint and open in the box topology [14]. Hence the set $F(\mathcal{I})$ can be rewritten as $F(\mathcal{I}) = P \cup Q$, where $P = F(\mathcal{I}) \cap A$ is the set of all bounded statistically convergent sequences of real numbers and $Q = F(\mathcal{I}) \cap B$ is the set of all unbounded statistically convergent sequences of real numbers. These sets P and Q are nonempty, disjoint and open in the subspace topology. Thus, $F(\mathcal{I})$ is not connected. □

After an extensive discussion regarding product topology and box topology we like to insight into Fort space.

Let X be an arbitrary infinite set and p be an arbitrary but a fixed point in X. A topological space (X, τ) is called the Fort space if τ consists of all those subsets of X which do not contain p and all those subsets of X which are complements of finite subsets.

Example 2.6. *The set $F(\mathcal{I})$ may or may not be closed but neither connected nor dense in the Fort space over $\mathbb{R}^{\mathbb{N}}$ with respect to a fixed point p of $\mathbb{R}^{\mathbb{N}}$.*

Proof. If $p \notin F(\mathcal{I})$ then $F(\mathcal{I})$ is neither closed nor connected infact it is an open set. So we assume $p \in F(\mathcal{I})$. Let us choose an element $q \in F(\mathcal{I})$ such that $p \neq q$. Then the singleton set $\{q\}$ is open as well as closed and thus $F(\mathcal{I})$ is a disconnected set. Next, let x be an element in $\mathbb{R}^{\mathbb{N}} \setminus F(\mathcal{I})$. Since $p \in F(\mathcal{I})$ so the singleton set $\{x\}$ is open. So x is not a limit point of the set $F(\mathcal{I})$. Thus, $F(\mathcal{I})$ is a closed set. Also we draw conclusion that the set $F(\mathcal{I})$ is not dense in the Fort space over $\mathbb{R}^{\mathbb{N}}$. □

If we consider $\mathbb{R}^{\mathbb{N}}$ with discrete topology then on a very obvious note either $F(\mathcal{I})$ or $F(\mathcal{I}) \cap \ell^{\infty}$ are both clopen sets but neither dense nor connected.

In this study we are also concerned about two notable topological spaces such as co-finite and co-countable topologies over $\mathbb{R}^{\mathbb{N}}$. We find $F(\mathcal{I})$ is dense and connected but not closed under these topologies.

Apart from topological spaces we like to draw attention over some sequence spaces. First we count Hilbert-Cube Space.

Let H_{∞} denote the set of all real sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$ and the distance function ρ be defined by

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - y_k|,$$

where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ belong to H_{∞}. This distance function ρ along with the set H_{∞} forms a metric space. The space is known as Hilbert-Cube Space and is denoted by the symbol H_{∞}.

We now sketch an example that shows the above Theorems 1.3 & 1.8 do not follow the closeness character while replacing ℓ^{∞} by H_{∞}.

Example 2.7. *The set $F(\mathcal{I}) \cap H_{\infty}$ is not a closed set in H_{∞}.*

Proof. Let $\mathcal{I}_{d^{\alpha}} = \{A \subset \mathbb{N} : d^{\alpha}(A) = 0\}$ in the range $\frac{1}{2} < \alpha \leq 1$. Then $\mathcal{I}_{d^{\alpha}}$ forms an admissible ideal in $\mathbb{N}$. We consider $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}} \in F(\mathcal{I}_{d^{\alpha}}) \cap H_{\infty}$ for $n = 1, 2, 3, ...$ and $x = \{x_k\}_{k\in\mathbb{N}}$ in H_{∞}.

Now we show that $x^{(n)} \to x$ as $n \to \infty$ but $x \notin F(\mathcal{I}_{d^{\alpha}}) \cap H_{\infty}$.

Let, $x^{(1)} = \{x_k^{(1)}\}_{k\in\mathbb{N}}$ where

$$x_k^{(1)} = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\}, \\ 0, & \text{otherwise}, \end{cases}$$

$x^{(2)} = \{x_k^{(2)}\}_{k\in\mathbb{N}}$ where

$$x_k^{(2)} = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{3\}, \\ 0, & \text{otherwise}, \end{cases}$$

$x^{(3)} = \{x_k^{(3)}\}_{k\in\mathbb{N}}$ where

$$x_k^{(3)} = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{3, 5\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\cdots$$

$x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$ where

$$x_k^{(n)} = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{3, 5, ..., 2n-1\}, \\ 0, & \text{otherwise} \end{cases}$$

and so on.

Clearly each $\mathcal{I}_{d^\alpha} - \lim x^{(n)} = 0$ where $n \in \mathbb{N}$. Setting the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ where

$$x_k = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{2m-1 : m \in \mathbb{N}\}, \\ 0, & \text{otherwise}. \end{cases}$$

Now,

$$\begin{aligned} \lim_{n\to\infty} \rho(x^{(n)}, x) &= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{2^k} |x_k^{(n)} - x_k| \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2^{2n+1}} + \frac{1}{2^{2n+3}} + \frac{1}{2^{2n+5}} + ... \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2^{2n+1}} \{1 + \frac{1}{2^2} + \frac{1}{2^4} + ...\} \right) = \lim_{n\to\infty} \frac{1}{2^{2n+1}} . \frac{4}{3} = 0. \end{aligned}$$

This implies $\lim_{n\to\infty} x^{(n)} = x$ in H_∞. For each $0 < \varepsilon < 1$ and for $x = \{x_k\}_{k\in\mathbb{N}}$,

$$A_x(\varepsilon) = \{k \in \mathbb{N} : |x_k - 0| \geq \varepsilon\} = \{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\} \notin \mathcal{I}_{d^\alpha},$$

$$B_x(\varepsilon) = \{k \in \mathbb{N} : |x_k - 1| \geq \varepsilon\} = \mathbb{N} \setminus (\{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\}) \notin \mathcal{I}_{d^\alpha}.$$

Hence $x \notin F(\mathcal{I}_{d^\alpha}) \cap H_\infty$ and our assertion is proved. □

Remark 2.8. *From the above example it comes out that the space $m_0^\alpha \cap H_\infty$ or $m_0^\alpha \cap H_\infty \cap \ell^\infty$ is not closed in H_∞. [13, Example 2.1]*

We approach onwards to the Fréchet metric space and like to continue our discussion.

Let F denote the set of all real sequences. Also let the distance function σ be defined by

$$\sigma(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|x_k-y_k|}{1+|x_k-y_k|},$$

where $x=\{x_k\}_{k\in\mathbb{N}}$ and $y=\{y_k\}_{k\in\mathbb{N}}$ are elements of F. This distance function σ forms a metric over F and hence F is called Fréchet sequence space or Fréchet metric space.

Example 2.9. *The set $F(\mathcal{I})$ does not form a closed set over F.*

Proof. To prove this example we find out a sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in $F(\mathcal{I})$ which converges to an element x in F but that x does not belong to $F(\mathcal{I})$.

Consider the sequence $x^{(n)}=\{x_k^{(n)}\}_{k\in\mathbb{N}}$ for all $n\in\mathbb{N}$, $x=\{x_k\}_{k\in\mathbb{N}}$ and the ideal $\mathcal{I}_{d^\alpha}=\{A\subset\mathbb{N}: d^\alpha(A)=0\}$ where $\frac{1}{2}<\alpha\leq 1$ as defined in Example 2.7.

Then

$$\lim_{n\to\infty}\sigma(x^{(n)},x)=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|x_k^{(n)}-x_k|}{1+|x_k^{(n)}-x_k|}\right)$$

$$=\lim_{n\to\infty}\left(\sum_{\substack{k=2n+1\\ k\in\{2m-1:m\in\mathbb{N}\}\cap(\mathbb{N}\setminus\{1^2,2^2,\ldots\})}}^{\infty}\frac{1}{2^k}\times\frac{1}{2}\right)$$

$$\leq\lim_{n\to\infty}\left(\frac{1}{2^{2n+1}}+\frac{1}{2^{2n+3}}+\frac{1}{2^{2n+5}}+\ldots\right)=\lim_{n\to\infty}\frac{1}{2^{2n+1}}.\frac{4}{3}=0.$$

This implies $x^{(n)}\to x$ in F as $n\to\infty$. But for each ε such that $0<\varepsilon<1$ and $x=\{x_k\}_{k\in\mathbb{N}}$ the set $A_x(\varepsilon)=\{k\in\mathbb{N}:|x_k-0|\geq\varepsilon\}=\{1^2,2^2,3^2,\ldots\}\cup\{2k-1:k\in\mathbb{N}\}\notin\mathcal{I}_{d^\alpha}$. Hence the result. □

Remark 2.10. *We could draw two conclusions from the above Example 2.9 as follows:*
(i) the set of all statistically convergent sequences of order α is not closed in Fréchet sequence space [13, Example 2.2].
(ii) the set of all bounded statistically convergent sequences of order α is not closed in Fréchet sequence space [13, Remark 2.3].

Suppose (X,ϱ) be any metric space. We introduce the set $F_X(\mathcal{I})$ as follows: $F_X(\mathcal{I})=\{x=\{x_n\}_{n\in\mathbb{N}}:$ each $x_n\in X$ and $\mathcal{I}-\lim x\in X\}$.

Finally at this stage we would like to impose certain conditions over any arbitrary sequence space under which the set $F_X(\mathcal{I})$ becomes closed.

Theorem 2.11. *Let (X,ϱ) and $(X^{\mathbb{N}},\sigma)$ be two complete metric spaces where ϱ and σ be chosen in such a manner such that $\varrho(x_k,y_k)\leq M\sigma(x,y)$ for all $k\in\mathbb{N}$, M is a fixed positive real number and $x=\{x_k\}_{k\in\mathbb{N}}$, $y=\{y_k\}_{k\in\mathbb{N}}$ are elements of $X^{\mathbb{N}}$. Then $F_X(\mathcal{I})$ forms a closed set in $X^{\mathbb{N}}$.*

Proof. In the sequence space $X^{\mathbb{N}}$, we define the sequences as $x^{(n)}=\{x_k^{(n)}\}_{k\in\mathbb{N}}\in F_X(\mathcal{I})$ for all $n\in\mathbb{N}$ and $x^{(n)}\to x=\{x_k\}_{k\in\mathbb{N}}$ in $X^{\mathbb{N}}$ as $n\to\infty$. Therefore $\sigma(x^{(n)},x)\to 0$ as $n\to\infty$. We establish that $x\in F_X(\mathcal{I})$.

Let $\mathcal{I}-\lim x^{(n)}=\xi_n\in X$ where $n\in\mathbb{N}$. We prove the theorem in two steps:

Step (i): To show that $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence in X.
Step (ii): Finally $\mathcal{I}-\lim x = \xi$, where $\lim\limits_{n\to\infty}\xi_n = \xi$.

Step (i): Since, $\sigma(x^{(n)}, x) \to 0$ as $n \to \infty$ so for each ε $(0 < \varepsilon < 1)$ there exists a natural number n_0 such that

$$\sigma(x^{(u)}, x^{(v)}) < \frac{\varepsilon}{3M} \text{ for all } u, v \geq n_0$$

$$\Rightarrow\ \varrho(x_k^{(u)}, x_k^{(v)}) < \frac{\varepsilon}{3} \text{ for all } u, v \geq n_0 \text{ and } k \in \mathbb{N}.$$

Let $U(\frac{\varepsilon}{3}) = \{k \in \mathbb{N} : \varrho(x_k^{(u)}, \xi_u) < \frac{\varepsilon}{3}\}$ and $V(\frac{\varepsilon}{3}) = \{k \in \mathbb{N} : \varrho(x_k^{(v)}, \xi_v) < \frac{\varepsilon}{3}\}$. Clearly $U(\frac{\varepsilon}{3})$ and $V(\frac{\varepsilon}{3})$ belong to $\mathfrak{F}(\mathcal{I})$ and $U(\frac{\varepsilon}{3}) \cap V(\frac{\varepsilon}{3}) \neq \varnothing$ since $\varnothing \notin \mathfrak{F}(\mathcal{I})$.

Now we choose a natural number k such that $k \in U(\frac{\varepsilon}{3}) \cap V(\frac{\varepsilon}{3})$ and hence

$$\varrho(\xi_u, \xi_v) \leq \varrho(\xi_u, x_k^{(u)}) + \varrho(x_k^{(u)}, x_k^{(v)}) + \varrho(x_k^{(v)}, \xi_v) < \varepsilon \text{ for all } u, v \geq n_0.$$

Therefore $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete metric space (X, ϱ). So there exists an element $\xi \in X$ such that $\lim\limits_{n\to\infty}\xi_n = \xi$.

Step (ii): Since $\sigma(x^{(n)}, x) \to 0$ and $\varrho(\xi_n, \xi) \to 0$ as $n \to \infty$, then there exists $m_0 \in \mathbb{N}$ such that

$$\sigma(x^{(p)}, x) < \frac{\varepsilon}{3M} \text{ and } \varrho(\xi_p, \xi) < \frac{\varepsilon}{3} \text{ for all } p \geq m_0$$

$$\Rightarrow\ \varrho(x_k^{(m_0)}, x_k) < \frac{\varepsilon}{3} \text{ and } \varrho(\xi_{m_0}, \xi) < \frac{\varepsilon}{3} \text{ for all } k \in \mathbb{N}.$$

Now $\varrho(x_k, \xi) \leq \varrho(x_k^{(m_0)}, x_k) + \varrho(x_k^{(m_0)}, \xi_{m_0}) + \varrho(\xi_{m_0}, \xi) \leq \varrho(x_k^{(m_0)}, \xi_{m_0}) + \frac{2\varepsilon}{3}$ for all $k \in \mathbb{N}$. So we get $\{k \in \mathbb{N} : \varrho(x_k, \xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : \varrho(x_k^{(m_0)}, \xi_{m_0}) \geq \frac{\varepsilon}{3}\} \in \mathcal{I}$. $\square$

Remark 2.12. *If we replace $X^{\mathbb{N}}$ by any complete subspace of $Y^{\mathbb{N}}$ of $X^{\mathbb{N}}$ in the Theorem 2.4 and assume the inequality holds over the space $Y^{\mathbb{N}}$ then $F_X(\mathcal{I}) \cap Y^{\mathbb{N}}$ forms a closed set over $Y^{\mathbb{N}}$.*

As an immediate consequence of the above theorem an important question arises that ”Does there exist such metric σ over $X^{\mathbb{N}}$ such that the hypothesis of the Theorem 2.11 is satisfied?”

Our following remark is the definite answer to this question.

Remark 2.13. *(i) Let us define a metric σ on $\mathbb{R}^{\mathbb{N}}$ by $\sigma(x, y) = \sup\{\overline{d}(x_k, y_k) : k \in \mathbb{N}\}$, where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. Then the metric σ is called the uniform metric on $\mathbb{R}^{\mathbb{N}}$ [14]. For any positive real number ε in $(0, 1)$, $|x_k - y_k| \leq \sigma(x, y) < \varepsilon$ satisfies for all $k \in \mathbb{N}$, $M = 1$ and $x = \{x_k\}_{k\in\mathbb{N}}, y = \{y_k\}_{k\in\mathbb{N}}$ are in $\mathbb{R}^{\mathbb{N}}$. Similarly $F(\mathcal{I})$ forms a closed set in $\mathbb{R}^{\mathbb{N}}$.*

(ii) The space ℓ^∞ consists of all bounded sequences of real numbers with that metric $\sigma(x, y) = \sup\limits_{k\in\mathbb{N}}|x_k - y_k|$ where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ belong to ℓ^∞. If we choose any positive real number ε however small then $|x_k - y_k| \leq \sigma(x, y) < \varepsilon$, for all $k \in \mathbb{N}$, $M = 1$, $x = \{x_k\}_{k\in\mathbb{N}}, y = \{y_k\}_{k\in\mathbb{N}}$ are in $X^{\mathbb{N}}$. Then $F(\mathcal{I}) \cap \ell^\infty$ is a closed set in ℓ^∞ [10, Theorem 2.3].

(iii) Let $p \geq 1$ be a fixed positive real number. By definition, each element in the space ℓ^p is a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers such that $\sum\limits_{k=1}^{\infty}|x_k|^p < \infty$ and the metric is defined by $\sigma(x, y) = \left(\sum\limits_{k=1}^{\infty}|x_k - y_k|^p\right)^{\frac{1}{p}}$ where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ belong to ℓ^p. Choosing any positive real number ε however small, $|x_k - y_k| \leq \sigma(x, y) < \varepsilon$, for all $k \in \mathbb{N}$, $M = 1$ and $x = \{x_k\}_{k\in\mathbb{N}}, y = \{y_k\}_{k\in\mathbb{N}}$ are in ℓ^p. Therefore $F(\mathcal{I}) \cap \ell^p$ is a closed set in ℓ^p.

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