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On the superposition operator in the space of functions of $H_q^w([0,1])$

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Abstract. In this paper, we obtained a necessary and sufficient condition for the embedding $H_q^{\omega}([0,1]) \subset IBV_p^q([0,1])$, where IBV_p^q denotes the set of functions of bounded q-integral p-variation. Additionally, the conditions for the composition and superposition operators were provided to map the space $H_q^{\omega}([0,1])$ into itself, by which these operators were bounded. Finally, we applied these results to examine the existence and uniqueness of solutions to Hammerstein integral equations in the space of $H_q^{\omega}([0,1])$.

1. Introduction

Given some domain $\Omega \subseteq \mathbb{R}^N$ and a Banach space $X = X(\Omega)$ of real function on Ω, find conditions on a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, possibly both necessary and sufficient, under which the superposition operator F_f defined by

$$F_f(x)(s) = f(s, x(s)), \quad s \in \Omega, x \in X,$$

maps the space of X into itself and has 'nice' analytic properties. Even in the much simpler form of an composition operator generated by some function $f : \mathbb{R} \to \mathbb{R}$, i.e.,

$$F_f(x)(s) = f(x(s)), \quad s \in \Omega, x \in X,$$

this problem is sometimes surprisingly difficult (see[2]).

These operators play a major role in various mathematical fields, especially in the theory of nonlinear integral equations. Rutitskii [19] studied the continuity of superposition operators on L^p. In other studies, the authors in [3,4,9,10] investigated some properties of superposition operators such as boundedness, continuity, etc. on the various function spaces. Bugajewska [7] provided sufficient conditions by which a superposition operator mapped the space of functions of bounded variation in the sense of Jordan or Young into itself, and then the results were applied to examine the existence and uniqueness of solutions to Hammerstein and Hammerstein-Volterra integral equations in this space. In this study, we are going to prove some theorems which describe sufficient conditions by which a superposition operator maps the space

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$H_q^{\omega}([0,1])$ into itself and proves the existence and uniqueness of solutions to the nonlinear Hammerstein integral equations in this space.

The present paper is organized with the following sections. In section 2 we collected some definitions and results which were needed for the sequel. In section 3, we obtained a necessary and sufficient condition for the embedding $H_q^{\omega}([0,1]) \subset IBV_p^q([0,1])$ under the certain natural assumptions imposed on the modulus of continuity. The section 4 provided conditions for the composition and superposition operators to map the space $H_q^{\omega}([0,1])$ into itself, showing the bounded of these operators. Finally, in section 5, we applied the theorems from the section 4 to prove the existence and uniqueness of solutions to the nonlinear Hammerstein integral equations in the space of $H_q^{\omega}([0,1])$.

2. Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}$ to be a real-valued function, f is satisfied in Lipschitz condition on $\mathbb{R}$, if there exists a positive real constant M such that, for all x_1 and $x_2 \in \mathbb{R}$, $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$. Also, function f is called locally Lipschitz condition if for each $x_0 \in \mathbb{R}$, there exists constant $M > 0$ and $\delta_0 > 0$ such that $|x - x_0| < \delta_0$ then $|f(x) - f(x_0)| \leq M|x - x_0|$.

For each bounded 1-periodic functions, Chanturia [8] introduced the concept of modulus of variation. Let $\omega(t)$ be a modulus of continuity, i.e., a continuous, subadditive and nondecreasing function on $[0,+\infty)$ is satisfied when $\omega(0) = 0$. For $1 \leq q < \infty$, $H_q^{\omega}([0,1])$ is denoted by the class of 1-periodic functions $f \in L^q([0,1])$ for which $\omega_q(\delta, f) = O(\omega(\delta))$ as $\delta \to 0^+$, where

$$\omega_q(\delta, f) = \sup_{0\leq h\leq\delta} \left(\int_0^{1-h} |f(t+h) - f(t)|^q dt \right)^{\frac{1}{q}}.$$

However, if f is defined on $\mathbb{R}$ instead of on $[0,1]$ and if f is 1-periodic, it is convenient to modify the definition and put

$$\omega_q(\delta, f) := \sup_{0\leq h\leq\delta} \left(\int_0^{1} |f(t+h) - f(t)|^q dt \right)^{\frac{1}{q}},$$

since the difference between the two definitions is then nonessential in all applications of the concept.
The space of $H_q^{\omega}([0,1])$ with the following norm is a Banach space:

$$\|f\|_{H_q^{\omega}} := \|f\|_q + \sup_{\delta>0} \frac{\omega_q(\delta, f)}{\omega(\delta)}.$$

Some properties of the space $H_q^{\omega}([0,1])$ are described in [11,14,16-18,21,22].
It is well know that for each modulus of continuity ω there exists a concave modulus of continuity ω^* such that $\omega(\delta) \leq \omega^*(\delta) \leq 2\omega(\delta)$ for $\delta \in [0,1]$. Then $H_q^{\omega^*} = H_q^{\omega}$. In what follows, we allways that ω is a concave modulus of continuity.
Let $f : I \to \mathbb{R}$ be a Lebesgue measurable function and let $[a,b] \subset I$ be a fixed interval, $a < b$. The value

$$\omega_q(f; a, b) = \sup_{0<h<b-a} \left(\int_a^{b-h} |f(t+h) - f(t)|^q \, \mathrm{d}t \right)^{\frac{1}{q}}$$

is called the L^q-modulus of continuity of the function f on the interval $[a,b]$.

Let $1 \leq p, q < \infty$ and $f : [a,b] \to \mathbb{R}$ be a Lebesgue measurable function. Let us set

$$\mathrm{ivar}_p^q(\delta, f) := \omega_{p,q}(\delta, f; a, b) = \sup \left(\sum_{i=1}^{N} \left(\omega_q(t_{i-1}, t_i, f) \right)^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all finite partitions of $[a,b]$ satisfying $\Delta x_i \leq \delta$. The quantity $\mathrm{ivar}_p^q(b-a, f)$ is called the q-integral p-variation of the function f on $[a,b]$. If $\mathrm{ivar}_p^q(b-a, f) < \infty$, then we say that f is a function of bounded q-integral p-variation. The class of all such functions is denoted by $IBV_p^q([a,b])$(see [12,20]).

Lemma 2.1. *The inequality*

$$w(c \cdot t) \leq (c+1) \cdot w(t)$$

holds for any positive c and for any modulus of continuity $w(t)$. (See[15])

3. On the embedding $H_q^{\omega}([0,1]) \subset IBV_p^q([0,1])$

In this section a necessary and sufficient condition for the embedding $H_q^{\omega}([0,1]) \subset IBV_p^q([0,1])$ are given.

Theorem 3.1. *Let $1 \leq p, q < +\infty$ and $\omega : [0,1] \to [0,+\infty)$ be a modulus of continuity. If $\omega(\delta) = O(\delta^{\frac{1}{p}})$ for $\delta \to 0^+$, then the embedding $H_q^{\omega}([0,1]) \subset IBV_p^q([0,1])$ holds.*

Proof. Let $f \in H_q^{\omega}([0,1])$. Then there exists a positive constant c such that $\omega_q(\delta, f) \leq c w(\delta)$. Also since $\omega(\delta) = O(\delta^{\frac{1}{p}})$, we have

$$\omega(\delta) \leq d \cdot \delta^{\frac{1}{p}},$$

where $\delta \in [0,1]$ and constant $d > 0$. Let's take an arbitrary finite partition $P = \{t_0, t_1, \cdots, t_n\}$ of the interval $[0,1]$ such that $t_i - t_{i-1} \leq \delta$. Then

$$\omega_q(t_{i-1}, t_i, f) \leq c\omega(\Delta t_i).$$

Therefore, it follows that

$$\begin{aligned}
\sum_{i=1}^{N} \omega_q^p(t_{i-1}, t_i, f) &\leq \sum_{i=1}^{N} c^p \cdot w^p(\Delta t_i) \\
&\leq \sum_{i=1}^{N} c^p \cdot d^p \, (t_i - t_{i-1})^{\frac{p}{p}} \\
&\leq (c \cdot d)^p \sum_{i=1}^{N} (t_i - t_{i-1})^{\frac{p}{p}} \\
&= (c \cdot d)^p.
\end{aligned}$$

Hence $f \in IBV_p^q([0,1])$. □

Lemma 3.2. *Let $1 \leq p, q < +\infty$ and $\omega : [0,1] \to [0,+\infty)$ be a modulus of continuity. If $\delta^{\frac{1}{p}} = o(\omega(\delta))$ for $\delta \to 0^+$, Then there exists a constant $M_0 > 0$ such that for the concave modulus of continuity $\omega(\delta)$ there exists an inverse function $\omega_{-1}(\delta)$, which is defined on $[0, M_0]$, and a sequence (δ_n), $\delta_1 \leq M_0$, $\delta \to 0^+$ such that*

$$\sum_{n=1}^{\infty} (\delta_n)^p = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} w_{-1}(\delta_n) = K,$$

where K is a some positive constant.

Proof. See [15, the proof of Theorem]. □

Theorem 3.3. *Let $1 \leq p, q < +\infty$ and $\omega : [0,1] \to [0,+\infty)$ be a modulus of continuity. If $\omega(\delta) \neq O(\delta^{\frac{1}{p}})$ and, in addition, $\delta^{\frac{1}{p}} = o(\omega(\delta))$ for $\delta \to 0^+$. Then there exists a function $f \in H_q^{\omega}([0,1])$ such that $f \notin IBV_p^q([0,1])$.*

Proof. By Lemma 3.2 there exists a sequence (δ_n), $\delta_1 \leq M_0$, $\delta \to 0^+$ such that

$$\sum_{n=1}^{\infty} (\delta_n)^p = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} w_{-1}(\delta_n) = K,$$

where K is a some positive constant.
Let $P = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0,1]$ such that

$$t_{2i} = \frac{1}{K}\sum_{k=1}^{i} \omega_{-1}(\delta_k) \quad \text{and} \quad t_{2i+1} = t_{2i} + \frac{1}{2K}\cdot \omega_{-1}(\delta_{i+1}),$$

for $i = 0, 1, \ldots$, putting $\sum_{k=1}^{0} \omega_{-1}(\delta_k) = 0$.
Consider the function f as follows

$$f(t) = \begin{cases} \frac{w(2K[t-t_{2i}])}{(\Delta t_{2i+1})^{\frac{1}{q}}} & t_{2i} \leq t \leq t_{2i+1}, \\ \frac{w(2K[t_{2i+2}-t])}{(\Delta t_{2i+2})^{\frac{1}{q}}} & t_{2i+1} \leq t \leq t_{2i+2}, \\ 0 & t = 1, \end{cases}$$

it is obvious that $f \in L^q([0,1])$.
Now we prove that there exits positive constant c such that

$$\omega_q(\delta, f) \leq c \cdot \omega(\delta).$$

We examine the following phrase for $h \in [0,1]$

$$\int_0^{1-h} \left|f(t+h) - f(t)\right|^q dt.$$

The supremum above phrase will be obtained in some interval of monotonicity of function f. It is equivalent to the investigation on the corresponding interval of increasity of this function f. Then there exists a natural number j_0 such that

$$t_{2j_0} \leq t \leq t_{2j_0+1} - h.$$

So we obtain

$$\begin{aligned} \sup_{0\leq t\leq 1-h} \left|f(t+h) - f(t)\right|^q &= \sup_{0\leq t\leq 1-h} \left|\frac{\omega(2K[t+h-t_{2j_0}]) - \omega(2K[t-t_{2j_0}])}{(\Delta t_{2j_0+1})^{\frac{1}{q}}}\right|^q \\ &= \sup_{0\leq t\leq 1-h} \left|\frac{\omega(2K[t+h-t_{2j_0}]) - \omega(2K[t-t_{2j_0}])}{(\frac{\omega_{-1}(t_{j_0+1})}{2K})^{\frac{1}{q}}}\right|^q \\ &\leq \frac{\omega^q(2Kh)}{(\frac{\omega_{-1}(t_{j_0+1})}{2K})}, \end{aligned}$$

and so according to Lemma 2.1, we have

$$\begin{aligned} \omega_q(\delta, f) &= \sup_{0\leq h\leq\delta} \Big(\int_0^{1-h} \left|f(t+h) - f(t)\right|^q dt\Big)^{\frac{1}{q}} \\ &\leq \Big(\frac{2K}{\omega_{-1}(t_{j_0+1})}\Big)^{\frac{1}{q}} \sup_{0\leq h\leq\delta} \omega(2Kh) \\ &= \Big(\frac{2K}{\omega_{-1}(t_{j_0+1})}\Big)^{\frac{1}{q}} \omega(2K\delta) \\ &\leq \Big(\frac{2K}{\omega_{-1}(t_{j_0+1})}\Big)^{\frac{1}{q}} (2K+1)\omega(\delta). \end{aligned}$$

Therefore $f \in H_q^{\omega}([0,1])$.
Eventually, we show that $f \notin IBV_p^q([0,1])$. An easy computation, shows that

$$\begin{aligned}\omega_q(t_{2i}, t_{2i+1}, f) &= \frac{\left|\omega(2K[t_{2i+1} - t_{2i}])\right|}{(\Delta t_{2i+1})^{\frac{1}{q}}} \sup_{0\le h\le \Delta t_{2i+1}} \Big((t_{2i+1} - t_{2i}) - h\Big)^{\frac{1}{q}} \\ &= \frac{\left|\omega(2K[t_{2i+1} - t_{2i}])\right|}{(\Delta t_{2i+1})^{\frac{1}{q}}}\Big(\Delta t_{2i+1}\Big)^{\frac{1}{q}} \\ &= \omega(2K[t_{2i+1} - t_{2i}]).\end{aligned}$$

By a similar argument we have

$$\omega_q(t_{2i+1}, t_{2i+2}, f) = \omega(2K[t_{2i+2} - t_{2i+1}]).$$

Therefore, for a given $1 \le p < \infty$ we have

$$\begin{aligned}\sum_{i=1}^{\infty} \omega_q^p(t_i, t_{i+1}, f) &= \sum_{i=1}^{\infty} [\omega_q^p(t_{2i}, t_{2i+1}, f) + \omega_q^p(t_{2i+1}, t_{2i+2}, f)] \\ &= \sum_{i=1}^{\infty} [\omega^p(2K[t_{2i+1} - t_{2i}]) + \omega^p(2K[t_{2i+2} - t_{2i+1}])] \\ &= \sum_{i=1}^{\infty} 2\delta_{i+1}^p.\end{aligned}$$

Then we obtain

$$ivar_p^q(f, 0, 1) \ge 2\sum_{i=1}^{\infty} (\delta_{i+1}^p)^{\frac{1}{p}} = \infty.$$

Consequently $f \notin IBV_p^q([0,1])$, which completes the proof. □

4. Superposition operator

We begin this section with the following theorem, which presents the conditions under which an composition operator maps the space of $H_q^{\omega}([0,1])$ into itself and it is bounded.

Theorem 4.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a given function such that satisfies the Lipschitz condition on $\mathbb{R}$ and $w : [0,1] \to [0,+\infty)$ to be a modulus of continuity. Then for $1 \le q < \infty$, the composition operator F_f, generated by f, maps the space $H_q^{\omega}([0,1])$ into itself, and it is bounded.*

Proof. We first show that F_f is well-defined, that is

$$\forall g \in H_q^{\omega}([0,1]), \quad F_f(g) \in H_q^{\omega}([0,1]).$$

Clearly $F_f(g)$ is well-defined, since $g \in H_q^{w}([0,1])$, then for every $t \in [0,1]$ we have $F_f(g)(t) = F_f(g)(t+1)$ therefore $F_f(g)$ is 1-periodic function. On the other hand, since f is satisfied in the Lipschitz condition on $\mathbb{R}$, then for $1 \le q < \infty$,

$$\begin{aligned}\|F_f(g)\|_q &= \Big(\int_0^1 |F_f(g)(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\ &= \Big(\int_0^1 |f(g(t))|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\ &\le \Big(\int_0^1 N^q \mathrm{d}t\Big)^{\frac{1}{q}} = N, \quad t \in [0,1], g \in H_q^{w}([0,1]),\end{aligned}$$

where N is some positive constant and hence $F_f(g) \in L^q\big([0,1]\big)$.

According to the given assumptions, since ω is the modulus of continuity for $0 \le \delta \le 1$ and $F_f(g) \in L^q\big([0,1]\big)$, $w(\delta)$ and $w_q(\delta, F_f(g))$ are meaningful. Therefore, since f is satisfied in the Lipschitz condition on $\mathbb{R}$, we have

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\omega_q(\delta, F_f(g))}{\omega(\delta)} &= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |F_f(g)(t+h) - F_f(g)(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |f(g(t+h)) - f(g(t))|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&\le \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(M\Big(\int_0^1 |g(t+h) - g(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&= M \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |g(t+h) - g(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&= M \lim_{\delta\to 0^+} \frac{\omega_q(\delta, g)}{\omega(\delta)} < \infty,
\end{aligned}$$

it follows that $F_f(g) \in H^\omega_q([0,1])$. Hence we have

$$\|F_f(g)\|_{H^\omega_q} \le N + M \sup_{\delta>0} \frac{w_q(\delta, g)}{w(\delta)}$$

which completes the proof. □

By Theorem 4.1, we can easily obtain locally bounded of composition operators for the spaces of $H^\omega_q([0,1])$.

Corollary 4.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a given function such that satisfies a local Lipschitz condition on $\mathbb{R}$ and $w : [0,1] \to [0,+\infty)$ to be a modulus of continuity. Then for $1 \le q < \infty$, the composition operator F_f, generated by f, maps the space $H^\omega_q([0,1])$ into itself, and it is locally bounded.*

In the following, we present the generalization of Theorem 4.1, which shows the bounded of superposition operator in the space of $H^\omega_q([0,1])$.

Theorem 4.3. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the given 1-periodic function on $[0,1]$ such that satisfies the Lipschitz condition on $\mathbb{R}$ and $w : [0,1] \to [0,+\infty)$ be a modulus of continuity such that for function $u : [0,1] \to \mathbb{R}$ and $1 \le q < \infty$, $\sup_{\delta>0} \dfrac{\omega_q(\delta, f_u)}{\omega(\delta)} < \infty$ $\Big($where $f_u(t+h) = f(t+h, u(t))$, for $t \in [0,1]$, $h \in [0,\delta]$ and $f_u \in L^q\big([0,1]\big)\Big)$. Then the superposition operator F_f, generated by f, maps the space $H^\omega_q([0,1])$ into itself, and it is bounded.*

Proof. In similar to the proof Theorem 4.1, for every $u \in H^\omega_q([0,1])$ well-defined of $F_f(u)$ is obvious, since f is a 1-periodic function on $[0,1]$, it follows that $F_f(u)$ is 1-periodic function. On the other hand, since f is satisfied in the Lipschitz condition on $\mathbb{R}$, then for $t \in [0,1]$ we have

$$\begin{aligned}
\|F_f(u)\|_q &= \Big(\int_0^1 |F_f(u)(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&= \Big(\int_0^1 |f(t, u(t))|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\le \Big(\int_0^1 t^q N^q \mathrm{d}t\Big)^{\frac{1}{q}} = N, \quad u \in H^\omega_q([0,1]),
\end{aligned}$$

where N is some positive constant and hence $F_f(u) \in L^q([0,1])$.
According to the given assumptions, since ω is the modulus of continuity for $0 \le \delta \le 1$ and $F_f(u) \in L^q([0,1])$, $w(\delta)$ and $w_q(\delta, F_f(u))$ are meaningful. Therefore, since f is satisfied the Lipschitz condition on $\mathbb{R}$ and assumption $\sup_{\delta>0} \dfrac{\omega_q(\delta, f_u)}{\omega(\delta)} < \infty$, we have

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\omega_q(\delta, F_f(u))}{\omega(\delta)} &= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\big(\int_0^1 |F_f(u)(t+h) - F_f(u)(t)|^q \mathrm{d}t\big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\big(\int_0^1 |f(t+h, u(t+h)) - f(t, u(t))|^q \mathrm{d}t\big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&\le \lim_{\delta\to 0^+} \frac{1}{\omega(\delta)}\Big(\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |f(t+h, u(t+h)) - f(t+h, u(t))|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \Big(\int_0^1 |f(t+h, u(t)) - f(t, u(t))|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)\Big) \\
&\le \lim_{\delta\to 0^+} \frac{1}{\omega(\delta)}\Big(\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 M^q |u(t+h) - u(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \Big(\int_0^1 |f_u(t+h) - f_u(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)\Big) \\
&= \lim_{\delta\to 0^+} \frac{1}{\omega(\delta)}\Big(M\omega_q(\delta, u)\Big) + \lim_{\delta\to 0^+} \frac{\omega_q(\delta, f_u)}{\omega(\delta)} \\
&= M \lim_{\delta\to 0^+} \frac{\omega_q(\delta, u)}{\omega(\delta)} + \lim_{\delta\to 0^+} \frac{\omega_q(\delta, f_u)}{\omega(\delta)} < \infty.
\end{aligned}$$

It follows that $F_f(u) \in H_q^w([0,1])$. Hence we have

$$\|F_f(u)\|_{H_q^\omega} \le N + M \sup_{\delta>0} \frac{\omega_q(\delta, u)}{\omega(\delta)} + \sup_{\delta>0} \frac{\omega_q(\delta, f_u)}{\omega(\delta)},$$

which completes the proof. □

By Theorem 4.3, we can easily obtain locally bounded of superposition operators for the spaces of $H_q^w([0,1])$.

Corollary 4.4. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the given 1-periodic function on $[0,1]$ such that satisfies the local Lipschitz condition on $\mathbb{R}$ and $w : [0,1] \to [0,+\infty)$ be a modulus of continuity such that for function $u : [0,1] \to \mathbb{R}$ and $1 \le q < \infty$, $\sup_{\delta>0} \dfrac{\omega_q(\delta, f_u)}{\omega(\delta)} < \infty$ $\Big($where $f_u(t+h) = f(t+h, u(t))$, for $t \in [0,1]$, $h \in [0,\delta]$ and $f_u \in L^q\big([0,1]\big)\Big)$. Then the superposition operator F_f, generated by f, maps the space $H_q^w([0,1])$ into itself, and it is locally bounded.*

We can illustrate Theorem 4.3 with the following example.

Example 4.5. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ can be defined by $f(t,u) = f_1(t)f_2(u)$, where*

$$f_1(t) = \begin{cases} t^2 & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{t}{2} & \dfrac{1}{2} < t \le 1, \end{cases}$$

and for every $0 \le u \le 1$, $f_2(u) = u$. Also let $f_1(t)$ and $f_2(u)$ are 1-periodic functions. It is clearly that $f_2(u) \in H_q^w([0,1])$. Now for every $t \in [0, \frac{1}{2}]$ and $u_1, u_2 \in [0,1]$, we have

$$|f(t,u_1) - f(t,u_2)| = |t^2 u_1 - t^2 u_2| = |t^2||u_1 - u_2| \le \frac{1}{4}|u_1 - u_2|,$$

also for every $t \in (\frac{1}{2}, 1]$ and $u_1, u_2 \in [0,1]$, we have

$$|f(t,u_1) - f(t,u_2)| = |\frac{t}{2}u_1 - \frac{t}{2}u_2| = |\frac{t}{2}||u_1 - u_2| \leq \frac{1}{2}|u_1 - u_2|,$$

therefore $M = max\{\frac{1}{4}, \frac{1}{2}\} = \frac{1}{2}$ is constant lipschitz for the function f. Define $w : [0,1] \to \mathbb{R}$ by $w(\delta) = \sqrt{\delta}$, in which w is the modulus of continuity. In the following, for $1 \leq q < \infty$, we have

$$\begin{aligned}
\|f_u\|_q &= \Big(\int_0^1 |f_u(t)|^q dt\Big)^{\frac{1}{q}} \\
&= \Big(\int_0^1 |f(t,u)|^q dt\Big)^{\frac{1}{q}} \\
&= \Big(\int_0^1 |f_1(t)f_2(u)|^q dt\Big)^{\frac{1}{q}} \\
&\leq \Big(\int_0^{\frac{1}{2}} |t^2 u|^q dt\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 |\frac{t}{2}u|^q dt\Big)^{\frac{1}{q}} \\
&= u\Big(\int_0^{\frac{1}{2}} t^{2q} dt\Big)^{\frac{1}{q}} + \frac{u}{2}\Big(\int_{\frac{1}{2}}^1 t^q dt\Big)^{\frac{1}{q}} < \infty.
\end{aligned}$$

Therefore $f_u \in L^q([0,1])$. On the other hand, we have

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\omega_q(\delta, f_u)}{\omega(\delta)} &= \lim_{\delta\to 0^+} \frac{1}{\omega(\delta)}\Big(\sup_{0\leq h\leq\delta}\Big(\int_0^1 |f_u(t+h) - f_u(t)|^q dt\Big)^{\frac{1}{q}}\Big) \\
&\leq \lim_{\delta\to 0^+} \frac{1}{w(\delta)}\Big(\sup_{0\leq h\leq\delta}\Big(\Big(\int_0^{\frac{1}{2}} |f_u(t+h) - f_u(t)|^q dt\Big)^{\frac{1}{q}} \\
&+ \Big(\int_{\frac{1}{2}}^1 |f_u(t+h) - f_u(t)|^q dt\Big)^{\frac{1}{q}}\Big)\Big) \\
&= \lim_{\delta\to 0^+} \frac{1}{w(\delta)}\Big(\sup_{0\leq h\leq\delta}\Big(\Big(\int_0^{\frac{1}{2}} |(t+h)^2 u - t^2 u|^q dt\Big)^{\frac{1}{q}} \\
&+ \Big(\int_{\frac{1}{2}}^1 |\frac{t+h}{2}u - \frac{t}{2}u|^q dt\Big)^{\frac{1}{q}}\Big)\Big) \\
&\leq \lim_{\delta\to 0^+} \frac{1}{w(\delta)}\Big(\sup_{0\leq h\leq\delta}\Big(2\mathrm{hu}\Big(\int_0^{\frac{1}{2}} t^q dt\Big)^{\frac{1}{q}} \\
&+ h^2 u\Big(\int_0^{\frac{1}{2}} dt\Big)^{1/q} + \frac{h}{2}u\Big(\int_{\frac{1}{2}}^1 dt\Big)^{\frac{1}{q}}\Big)\Big) \\
&= \lim_{\delta\to 0^+} \frac{1}{w(\delta)}\Big(\sup_{0\leq h\leq\delta}\Big(2huN_1 + h^2 uN_2 + \frac{h}{2}uN_3\Big)\Big) \\
&= \lim_{\delta\to 0^+} \frac{2\delta u N_1}{\sqrt{\delta}} + \lim_{\delta\to 0^+} \frac{\delta^2 u N_2}{\sqrt{\delta}} + \lim_{\delta\to 0^+} \frac{\delta u N_3}{2\sqrt{\delta}} \\
&= 0 + 0 + 0 = 0,
\end{aligned}$$

where $N_1 = (\int_0^{\frac{1}{2}} t^q dt)^{\frac{1}{q}}$, $N_2 = (\int_0^{\frac{1}{2}} dt)^{\frac{1}{q}}$ and $N_3 = (\int_{\frac{1}{2}}^1 dt)^{\frac{1}{q}}$, so the assumptions of Theorem 4.3 are satisfied.

5. Applications to linear and nonlinear integral equations

The present section was aimed to investigate solutions of linear integral equations and nonlinear Hammerstein integral equations in the class of $H_q^w([0,1])$-functions.
Consider the nonlinear Hammerstein integral equation

$$x(t) = g(t) + \lambda \int_0^1 K(t,s)f(s,x(s))\mathrm{d}s \quad \text{for } t \in [0,1], \text{ and } \lambda \in \mathbb{R}. \quad (1)$$

Assume that:
(i) $g : [0,1] \to \mathbb{R}$ is a H_q^w-function;
(ii) $f : [0,1] \times \mathbb{R} \to \mathbb{R}, (t,v) \to f(t,v)$, is the given 1-periodic function on $[0,1]$ such that satisfies the Lipschitz condition on $\mathbb{R}$ and $w : [0,1] \to [0,+\infty)$ is a modulus of continuity such that for every $u : [0,1] \to \mathbb{R}$ and $1 < q < \infty$, $\sup_{\delta>0} \dfrac{\omega_q(\delta, f_u)}{\omega(\delta)} < \infty$;
(iii) $K : [0,1] \times [0,1] \to \mathbb{R}, (t,s) \to K(t,s)$, is a function such that $\int_0^1 |K(t+h,s) - K(t,s)|^q \mathrm{d}s \le (M(t+h) - M(t))^q$ for $h \in [0,\delta]$, where $M : [0,1] \to \mathbb{R}^+$ is belong to $H_q^w([0,1])$ and $K(t,\cdot)$ is belong to $L^q[0,1]$ for every $t \in [0,1]$ (to find out more about the nonlinear Hammerstein integral equation, see [1]).

Theorem 5.1. *Given the assumptions mentioned above, there is a number $\eta > 0$ such that for every λ with $|\lambda| < \eta$, Eq.(1) has a unique H_q^w-solution, defining on $[0,1]$.*

Proof. Let $g \in H_q^w([0,1])$ for $1 < q < \infty$ and $r > 0$ be such that $\|g\|_{H_q^\omega} < r$ and let L_r denote the Lipschitz constant which corresponds to the function f and interval $[-r,r]$. Define $I_r = [0,1] \times [-r,r]$. Choose a number $\eta > 0$ such that

$$\|g\|_{H_q^\omega} + \eta \sup_{(s,z)\in I_r} |f(s,z)| \Big(\Big(\int_0^1 N_t \mathrm{d}t \Big)^{\frac{1}{q}} + \sup_{\delta>0} \frac{\omega_q(\delta, M)}{\omega(\delta)} \Big) < r$$

and

$$\eta L_r \Big(\Big(\int_0^1 N_t \mathrm{d}t \Big)^{\frac{1}{q}} + \sup_{\delta>0} \frac{\omega_q(\delta, M)}{\omega(\delta)} \Big) < 1,$$

where $N_t = \int_0^1 |K(t,s)|^q \mathrm{d}s$ for $K(t,\cdot) \in L^q[0,1]$.
Denote by $\bar{B}_r$ the closed ball of center zero and radius r in the space $H_q^w([0,1])$. Put $|\lambda| < \eta$. Define $D(x)(t) = g(t) + \lambda P(x)(t)$, where

$$P(x)(t) = \int_0^1 K(t,s)f(s,x(s))\mathrm{d}s, \quad x \in \bar{B}_r, t \in [0,1].$$

According to Theorem 4.3, it is obvious that the superposition operator generated by the function f acts in the space $H_q^w([0,1])$ and it is bounded, therefore the operators D, P are well defined. For $u \in \bar{B}_r$, We have

$$\begin{aligned}
\|P(u)\|_q &= \Big(\int_0^1 |P(u)(t)|^q \mathrm{d}t \Big)^{\frac{1}{q}} \\
&= \Big(\int_0^1 \Big| \int_0^1 K(t,s)f(s,u(s))\mathrm{d}s \Big|^q \mathrm{d}t \Big)^{\frac{1}{q}} \\
&\le \sup_{s\in[0,1]} \big| f(s,u(s)) \big| \Big(\int_0^1 \Big(\int_0^1 \big| K(t,s) \big| \mathrm{d}s \Big)^q \mathrm{d}t \Big)^{\frac{1}{q}} \\
&\le \sup_{s\in[0,1]} \big| f(s,u(s)) \big| \Big(\int_0^1 \int_0^1 \big| K(t,s) \big|^q \mathrm{d}s\mathrm{d}t \Big)^{\frac{1}{q}} \\
&= \sup_{s\in[0,1]} \big| f(s,u(s)) \big| \Big(\int_0^1 N_t \mathrm{d}t \Big)^{\frac{1}{q}},
\end{aligned}$$

so $P(u) \in L^q([0,1]$. In addition, we have

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\omega_q(\delta, P(u))}{\omega(\delta)} &= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |P(u)(t+h) - P(u)(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&= \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |\int_0^1 (K(t+h,s)f(s,u(s)) - K(t,s)f(s,u(s)))\mathrm{d}s|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&\le \sup_{s\in[0,1]} |f(s,u(s))| \lim_{\delta\to 0^+} \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1\int_0^1 |K(t+h,s) - K(t,s)|^q \mathrm{d}s\mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)} \\
&\le \sup_{s\in[0,1]} |f(s,u(s))| \lim_{\delta\to 0^+} \frac{1}{\omega(\delta)}\Big(\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 \Big(M(t+h) - M(t)\Big)^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)\Big) \\
&= \sup_{s\in[0,1]} |f(s,u(s))| \lim_{\delta\to 0^+} \frac{\omega_q(\delta, M)}{\omega(\delta)} < \infty.
\end{aligned}$$

Therefore $P(u) \in H_q^w([0,1])$. Moreover, we obtain

$$\begin{aligned}
\|D(u)\|_{H_q^\omega} &= \|g + \lambda P(u)\|_{H_q^\omega} \\
&\le \|g\|_{H_q^\omega} + |\lambda|\|P(u)\|_{H_q^\omega} \\
&= \|g\|_{H_q^\omega} + |\lambda|\Big(\|P(u)\|_q + \sup_{\delta>0} \frac{\omega_q(\delta, P(u))}{\omega(\delta)}\Big) \\
&\le \|g\|_{H_q^\omega} + |\lambda|\Big(\sup_{s\in[0,1]} |f(s,u(s))|\Big(\int_0^1 N_t \mathrm{d}t\Big)^{\frac{1}{q}} + \sup_{s\in[0,1]} |f(s,u(s))| \sup_{\delta>0} \frac{\omega_q(\delta, M)}{\omega(\delta)}\Big) \\
&= \|g\|_{H_q^\omega} + |\lambda| \sup_{(s,z)\in I_r} |f(s,z)|\Big(\Big(\int_0^1 N_t \mathrm{d}t\Big)^{\frac{1}{q}} + \sup_{\delta>0} \frac{\omega_q(\delta, M)}{\omega(\delta)}\Big) < r,
\end{aligned}$$

therefore $D(\bar{B}_r) \subset \bar{B}_r$.
Now we show that D is a contraction. For any $u, v \in \bar{B}_r$ we have

$\|D(u) - D(v)\|_{H_q^\omega}$

$$\begin{aligned}
&= \|g + \lambda P(u) - g - \lambda P(v)\|_{H_q^\omega} \\
&= |\lambda|\|P(u) - P(v)\|_{H_q^\omega} \\
&= |\lambda|\Big(\|P(u) - P(v)\|_q + \sup_{\delta>0} \frac{\omega_q(\delta, P(u) - P(v))}{\omega(\delta)}\Big) \\
&= |\lambda|\Big(\Big(\int_0^1 \Big|(P(u) - P(v))(t)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |(P(u) - P(v))(t+h) - (P(u) - P(v))(t)|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)}\Big)
\end{aligned}$$

$$= |\lambda|\Big(\int_0^1 \Big|\int_0^1 K(t,s)f(s,u(s)) - K(t,s)f(s,v(s))\mathrm{d}s\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+ \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 |\int_0^1 K(t+h,s)(f(s,u(s)) - f(s,v(s))) - K(t,s)(f(s,u(s)) - f(s,v(s)))\mathrm{d}s|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)}\Big)$$
$$\le |\lambda|\Big(\sup_{s\in[0,1]} |f(s,u(s)) - f(s,v(s))|\Big(\int_0^1\int_0^1 |K(t,s)|^q \mathrm{d}s\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+ \sup_{s\in[0,1]} |f(s,u(s)) - f(s,v(s))| \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1\int_0^1 |K(t+h,s) - K(t,s)|^q \mathrm{d}s\mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)}\Big)$$
$$\le |\lambda|\Big(\sup_{s\in[0,1]} |f(s,u(s)) - f(s,v(s))|\Big(\Big(\int_0^1 N_t \mathrm{d}t\Big)^{\frac{1}{q}} + \frac{\sup_{0\le h\le\delta}\Big(\Big(\int_0^1 \big(M(t+h) - M(t)\big)^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big)}{\omega(\delta)}\Big)\Big)$$
$$\le |\lambda| L_r \sup_{s\in[0,1]} |u(s)) - v(s)|\Big(\Big(\int_0^1 N_t \mathrm{d}t\Big)^{\frac{1}{q}} + \sup_{\delta>0}\frac{\omega_q(\delta,M)}{\omega(\delta)}\Big)$$
$$\le |\lambda| L_r \|u - v\|_{H_q^\omega}\Big(\Big(\int_0^1 N_t \mathrm{d}t\Big)^{\frac{1}{q}} + \sup_{\delta>0}\frac{\omega_q(\delta,M)}{\omega(\delta)}\Big) \le \|u - v\|_{H_q^\omega}.$$

Considering the Banach contraction principle it is concluded that D has a unique fixed point in $\bar{B}_r$, which is a H_q^w-solution of *Eq*.(1), defined on $[0,1]$. □

Now let us consider the nonlinear Hammerstein integral equation

$$x(t) = g(t) + \lambda\int_0^1 K(t,s)f(x(s))\mathrm{d}s \quad \text{for } t\in[0,1], \text{ and } \lambda\in\mathbb{R}. \quad (2)$$

Suppose that (i) and (iii) are satisfied. Assume also that:
(iv) $f:[0,1]\to\mathbb{R}, t\to f(t)$, is a given function such that satisfies the Lipschitz condition on $\mathbb{R}$ and $w:[0,1]\to[0,+\infty)$ to be a modulus of continuity.
Similarly, Theorems 4.1 is valid for nonlinear integral equations of Hammerstein type.

Theorem 5.2. *Given the assumptions mentioned above, there is a number $\eta > 0$ such that for every λ with $|\lambda| < \eta$, Eq.*(2) *has a unique H_q^w-solution, defining on* $[0,1]$.

Let K be a linear integral operator generated by the kernel $k:[0,1]\times[0,1]\to\mathbb{R}$ as follows

$$Kx(t) = \int_0^1 k(t,s)x(s)\mathrm{d}s, \quad t\in[0,1].$$

These operators, considered on spaces of functions of bounded variation in the sense of Jordan and Wiener, have been extensively studied in [5,6]. Gulgowski [12] has recently studied these linear operators on spaces of $IBV_1^q(I)$.
In the following theorem, we are going to present sufficient conditions for the continuity of the integral operator $K: L^q([0,1])\to H_q^w([0,1])$.

Theorem 5.3. *Let $1 < q < +\infty$ and $\frac{1}{q}+\frac{1}{q'} = 1$. Assume that the function $k:[0,1]\times[0,1]\to\mathbb{R}$ satisfies the following conditions:*
(i) k is a Lebesgue measurable function on $[0,1]\times[0,1]$;
(ii) there exists a function $m_0\in L^{q'}([0,1])$ such that $\|k(\cdot,s)\|_{L^q}\le m_0(s)$ for a.e. $s\in[0,1]$;
(iii) there exists a function $m\in L^{q'}([0,1])$ such that $\Big(\int_0^1 |k(t+h,s)-k(t,s)|^q dt\Big)^{\frac{1}{q}} \le \Big(m(s+h)-m(s)\Big)$ and $\sup_{\delta>0}\frac{w_{q'}(\delta,m)}{w(\delta)} < \infty$ for a.e. $s\in[0,1]$.
Then the linear operator K, generated by k, maps $L^q([0,1])$ to $H_q^w([0,1])$ and it is continuous.

Proof. First, we show that K is well-defined. For do this, according to the generalized Minkowski inequality (see [13, Theorem 202]) for each $x \in L^q([0,1])$ and a.e. $t \in [0,1]$ we have

$$\begin{aligned}
\|K(x)\|_{L^q} &= \left(\int_0^1 |K(x)(t)|^q dt\right)^{\frac{1}{q}} \\
&= \Big(\int_0^1 \Big|\int_0^1 k(t,s)x(s)ds\Big|^q dt\Big)^{\frac{1}{q}} \\
&\leq \int_0^1 |x(s)| \left(\int_0^1 |k(t,s)|^q dt\right)^{\frac{1}{q}} ds \\
&\leq \int_0^1 |x(s)| m_0(s) ds \\
&\leq \|x\|_{L^q} \, \|m_0\|_{L^{q'}} .
\end{aligned}$$

Therefore, $K(x) \in L^q([0,1])$.
Now, we apply the generalized Minkowski once more and we have

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{w_q(\delta, K(x))}{w(\delta)} &= \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 |K(x)(t+h) - K(x)(t)|^q dt\right)^{\frac{1}{q}}}{w(\delta)} \\
&= \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 \Big|\int_0^1 k(t+h,s)x(s) - k(t,s)x(s)ds\Big|^q dt\right)^{\frac{1}{q}}}{w(\delta)} \\
&\leq \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 \Big(\int_0^1 \Big|k(t+h,s)x(s) - k(t,s)x(s)\Big|^q dt\Big)^{\frac{1}{q}} ds\right)}{w(\delta)} \\
&\leq \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 |x(s)|\Big(\int_0^1 \Big|k(t+h,s) - k(t,s)\Big|^q dt\Big)^{\frac{1}{q}} ds\right)}{w(\delta)} \\
&\leq \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 |x(s)|(m(s+h) - m(s))ds\right)}{w(\delta)} \\
&\leq \|x\|_{L^q} \lim_{\delta\to 0^+} \frac{\sup_{0\leq h\leq\delta}\left(\int_0^1 (m(s+h) - m(s))^{q'} ds\right)^{\frac{1}{q'}}}{w(\delta)} \\
&= \|x\|_{L^q} \lim_{\delta\to 0^+} \frac{w_{q'}(\delta, m)}{w(\delta)} < \infty.
\end{aligned}$$

Therefore, $K(x) \in H_q^w([0,1])$.
In the following, let $u \in \bar{B}_r$, where $\bar{B}_r$ denotes a closed ball of radius $r > 0$ and center 0 in the space $L^q([0,1])$. We have

$$\begin{aligned}
\|K(u)\|_{H_q^w} &= \|K(u)\|_{L^q} + \sup_{\delta>0} \frac{w_q(\delta, K(u)}{w(\delta)} \\
&\leq \|u\|_{L^q} \, \|m_0\|_{L^{q'}} + \|u\|_{L^q} \sup_{\delta>0} \frac{w_{q'}(\delta, m)}{w(\delta)} \\
&= \|u\|_{L^q} \left(\|m_0\|_{L^{q'}} + \sup_{\delta>0} \frac{w_{q'}\,(\delta, m))}{w(\delta)}\right).
\end{aligned}$$

Which completes the proof. $\square$

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