Filomat 37:5 (2023), 1673–1686
https://doi.org/10.2298/FIL2305673I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Compact Operators on Cesàro sequence spaces and norms of Cesàro operators

Merve İlkhan Kara[a], **Hadi Roopaei**[b]

[a] *Düzce University, Faculty of Arts and Sciences, Department of Mathematics, Düzce, Turkey*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, Canada*

Abstract. This paper deals with the characterization of compact operators on Cesàro sequence spaces as an application of Hausdorff measure of noncompactness. Further, the norms of Cesàro operators on certain spaces are investigated.

1. Introduction and background

In the realm of functional analysis, the method constructing a new sequence space by the aid of matrix domain of a particular summability matrix has recently been studied by several authors. For the relevant literature, see [1–6].

Characterization of compact operators on matrix domains is one of the fundamental application of Hausdorff measure of non-compactness in the theory of sequence spaces. Recently, many fascinating results have been presented in this theory (see [7–12]).

The results related to norms of matrix operators on sequence spaces go back to the theorems of Hardy, Copson and Hilbert. The problem of finding the norm and the upper bounds of certain matrix operators on different sequence spaces are studied by [13–17].

The main purpose of this study is to characterize the compact operators on the matrix domain of the Cesàro matrix of order n by using the concept of the Hausdorff measure of non-compactness. Moreover, it is aimed to compute the norms of Cesàro operators on Hilbert sequence space, difference sequence space and Hausdorff matrix domains.

2. Known Results

Let ω be the space of all real or complex valued sequences. If $X \subset \omega$, then X is called a sequence space. The most used classical sequence spaces are the space of all p-absolutely summable sequences ℓ_p $(1 \leq p < \infty)$, all convergent sequences c, all convergent to zero sequences c_0 and all bounded sequences ℓ_∞. These spaces are Banach spaces endowed with the norms

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p}, \quad \|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_k |x_k|.$$

2020 *Mathematics Subject Classification*. Primary 46B45; Secondary 46A45, 40G05, 47B37
Keywords. Cesàro sequence space, compact operators, Hausdorff matrix, Hilbert matrix
Received: 23 October 2020; Accepted: 20 November 2020
Communicated by In Sung Hwang
Email addresses: merveilkhan@gmail.com (Merve İlkhan Kara), h.roopaei@gmail.com (Hadi Roopaei)

Also, ψ denotes the set of all finite sequences.

Through this study we suppose that $1 < p < \infty$.

A Banach space X is called a BK-space if the mapping $\tilde{I}_j : X \to \mathbb{R}$ defined by $\tilde{I}_j(x) = x_j$ is continuous for each $j \in \mathbb{N}$. Let $e^0 = (1,0,0,...)$, $e^1 = (0,1,0,...)$, $e^2 = (0,0,1,...)$, $\cdots$, $e^k = (0,0,...,0,1,0,...)$, $\cdots$. If $x = (x_k) \in X$ is written uniquely as $x = \sum_k x_k e^k$, then it is said that the BK-space X satisfies the AK-property. ℓ_p $(1 \leq p < \infty)$ and c_0 satisfies AK-property but c and ℓ_∞ do not satisfy this property.

The β-dual of a sequence space X is defined by

$$X^\beta = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k u_k \text{ converges for all } u = (u_k) \in X \right\}.$$

Let X and Y be sequence spaces and $T = (t_{j,k})$ be an infinite matrix of real or complex numbers $t_{j,k}$. Then T gives a matrix transformation from X into Y and we write $T : X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $Tx = (T_j(x))$, the $T-$transform of x, is in Y, where

$$T_j(x) = \sum_k t_{j,k} x_k \quad (j \in \mathbb{N}).$$

Throughout the study, T_j will be the sequence of j^{th} row of an infinite matrix $T = (t_{j,k})$. By (X, Y), we denote the class of all infinite matrices that map X into Y. Hence, $T \in (X, Y)$ if and only if $T_j \in X^\beta$ for all $j \in \mathbb{N}$.

The matrix domain of an infinite matrix T in the sequence space ℓ_p is defined as

$$T_p = \{x \in \omega : Tx \in \ell_p\}$$

which is also a sequence space. If T is a triangle, then this new sequence space is also a normed space by the induced norm $\|x\|_{T_p} = \|Tx\|_{\ell_p}$, ([18], Theorem 4.3.12]). It is easy to see that for any bounded matrix T the inclusion $\ell_p \subset T_p$ holds.

Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^{\infty}$, with entries of the form:

$$h_{j,k} = \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) & \text{if} \quad 0 \leq k \leq j \\ 0 & \text{if} \quad k > j. \end{cases}$$

where μ is a probability measure on $[0, 1]$. The Hausdorff matrix contains the famous classes of matrices. For real $\alpha > 0$, these classes are as follow:

(i) The choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1} d\theta$ gives the Cesàro matrix of order α;
(ii) The choice $d\mu(\theta) = \alpha\theta^{\alpha-1} d\theta$ gives the Gamma matrix of order α;
(iii) The choice $d\mu(\theta) = \frac{|\log \theta|^{\alpha-1}}{\Gamma(\alpha)} d\theta$ gives the Hölder matrix of order α;
(iv) The choice $d\mu(\theta) =$ *point evaluation at* $\theta = \alpha$ gives the Euler matrix of order α.

Hardy's formula ([19], Theorem 216) states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta). \tag{1}$$

By letting $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix $C^n = (c_{j,k}^n)$ of order n is defined as follows

$$c_{j,k}^n = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leq k \leq j \\ 0 & \textit{otherwise}, \end{cases} \tag{2}$$

which has the norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}, \tag{3}$$

where p^* is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$. This follows from (1) and (i) using a known formula for Euler's Beta function. Note that, C^1 is the well-known Cesàro matrix C with $\|C\|_{\ell_p} = p^*$.

The following matrix domains are the sequence spaces associated with the Cesàro matrix of order n.

$$C_p^n = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\}$$

and

$$C_\infty^n = \left\{ x = (x_j) \in \omega : \sup_j \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right| < \infty \right\}.$$

The sequence $y = (y_j)$ will denote the C^n-transform of a sequence $x = (x_j)$; that is,

$$y_j = (C^n x)_j = \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \tag{4}$$

for all $j \in \mathbb{N}$.

The Hausdorff measure of noncompactness of a bounded set A is denoted by $\chi(A)$ and defined as

$$\chi(A) = \inf\{r > 0 : A \subset \cup_{k=1}^{j} B(x_k, r_k), x_k \in X, r_k < r, j \in \mathbb{N}\},$$

where $B(x_k, r_k)$ is the open ball centered at x_k and radius r_k for each $k = 1, 2, ..., j$. For basic properties of Hausdorff measure of noncompactness, we refer to [20] and references therein.

Let $L : X \to Y$ be a linear operator. We call L as compact if the domain of L is whole of X and for any bounded sequence $x = (x_j)$ in X, the sequence $(L(x_j))$ has a convergent subsequence in Y. If L is a bounded linear operator, then the value

$$\|L\|_\chi = \chi(L(\{x \in X : \|x\| = 1\}))$$

is called the Hausdorff measure of noncompactness of the operator L. There is a close relation between the concepts of the Hausdorff measure of noncompactness and compact operators. Also, we have by [21, Corollary 1.15] that

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{5}$$

Let $(X, \|.\|)$ be a BK-space and $u = (u_k) \in \omega$. The notation $\|.\|_X^*$ means that

$$\|u\|_X^* = \sup_{x \in X, \|x\|=1} \left| \sum_k u_k x_k \right| < \infty.$$

Further, this implies that $u \in X^\beta$.

Lemma 2.1. *[20, Theorem 1.29]*

(a) $\ell_p^\beta = \ell_{p^*}$ *and* $\|u\|_{\ell_p}^* = \|u\|_{\ell_{p^*}}$ *for* $1 < p < \infty$.

(b) $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ *and* $\|u\|_{\ell_\infty}^* = \|u\|_c^* = \|u\|_{c_0}^* = \|u\|_{\ell_1}$.

(c) $\ell_1^\beta = \ell_\infty$ *and* $\|u\|_{\ell_1}^* = \|u\|_{\ell_\infty}$.

By $\mathcal{B}(X, Y)$, we mean the family of all bounded linear operators $L : X \to Y$.

Lemma 2.2. *[20, Theorem 1.23 (a)] Let X and Y be BK-spaces. Then every $T \in (X, Y)$, defines an operator $L_T \in \mathcal{B}(X, Y)$ such that $L_T(x) = Tx$ for all $x \in X$.*

The following result is used to determine the Hausdorff measure of noncompactness in the spaces ℓ_p.

Theorem 2.3. *[22, Theorem 2.8] Let A be a bounded subset in ℓ_p and $P_l : \ell_p \to \ell_p$ be the operator defined by $P_l(x) = (x_1, x_2, ..., x_l, 0, 0, ...)$ for all $x = (x_k) \in \ell_p$ and each $l \in \mathbb{N}$. Then, we have*

$$\chi(A) = \lim_{l\to\infty}\left(\sup_{x\in A} \|(I - P_l)(x)\|_{\ell_p}\right),$$

where I is the identity operator on ℓ_p.

Lemma 2.4. *[7, Theorem 3.7] If $X \supset \psi$ is a BK-space, then the following statements hold.*
(a) *$T \in (X, \ell_\infty)$, then $0 \leq \|L_T\|_\chi \leq \limsup_j \|T_j\|^*_X$.*
(b) *$T \in (X, c_0)$, then $\|L_T\|_\chi = \limsup_j \|T_j\|^*_X$.*
(c) *If X has AK or $X = \ell_\infty$ and $T \in (X, c)$, then*

$$\frac{1}{2}\limsup_n \|T_j - t\|^*_X \leq \|L_T\|_\chi \leq \limsup_j \|T_j - t\|^*_X,$$

where $t = (t_k)$ and $t_k = \lim_j t_{j,k}$ for each $k \in \mathbb{N}$.

By $\mathcal{N}$, we denote the collection of all finite subsets of $\mathbb{N}$ and by $\mathcal{N}_l$, we denote the sub-collection of $\mathcal{N}$ with elements that are greater than l.

Lemma 2.5. *[7, Theorem 3.11] Let $X \supset \psi$ be a BK-space. If $T \in (X, \ell_1)$, then*

$$\lim_l\left(\sup_{N\in\mathcal{N}_l}\left\|\sum_{j\in N} T_j\right\|^*_X\right) \leq \|L_T\|_\chi \leq 4\lim_l\left(\sup_{N\in\mathcal{N}_l}\left\|\sum_{j\in N} T_j\right\|^*_X\right)$$

and L_T is compact if and only if $\lim_l\left(\sup_{N\in\mathcal{N}_l}\|\sum_{j\in N} T_j\|^*_X\right) = 0$.

3. Compact Operators on Cesàro Sequence Spaces

The following auxiliary results are required in order to prove our main results.

Lemma 3.1. *If $u = (u_k) \in \{C^n_p\}^\beta$ with $1 \leq p \leq \infty$, then we have*

$$\sum_k u_k x_k = \sum_k \tilde{u}_k y_k \tag{6}$$

for all $x = (x_k) \in C^n_p$ and also $\tilde{u} = (\tilde{u}_k) \in \ell^\beta_p$, where

$$\tilde{u}_k = \sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}u_i \quad (k \in \mathbb{N}). \tag{7}$$

In the rest of the study, the infinite matrix $\tilde{T} = (\tilde{t}_{j,k})$ with entries

$$\tilde{t}_{j,k} = \sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}$$

is used under the assumption that the series is convergent, where $T = (t_{j,k})$ is a given matrix.

Lemma 3.2. *Let Y be an arbitrary subset of ω. If $T \in (C_p^n, Y)$, then $\tilde{T} \in (\ell_p, Y)$ and $Tx = \tilde{T}y$ for all $x \in C_p^n$, where $1 \le p \le \infty$.*

Proof. It follows from Lemma 3.1. □

Theorem 3.3. *Consider the sequence $\tilde{u} = (\tilde{u}_k)$ given in (7).*

(a) $\|u\|^*_{C_p^n} = \left(\sum_k |\tilde{u}_k|^{p^*}\right)^{1/p^*} < \infty$ *for all $u = (u_k) \in \{C_p^n\}^\beta$ and $1 < p < \infty$.*

(b) $\|u\|^*_{C_\infty^n} = \sum_k |\tilde{u}_k| < \infty$ *for all $u = (u_k) \in \{C_\infty^n\}^\beta$.*

(c) $\|u\|^*_{C_1^n} = \sup_k |\tilde{u}_k| < \infty$ *for all $u = (u_k) \in \{C_1^n\}^\beta$.*

Proof.

(a) Let $u = (u_k) \in \{C_p^n\}^\beta$. Then, from Lemma 3.1, we have $\tilde{u} = (\tilde{u}_k) \in \ell_{p^*}$ and the equality (6) holds. Since $\|x\|_{C_p^n} = \|y\|_{\ell_p}$ holds, it follows that

$$\|u\|^*_{C_p^n} = \sup_{x \in C_p^n, \|x\|_{C_p^n}=1} \left|\sum_k u_k x_k\right| = \sup_{y \in \ell_p, \|y\|_{\ell_p}=1} \left|\sum_k \tilde{u}_k y_k\right| = \|\tilde{u}\|^*_{\ell_p}.$$

Hence, from Lemma 2.1 (a), we deduce that $\|u\|^*_{C_p^n} = \|\tilde{u}\|^*_{\ell_p} = \|\tilde{u}\|_{\ell_{p^*}} = \left(\sum_k |\tilde{u}_k|^{p^*}\right)^{1/p^*} < \infty$.

(b) If $u = (u_k) \in \{C_\infty^n\}^\beta$, we deduce from Lemma 2.1 (b) that $\|u\|^*_{C_\infty^n} = \|\tilde{u}\|^*_{\ell_\infty} = \|\tilde{u}\|_{\ell_1} = \sum_k |\tilde{u}_k| < \infty$.

(c)If $u = (u_k) \in \{C_1^n\}^\beta$, we deduce from Lemma 2.1 (c) that $\|u\|^*_{C_1^n} = \|\tilde{u}\|^*_{\ell_1} = \|\tilde{u}\|_{\ell_\infty} = \sup_k |\tilde{u}_k| < \infty$. □

Now, we are ready to characterize compact operators.

Theorem 3.4. *Let $1 < p < \infty$.*

(a) If $T \in (C_p^n, \ell_\infty)$, then we have

$$0 \le \|L_T\|_\chi \le \limsup_j \left(\sum_k \left|\sum_{i=k}^{\infty} (-1)^{(i-k)} \binom{n}{i-k} \binom{n+k}{k} t_{j,i}\right|^{p^*}\right)^{1/p^*}.$$

(b) If $T \in (C_p^n, c_0)$, then we have

$$\|L_T\|_\chi = \limsup_j \left(\sum_k \left|\sum_{i=k}^{\infty} (-1)^{(i-k)} \binom{n}{i-k} \binom{n+k}{k} t_{j,i}\right|^{p^*}\right)^{1/p^*}.$$

(c) If $T \in (C_p^n, \ell_1)$, then we have

$$\lim_l \|T\|^{(l)}_{(C_p^n, \ell_1)} \le \|L_T\|_\chi \le 4 \lim_l \|T\|^{(l)}_{(C_p^n, \ell_1)},$$

where $\|T\|^{(l)}_{(C_p^n, \ell_1)} = \sup_{N \in \mathcal{N}_l} \left(\sum_k \left|\sum_{j \in N} \sum_{i=k}^{\infty} (-1)^{(i-k)} \binom{n}{i-k} \binom{n+k}{k} t_{j,i}\right|^{p^*}\right)^{1/p^*}$ *for each $l \in \mathbb{N}$.*

Proof.

(a) If T is a mapping from C_p^n to ℓ_∞ and $x = (x_k) \in C_p^n$, then the series $\sum_k t_{j,k} x_k$ converges. This means $T_j \in \{C_p^n\}^\beta$ for each $j \in \mathbb{N}$. By Theorem 3.3 (a), we write $\|T_j\|^*_{C_p^n} = \|\tilde{T}_j\|^*_{\ell_p} = \|\tilde{T}_j\|_{\ell_{p^*}} = \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}$ for each $j \in \mathbb{N}$, where $\tilde{t}_{j,k} = \sum_{i=k}^{\infty} (-1)^{(i-k)} \binom{n}{i-k} \binom{n+k}{k} t_{j,i}$. Thus, Lemma 2.4 (a) yields that

$$0 \le \|L_T\|_\chi \le \limsup_j \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}.$$

(b) Let T be a mapping from C_p^n to c_0. Since we have $\|T_j\|_{C_p^n}^* = \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}$ for each $j \in \mathbb{N}$, Lemma 2.4 (b) yields that

$$\|L_T\|_\chi = \limsup_j \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}.$$

(c) Let T be a mapping from C_p^n to ℓ_1. By Lemma 3.2, we have $\tilde{T} \in (\ell_p, \ell_1)$. It follows from Lemma 2.5 that

$$\lim_l \left(\sup_{N \in \mathcal{N}_l} \left\| \sum_{j \in N} \tilde{T}_j \right\|_{\ell_p}^*\right) \leq \|L_T\|_\chi \leq 4 \lim_l \left(\sup_{N \in \mathcal{N}_l} \left\| \sum_{j \in N} \tilde{T}_j \right\|_{\ell_p}^*\right).$$

Finally, from Lemma 2.1, we conclude that $\|\sum_{j \in N} \tilde{T}_j\|_{\ell_p}^* = \|\sum_{j \in N} \tilde{T}_j\|_{\ell_{p^*}} = \left(\sum_k |\sum_{j \in N} \tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}$.
□

Theorem 3.5. *If $T \in (C_p^n, c)$, then we have*

$$\frac{1}{2} \limsup_j \left(\sum_k |\tilde{t}_{j,k} - \tilde{t}_k|^{p^*}\right)^{1/p^*} \leq \|L_T\|_\chi \leq \limsup_j \left(\sum_k |\tilde{t}_{j,k} - \tilde{t}_k|^{p^*}\right)^{1/p^*},$$

where $\tilde{t} = (\tilde{t}_k)$ and $\tilde{t}_k = \lim_j \tilde{t}_{j,k}$ for each $k \in \mathbb{N}$.

Proof. If $T \in (C_p^n, c)$, Lemma 3.2 implies that $\tilde{T} \in (\ell_p, c)$. Hence, it follows from Lemma 2.4 (c) that

$$\frac{1}{2} \limsup_j \|\tilde{T}_j - \tilde{t}\|_{\ell_p}^* \leq \|L_T\|_\chi \leq \limsup_j \|\tilde{T}_j - \tilde{t}\|_{\ell_p}^*,$$

where $\tilde{t} = (\tilde{t}_k)$ and $\tilde{t}_k = \lim_j \tilde{t}_{j,k}$ for each $k \in \mathbb{N}$. From Lemma 2.1 (a), we conclude that $\|\tilde{T}_j - \tilde{t}\|_{\ell_p}^* = \|\tilde{T}_j - \tilde{t}\|_{\ell_{p^*}} = \left(\sum_k |\tilde{t}_{j,k} - \tilde{t}_k|^{p^*}\right)^{1/p^*}$ for each $j \in \mathbb{N}$. □

From (5), we have the following result.

Corollary 3.6. *Let $1 < p < \infty$.*

(a) L_T is compact for $T \in (C_p^n, \ell_\infty)$ if

$$\lim_j \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*} = 0.$$

(b) L_T is compact for $T \in (C_p^n, c)$ if and only if

$$\lim_j \left(\sum_k |\tilde{t}_{j,k} - \tilde{t}_k|^{p^*}\right)^{1/p^*} = 0.$$

(c) L_T is compact for $T \in (C_p^n, c_0)$ if and only if

$$\lim_j \left(\sum_k |\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*} = 0.$$

(d) *L_T is compact for $T \in (C_p^n, \ell_1)$ if and only if*

$$\lim_l \|T\|^{(l)}_{(C_p^n,\ell_1)} = 0,$$

where $\|T\|^{(l)}_{(C_p^n,\ell_1)} = \sup_{N\in\mathcal{N}_l}\left(\sum_k |\sum_{j\in N}\tilde{t}_{j,k}|^{p^*}\right)^{1/p^*}$.

Theorem 3.7.

1. *If $T \in (C_\infty^n, \ell_\infty)$, then we have*

$$0 \le \|L_T\|_\chi \le \limsup_j \sum_k \left|\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|.$$

2. *If $T \in (C_\infty^n, c_0)$, then we have*

$$\|L_T\|_\chi = \limsup_j \sum_k \left|\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|.$$

3. *If $T \in (C_\infty^n, \ell_1)$, then we have*

$$\lim_l \|T\|^{(l)}_{(C_\infty^n,\ell_1)} \le \|L_T\|_\chi \le 4\lim_l \|T\|^{(l)}_{(C_\infty^n,\ell_1)},$$

where $\|T\|^{(l)}_{(C_\infty^n,\ell_1)} = \sup_{N\in\mathcal{N}_l}\left(\sum_k \left|\sum_{j\in N}\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|\right)$ $(l \in \mathbb{N})$.

Proof. It follows in the same manner if one consider Theorem 3.3 (b) instead of (a) in the proof of Theorem 3.4. □

Theorem 3.8. *If $T \in (C_\infty^n, c)$, then we have*

$$\frac{1}{2}\limsup_j \sum_k |\tilde{t}_{j,k} - \tilde{t}_k| \le \|L_T\|_\chi \le \limsup_j \sum_k |\tilde{t}_{j,k} - \tilde{t}_k|,$$

where $\tilde{t} = (\tilde{t}_k)$ and $\tilde{t}_k = \lim_j \tilde{t}_{j,k}$ for each $k \in \mathbb{N}$.

Proof. By Lemma 2.1 (b), we have $\|\tilde{T}_j - \tilde{t}\|^*_{\ell_\infty} = \|\tilde{T}_j - \tilde{t}\|_{\ell_1} = \sum_k |\tilde{t}_{j,k} - \tilde{t}_k|$ for each $j \in \mathbb{N}$. Hence, the proof follows in the same manner with the proof of Theorem 3.5. □

Similarly, the following result is given.

Corollary 3.9.

1. *L_T is compact for $T \in (C_\infty^n, \ell_\infty)$ if*

$$\lim_j \sum_k |\tilde{t}_{j,k}| = 0.$$

2. *L_T is compact for $T \in (C_\infty^n, c)$ if and only if*

$$\lim_j \sum_k |\tilde{t}_{j,k} - \tilde{t}_k| = 0.$$

3. *L_T is compact for $T \in (C_\infty^n, c_0)$ if and only if*

$$\lim_j \sum_k |\tilde{t}_{j,k}| = 0.$$

4. *L_T is compact for $T \in (C_\infty^n, \ell_1)$ if and only if*

$$\lim_l \|T\|^{(l)}_{(C_\infty^n,\ell_1)} = 0,$$

where $\|T\|^{(l)}_{(C_\infty^n,\ell_1)} = \sup_{N\in\mathcal{N}_l}\left(\sum_k |\sum_{j\in N}\tilde{t}_{j,k}|\right)$.

Theorem 3.10.

1. *If $T \in (C_1^n, \ell_\infty)$, then we have*

$$0 \leq \|L_T\|_\chi \leq \limsup_j \left(\sup_k \left|\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|\right).$$

2. *If $T \in (C_1^n, c_0)$, then we have*

$$\|L_T\|_\chi = \limsup_j \left(\sup_k \left|\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|\right).$$

3. *If $T \in (C_1^n, \ell_1)$, then we have*

$$\|L_T\|_\chi = \lim_l \left(\sup_k \sum_{j=l}^{\infty}\left|\sum_{i=k}^{\infty}(-1)^{(i-k)}\binom{n}{i-k}\binom{n+k}{k}t_{j,i}\right|\right).$$

Proof. It follows in the same manner if one consider Theorem 3.3 (c) instead of (a) in the proof of Theorem 3.4.
□

Theorem 3.11. *If $T \in (C_1^n, c)$, then we have*

$$\frac{1}{2}\limsup_j \left(\sup_k |\tilde{t}_{j,k} - \tilde{t}_k|\right) \leq \|L_T\|_\chi \leq \limsup_j \left(\sup_k |\tilde{t}_{j,k} - \tilde{t}_k|\right).$$

Proof. By Lemma 2.1 (c), we have $\|\tilde{T}_j - \tilde{t}\|^*_{\ell_1} = \|\tilde{T}_j - \tilde{t}\|_{\ell_\infty} = \sup_k |\tilde{t}_{j,k} - \tilde{t}_k|$ for each $j \in \mathbb{N}$. Hence, the proof follows in the same manner with the proof of Theorem 3.5. □

Similarly, the following result is given.

Corollary 3.12.

1. *L_T is compact for $T \in (C_1^n, \ell_\infty)$ if*

$$\lim_j \left(\sup_k |\tilde{t}_{j,k}|\right) = 0.$$

2. *L_T is compact for $T \in (C_1^n, c)$ if and only if*

$$\lim_j \left(\sup_k |\tilde{t}_{j,k} - \tilde{t}_k|\right) = 0.$$

3. *L_T is compact for $T \in (C_1^n, c_0)$ if and only if*

$$\lim_j \left(\sup_k |\tilde{t}_{j,k}|\right) = 0.$$

4. *L_T is compact for $T \in (C_1^n, \ell_1)$ if and only if*

$$\lim_l \left(\sup_k \sum_{j=l}^{\infty} |\tilde{t}_{j,k}|\right) = 0.$$

4. Norm of Cesàro operator on some sequence spaces

In this part of study, we investigate the problem of finding the norm of Cesàro operator of order n on some sequence spaces. The following lemma has the key role in finding the norm of operators between matrix domains.

Lemma 4.1. *Let U is a bounded operator on ℓ_p and A_p and B_p are two matrix domains such that $A_p \simeq \ell_p$.*
(*a*) *If $BT = UA$, then T is a bounded operator from the matrix domain A_p into B_p and*

$$\|T\|_{A_p,B_p} = \|U\|_{\ell_p}.$$

In particular, if T is a bounded operator on ℓ_p and $AT = TA$, then T is a bounded operator on the matrix domain A_p and

$$\|T\|_{A_p} = \|T\|_{\ell_p}.$$

(*b*) *If T has a factorization of the form $T = UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and*

$$\|T\|_{A_p,\ell_p} = \|U\|_{\ell_p}.$$

Proof. (*a*) Since A_p and ℓ_p are isomorphic, hence

$$\begin{aligned}\|T\|_{A_p,B_p} &= \sup_{x\in A_p} \frac{\|Tx\|_{B_p}}{\|x\|_{A_p}} = \sup_{x\in A_p} \frac{\|BTx\|_{\ell_p}}{\|Ax\|_{\ell_p}} = \sup_{x\in A_p} \frac{\|UAx\|_{\ell_p}}{\|Ax\|_{\ell_p}} \\ &= \sup_{y\in \ell_p} \frac{\|Uy\|_{\ell_p}}{\|y\|_{\ell_p}} = \|U\|_{\ell_p},\end{aligned}$$

(*b*) It is sufficient to let $B = I$ in the part (*a*). □

4.1. Norm of Cesàro operator on the Hilbert sequence spaces

Let us recall the Hilbert matrix $H = (h_{j,k})$, which is defined by

$$h_{j,k} = \frac{1}{j+k+1} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad j,k = 0,1,\ldots,$$

and is a bounded operator on ℓ_p with ℓ_p-norm $\|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi\csc(\pi/p)$. by [23], Theorem 323.

For a positive integer n, we define the Hilbert matrix of order n, $H^n = (h^n_{j,k})$, by

$$h^n_{j,k} = \frac{1}{j+k+n+1} \qquad (j,k = 0,1,\cdots).$$

Note that for $n = 0$, $H^0 = H$ is the Hilbert matrix. For more examples:

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For non-negative integers n, j and k, let us define the matrix $B^n = (b^n_{j,k})$ by

$$\begin{aligned} b^n_{j,k} &= \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)} \\ &= \binom{n+k}{k}\beta(j+k+1, n+1) \qquad (j,k = 0,1,\ldots), \end{aligned}$$

where the β function is

$$\beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}dz \qquad (m,n = 1,2,\ldots).$$

Consider that for $n = 0$, $B^0 = H$, where H is the Hilbert matrix.

For computing the norm of Cesàro operator on the Hilbert matrix domains we need the following lemma.

Lemma 4.2. *The Hilbert matrix H and the Hilbert matrix of order n, H^n, have the following factorizations based on the Cesàro matrix of order n:*

(a) *$H = B^nC^n$, where B^n defined in relation (8) and is a bounded operator on ℓ_p and*

$$\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.$$

(b) *$H^n = C^nB^n$,*

(c) *$C^nH = H^nC^n$,*

Proof. (a) This part is Corollary 2.3 of [24]. (b) This is Lemma 3.18 of [25]. (c) This part is the result of the two previous parts. □

The sequence space associated with the Hilbert matrix of order n, H^n_p, is defined by

$$H^n_p = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p < \infty \right\},$$

which has the norm

$$\|x\|_{H^n_p} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p\right)^{\frac{1}{p}}.$$

In Particular, for $n = 0$, we have

$$H_p = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p < \infty \right\},$$

with the norm $\|x\|_{H_p} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p\right)^{\frac{1}{p}}$.

Theorem 4.3. *The Cesàro operator of order n, C^n, is a bounded operator from the Hilbert space H_p into the Hilbert space H^n_p and*

$$\|C^n\|_{H_p,H^n_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

In particular, the Cesàro operator C, is a bounded operator from the Hilbert space H_p into the Hilbert space H^1_p and

$$\|C\|_{H_p,H^1_p} = p^*.$$

Proof. Since H_p and ℓ_p are isomorphic spaces, hence according to Lemma 4.2 we have

$$\|C^n\|_{H_p,H_p^n} = \|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

□

4.2. Norm of Cesàro operator on the difference sequence space bv_p

The idea of difference sequence spaces was introduced by Kizmaz [26]. The backward difference matrix $\Delta = (\delta_{j,k})$ and its inverse $\Delta^{-1} = (\delta^{-1}_{j,k})$ are

$$\delta_{j,k} = \begin{cases} 1 & k = j \\ -1 & k = j-1 \\ 0 & otherwise, \end{cases} \qquad and \qquad \delta^{-1}_{j,k} = \begin{cases} 1 & 0 \le k \le j \\ 0 & otherwise, \end{cases}$$

and the matrix representation as follows

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Delta^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence space associated with the matrix Δ is called bv_p, which is defined by

$$bv_p = \left\{ x = (x_n) \ : \ \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\},$$

and has the norm

$$\|x\|_{bv_p} = \left(\sum_{n=1}^{\infty} |x_n - x_{n-1}|^p \right)^{\frac{1}{p}}.$$

Note that, we use the notation $\|T\|_{bv_p}$ as the norm of operator T from the sequence space bv_p into itself, i,e: $\|T\|_{bv_p} = \|T\|_{bv_p,bv_p}$.

We say that $T = (t_{j,k})$ is a lower triangular, if $t_{j,k} = 0$ for $k > j$. A non-negative lower triangular matrix is called a summability matrix if $\sum_{k=0}^{j} t_{j,k} = 1$ for all j. In sequel, we need the Schur's theorem which is

Theorem 4.4 ([23], Theorem 275). *Let $p > 1$ and $T = (t_{m,k})$ be a matrix operator with $t_{m,k} \ge 0$ for all m, k. Suppose that C, R are two strictly positive numbers such that*

$$\sum_{m=0}^{\infty} t_{m,k} \le C \quad for\ all\ k, \qquad \sum_{k=0}^{\infty} t_{m,k} \le R \quad for\ all\ m,$$

(bounds for column and row sums respectively). Then

$$\|T\|_{\ell_p \to \ell_p} \le R^{1/p^*} C^{1/p}.$$

Lemma 4.5. *Let $T = (t_{j,k})$ be a summability matrix and $R_j = \sum_{k=0}^{j} (k+1) t_{j,k}$. If $\sup_j (R_j - R_{j-1}) \le 1$ for all j, then T is a bounded operator on bv_p and*

$$\|T\|_{bv_p} = 1.$$

Proof. By letting $A = B = \Delta$ in Lemma 4.1, we have $\|T\|_{bv_p} = \|U\|_{\ell_p}$, where $U = \Delta T\Delta^{-1}$. If $S = T\Delta^{-1}$, by assuming $S = (s_{i,j})$ and $U = (u_{i,j})$, we have $s_{i,j} = \sum_{k=j}^{i} t_{i,k}$ and $u_{i,j} = (\Delta S)_{i,j} = s_{i,j} - s_{i-1,j}$. Thus

$$\sum_{i=j}^{k} u_{i,j} = \sum_{i=j}^{k}(s_{i,j} - s_{i-1,j}) = s_{k,j} = \sum_{t=j}^{k} t_{k,t} = t_{k,j} + \cdots + t_{k,k} \leq 1 \quad (k = 0, 1, \cdots),$$

hence $\sum_{i=0}^{\infty} u_{i,j} \leq 1$. Also

$$\sum_{k=0}^{j} s_{j,k} = \sum_{k=0}^{j}\sum_{l=k}^{j} t_{j,l} = \sum_{k=0}^{j}(k+1)t_{j,k} = R_j,$$

and

$$\sum_{k=0}^{\infty} u_{j,k} = \sum_{k=0}^{j} u_{j,k} = \sum_{k=0}^{j}(s_{j,k} - s_{j-1,k}) = R_j - R_{j-1}.$$

Now since $\sup_j(R_j - R_{j-1}) \leq 1$, we have that $\|T\|_{bv_p} \leq 1$. Also letting $x = (1, 1, \cdots)$ result that $Tx = x$, and therefore $\|T\|_{bv_p} = 1$. □

Lemma 4.6. *For non-negative integers n, j and k we have*

(a) $\sum_{k=0}^{j}\binom{n+k-1}{k} = \binom{n+j}{j}$,

(b) $\sum_{k=0}^{j}(k+1)\binom{n+k-1}{k} = (j+1)\binom{n+j}{j} - \binom{n+j}{j-1}$.

Proof. Proof of part (a) is obvious. (b) Let $\binom{n+k-1}{k} = a_k^n$ and $A = a_0^n + a_1^n + \cdots + a_j^n$. By the part (i) $A = \binom{n+j}{j}$. Now

$$\begin{aligned}
\sum_{k=0}^{j}(k+1)\binom{n+k-1}{k} &= \sum_{k=0}^{j}(k+1)a_k^n \\
&= A + \{A - a_0^n\} + \{A - (a_0^n + a_1^n)\} + \cdots + \{A - (a_0^n + a_1^n + \cdots + a_{j-1}^n)\} \\
&= A + \{A - a_0^{n+1}\} + \{A - a_1^{n+1}\} + \cdots + \{A - a_{j-1}^{n+1}\} \\
&= (j+1)A - \{a_0^{n+1} + a_1^{n+1} + \cdots + a_{j-1}^{n+1}\} \\
&= (j+1)A - a_{j-1}^{n+2} = (j+1)\binom{n+j}{j} - \binom{n+j}{j-1}.
\end{aligned}$$

□

Theorem 4.7. *The Cesàro operator of order n, C^n, is a bounded operator on the sequence space bv_p and*

$$\|C^n\|_{bv_p} = 1.$$

In particular, the Cesàro operator is a bounded operator on the sequence space bv_p and $\|C\|_{bv_p} = 1$.

Proof. For the Cesàro matrix of order n let $s = j - k$, we have the following identity

$$\begin{aligned}
\sum_{k=0}^{j}(k+1)\binom{n+j-k-1}{j-k} &= \sum_{s=0}^{j}(j-s+1)\binom{n+s-1}{s} \\
&= \sum_{s=0}^{j}[(j+2)-(s+1)]\binom{n+s-1}{s} \\
&= (j+2)\sum_{s=0}^{j}\binom{n+s-1}{s} - \sum_{s=0}^{j}(s+1)\binom{n+s-1}{s}.
\end{aligned}$$

Now by the notation of Lemma 4.5

$$R_j = \sum_{k=0}^{j}(k+1)c_{j,k}^n = \frac{1}{\binom{n+j}{j}}\sum_{k=0}^{j}(k+1)\binom{n+j-k-1}{j-k}$$
$$= 1+\frac{\binom{n+j}{j-1}}{\binom{n+j}{j}} = 1+\frac{j}{n+1},$$

and $\sup_j(R_j - R_{j-1}) = \frac{1}{n+1} \leq 1$. Hence according to Lemma 4.5, $\|C^n\|_{bv_p} = 1$. □

4.3. Norm of Cesàro operator on the Hausdorff matrix domains

Theorem 4.8 ([27], Theorem 9). *Let $p \geq 1$ and H^μ, H^ω and H^ν be Hausdorff matrices related by $H^\mu = H^\omega H^\nu$. Then H^μ is bounded on ℓ_p if and only if both H^ω and H^ν are bounded on ℓ_p. Moreover, we have*

$$\|H^\mu\|_{\ell_p} = \|H^\omega\|_{\ell_p}\|H^\nu\|_{\ell_p}.$$

Theorem 4.9. *Let C^n be the Cesàro operator of order n. Then*

(a) *C^n is a bounded operator from ℓ_p into H_p^μ and*

$$\|C^n\|_{\ell_p,H_p^\mu} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta),$$

(b) *C^n is a bounded operator from H_p^μ into ℓ_p and*

$$\|C^n\|_{H_p^\mu,\ell_p} = \frac{\Gamma(n+1/p^*)}{\Gamma(n+1)\Gamma(1/p^*)}\left(\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta)\right)^{-1},$$

(c) *C^n is a bounded operator on H_p^μ and*

$$\|C^n\|_{H_p^\mu} = \|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

Proof. (a) By letting $A = I$ in Lemma 4.1 part (a), applying Theorem 4.8 and Hardy's formula we have

$$\|C^n\|_{\ell_p,H_p^\mu} = \|H^\mu C^n\|_{\ell_p} = \|H^\mu\|_{\ell_p}\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}\int_0^1 \theta^{\frac{-1}{p}}d\mu(\theta),$$

(b) According to Bennett ([28], page 120), C^n has a factorization of the form $C^n = H^\omega H^\mu$, where ω is a quotient measure. So Lemma 4.1 part (b) implies that

$$\|C^n\|_{H_p^\mu,\ell_p} = \|H^\omega\|_{\ell_p} = \|C^n\|_{\ell_p}/\|H^\mu\|_{\ell_p}.$$

(c) Since two Hausdorff matrices commute, hence Lemma 4.1 part (a) gives the result. □

References

[1] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.

[2] M. Et, On Some Difference Sequence Spaces, *Turkish J. Math.* **17** (1993), 18-24.

[3] M. Kirişçi and F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, *Comput. Math. Appl.* **60** (2010), 1299-1309.

[4] M. Başarır and E. E. Kara, On some difference sequence spaces of weighted means and compact operators, *Ann. Func. Anal.* **2** (2011), 114-129.

[5] P. Zengin Alp and M. İlkhan, On the difference sequence space $\ell_p(\hat{T}^q)$, *Math. Sci. Appl. E-Notes* **7**(2) (2019), 161-173.

[6] M. İlkhan and E. E. Kara, A new Banach space defined by Euler totient matrix operator, *Oper. Matrices* **13**(2) (2019), 527-544.

[7] M. Mursaleen and A. K. Noman, Compactness by the Hausdorff measure of noncompactness, *Nonlinear Analysis* **73** (2010), 2541–2557.

[8] M. Başarır and E. E. Kara, On compact operators on the Riesz B(m)-difference sequence spaces, *Iranian J. Sci. Technol.* **35** (2011), 279-285.

[9] E. E. Kara and M. Başarır, On compact operators and some Euler $B^{(m)}$-difference sequence spaces, *J. Math. Anal. Appl.* **379** (2011), 499-511.

[10] M. Başarır and E. E. Kara, On the B-difference sequence space derived by generalized weighted mean and compact operators, *J. Math. Anal. Appl.* **391** (2012), 67–81.

[11] M. Mursaleen and S.A. Mohiuddine, Applications of measures of non-compactness to the infinite system of differential equations in ℓ_p spaces, *Nonlinear Analysis* **75**(4) (2012), 2111-2115.

[12] A. Alotaibi, M. Mursaleen, BAS Alamri and S.A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, *J. Inequal. Appl.* 2015 (1), 203.

[13] H. Roopaei and D. Foroutannia, The norm of matrix operators on Cesàro weighted sequence space, *Linear Multilinear Algebra* **67**(1)(2019), 175-185.

[14] H. Roopaei and D. Foroutannia, The norms of certain matrix operators from ℓ_p spaces into $\ell_p(\Delta^n)$ spaces, *Linear Multilinear Algebra* **67**(4) (2019), 767-776.

[15] H. Roopaei, Norms of summability and Hausdorff mean matrices on difference sequence spaces, *Math. Inequal. Appl.* **22**(3) (2019), 983-987.

[16] D. Foroutannia and H. Roopaei, The norms and the lower bounds for matrix operators on weighted difference sequence spaces, *UPB Sci. Bull., Ser. A* **79**(2) (2017), 151-160.

[17] M. İlkhan, Norms and lower bounds of some matrix operators on Fibonacci weighted difference sequence space, *Math. Methods Appl. Sci.* **42**(16) (2019), 5143-5153.

[18] A. Wilansky, *Sumability Through Functional Analysis*, Vol. 85, NorthHolland, Amsterdam, 1984. Mathematical Studies.

[19] G. H. Hardy, *Divergent Series*, Oxford University Press, 1973.

[20] E. Malkowsky and V. Rakocevic, An introduction into the theory of sequence spaces and measure of noncompactness, Zbornik radova, Matematicki Inst. SANU, Belgrade, **9**(17) (2000), 143-234.

[21] A.M. Jarrah and E. Malkowsky, Ordinary, absolute and strong summability and matrix transformations, *Filomat* **17** (2003), 59-78.

[22] V. Rakocevic, Measures of noncompactness and some applications, *Filomat* **12** (1998), 87-120.

[23] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 2001.

[24] H. Roopaei, Factorization of the Hilbert matrix based on Cesàro and Gamma matrices, *Results Math.* **75**, 3 (2020).

[25] H. Roopaei, D. Foroutannia, M. İlkhan and E. E. Kara, Cesàro Spaces and Norm of Operators on These Matrix Domains, *Mediterr. J. Math.* **17** 121 (2020).

[26] H. Kızmaz, On certain sequence spaces I, *Canad. Math. Bull.* **25**(2)(1981), 169-176.

[27] G. Bennett, Lower bounds for matrices II, *Canad. J. Math.* **44** (1992), 54-74.

[28] G. Bennett, Factorizing the classical inequalities, *Mem. Amer. Math. Soc.* **576** (1996).