Filomat 37:5 (2023), 1635–1651
https://doi.org/10.2298/FIL2305635K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Steklov averages in operator cosine function framework

Andi Kivinukk[a], Anna Šeletski[a]

[a]*School of Digital Technologies, Tallinn University, Narva Road 25, 10120 Tallinn, Estonia*

Abstract. The Steklov averages (Steklov or integral means) are used in approximation theory of functions in different aspects. This article concerns the Steklov averages by using the operator cosine function framework. The operator cosine function offers a counterpart of the translation operator, which forms the basic concept for the modulus of continuity and for some approximation processes as well. We will show that the operator cosine function concept allows to define very general Steklov averages in an abstract Banach space. The approximation properties of these generalized Steklov averages appear to be quite similar to the properties of the Steklov averages in trigonometric approximation.

1. Introduction

The Steklov averages are used in approximation theory of functions in different aspects [1], [8], [13], [14]. This article concerns the Steklov averages by using the operator cosine function framework. The operator cosine function [6], [7], [11] offers a counterpart of the translation operator, which forms the basic concept for the modulus of continuity and for some approximation processes as well. We will show that the operator cosine function concept allows to define very generally Steklov averages in an abstract Banach space. The approximation properties of these generalized Steklov averages appear to be quite similar to the properties of the Steklov averages in trigonometric approximation ([1], [13]).

Let X be an arbitrary (real or complex) Banach space and $[X]$ be the Banach algebra of all bounded linear operators $U : X \to X$. We start with the definition (compare [6], [7]).

Definition 1.1. *An **equibounded cosine operator function** $C_h \in [X]$ $(h \geq 0)$ is defined by the properties:*

(i) $C_0 = I$*(identity operator),*

(ii) $C_{h_1} \cdot C_{h_2} = \frac{1}{2}(C_{h_1+h_2} + C_{|h_1-h_2|})$,

(iii) $\|C_h f\| \leq T\|f\|$, $0 < T-$ *not depending on* $h > 0$.

We denote by $T_h \in [X]$, $h \in \mathbb{R}$, a translation operator, which is defined by the properties

2020 *Mathematics Subject Classification*. Primary 41A65; Secondary 41A17
Keywords. cosine operator function, Steklov averages, modulus of continuity, degree of approximation
Received: 06 April 2021; Revised: 29 April 2021; Accepted: 07 May 2021
Communicated by Miodrag Spalević
Research supported partially by the EU, European Regional Development Fund (ASTRA project for 2016- 2022 of Estonian Doctoral School in Mathematics and Statistics; Tallinn University TLU TEE).
Email addresses: andik@tlu.ee (Andi Kivinukk), annat@tlu.ee (Anna Šeletski)

(i) $T_0 = I$,

(ii) $T_{h_1} \cdot T_{h_2} = T_{h_1+h_2}$,

(iii) $\|T_h f\| \leq T\|f\|, 0 < T-$ not depending on $h \in \mathbb{R}$.

Then $C_h := \frac{1}{2}(T_h + T_{-h}),\ h \geq 0$, is a cosine operator function. It means that if we can define a translation operator, then we have also the cosine operator function.

For example, a less trivial cosine operator function is related with the Fourier-Chebyshev series ([3] and [2], where certain general cosine operator functions can be find), where $x \in [-1, 1],\ 0 \leq h \leq \pi$ and

$$(C_h^C f)(x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1-x^2}\sin h) + f(x\cos h - \sqrt{1-x^2}\sin h)\right\}.$$

But for some spaces we cannot define the translation operator $T_h \in [X],\ h \in \mathbb{R}$, nevertheless the cosine operator function does exist.

Example. Let $X = C_{2\pi}^-$ be the space of π-symmetric and 4π-periodic continuous functions, i.e. $f(\pi - x) = f(\pi + x)$ and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$. Let us discuss the functions $f(x) = \sin(k - \frac{1}{2})x$, $k \in \mathbb{N}$ in space $C_{2\pi}^-$. Here $T_h\left(\sin\left(\frac{1}{2}\circ\right), x\right) = \sin\frac{1}{2}(x + h) \notin C_{2\pi}^-$ for some $h \in \mathbb{R}$, but $C_h f \in C_{2\pi}^-$, where $C_h := \frac{1}{2}(T_h + T_{-h})$ and T_h is the ordinary translation operator.

2. Steklov averages

We start with basic notion of this article.

Definition 2.1. *It is said that $S_{h,1}f := \int_0^1 C_{ht} f dt$ for every $h > 0$ and $S_{h,r}f := S_{h,1}(S_{h,r-1}f)$ for $r = 2, 3, \ldots$ are **Steklov averages** for an element $f \in X$.*

For instance, for $r = 2$ we have

$$S_{h,2}f = S_{h,1}(S_{h,1}f) = S_{h,1}\left(\int_0^1 C_{ht} f dt\right) = \int_0^1 C_{ht}(S_{h,1}f)dt. \tag{1}$$

If $0 \leq t \leq 1$ then we have by Definition 1.1, (ii)

$$C_{ht}(S_{h,1}f) = \int_0^1 C_{ht}(C_{hu}f)du = \frac{1}{2}\int_0^1 \left(C_{h(u+t)}f + C_{h|u-t|}f\right)du. \tag{2}$$

Let us consider for $0 \leq t \leq 1$

$$\int_0^1 C_{h|u-t|} f du = \int_0^t C_{h(t-u)} f du + \int_t^1 C_{h(u-t)} f du = \int_0^t C_{hv} f dv + \int_0^{1-t} C_{hv} f dv,$$

then by (2) for $0 < t < 1$

$$C_{ht}(S_{h,1}f) = \frac{1}{2}\left(\int_0^{t+1} C_{hv} f dv + \int_0^{1-t} C_{hv} f dv\right), \tag{3}$$

and by (1)

$$\begin{aligned}
S_{h,2}f &= \frac{1}{2}\left(\int_0^1 dt \int_0^{t+1} C_{hv} f dv + \int_0^1 dt \int_0^{1-t} C_{hv} f dv\right) \\
&= \frac{1}{2}\left(\int_0^1 dv \int_0^1 C_{hv} f dt + \int_1^2 dv \int_{v-1}^1 C_{hv} f dt + \int_0^1 dv \int_0^{1-v} C_{hv} f dt\right) \\
&= \frac{1}{2}\left(\int_0^1 C_{hv} f dv + \int_1^2 (2-v) C_{hv} f dv + \int_0^1 (1-v) C_{hv} f dv\right).
\end{aligned}$$

So we have

$$S_{h,2}f = \frac{1}{2}\int_0^2 (2-v)C_{hv}f\,dv. \tag{4}$$

Remark 2.2. *If the cosine operator function is defined in a functional space $X = C(\mathbb{R})$ by*

$$C_h f(x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right),\ h \geq 0, \tag{5}$$

then the Steklov averages of Definition 2.1 coincide with the usual Steklov averages (see, e.g. [8], [13])

$$L_{0,h}f := f,\ \ L_{r,h}f(x) := \frac{1}{2h}\int_{x-h}^{x+h} L_{r-1,h}f(t)dt,\ h \geq 0,$$

i. e.

$$S_{h,r}f(x) = L_{r,h}f(x).$$

Remark 2.3. *If the cosine operator function is defined by (20) in $X = C_{[-1,1]}$ (continuous real-valued functions on $[-1,1]$) or in $X = L^p_w, 1 \leq p < \infty$ (measurable real-valued functions on $[-1,1]$ for which the norm*

$$\|f\|_p := (\frac{1}{\pi}\int_{-1}^1 |f(u)|^p w(u)du)^{1/p}, \quad w(u) = \frac{1}{\sqrt{1-u^2}}$$

is finite), then the Steklov averages of Definition 2.1 coincide with the Steklov averages given in [3], Definition 3. Let us set by Definition 3 for $f \in X, h \in [-1,1)$ and $h' = \sqrt{(1+h)/2}$

$$(\bar{A}_h^{1/2}f)(x) := \frac{2}{\arccos h}\int_{h'}^1 (C^C_{\arccos u}f)(x)\frac{du}{\sqrt{1-u^2}} \quad (x \in [-1,1]).$$

Indeed, changing of variables gives us

$$\bar{A}^{1/2}_{\cos h}f = S_{h/2,1}f,$$

and by equation (1) we have for

$$\bar{A}^1_{\cos h}f := \bar{A}^{1/2}_{\cos h}(\bar{A}^{1/2}_{\cos h}f) = S_{h/2,2}f.$$

Lemma 2.4. *Let φ_r ($r \in \mathbb{N}$) be the kernel function of $S_{h,r}$, i.e.*

$$S_{h,r}f = \int_0^r \varphi_r(u)C_{hu}f\,du.$$

Then the kernel function φ_{r+1} has the form ($r \geq 2$):

1. *$\varphi_{r+1}(v) = \frac{1}{2}\int_0^{1-v}\varphi_r(u)du + \frac{1}{2}\int_0^{1+v}\varphi_r(u)du$ on the interval $[0,1]$;*

2. *$\varphi_{r+1}(v) = \frac{1}{2}\int_{v-1}^{v+1}\varphi_r(u)du$ on the interval $[1, r-1]$;*

3. *$\varphi_{r+1}(v) = \frac{1}{2}\int_{v-1}^{r}\varphi_r(u)du$ on the interval $[r-1, r+1]$.*

Proof. The case $r = 2$ is given by (4). By Definition 2.1, and since $S_{h,1}$ and C_{hu} are commutative, we have

$$S_{h,r+1}f = S_{h,1}\left(\int_0^r \varphi_r(u)C_{hu}f\,du\right) = \int_0^1 \varphi_r(u)C_{hu}(S_{h,1}f)du + \int_1^r \varphi_r(u)C_{hu}(S_{h,1}f)du. \tag{6}$$

By (3) the first integral in (6) has the form

$$I_1 := \int_0^1 \varphi_r(u)C_{hu}(S_{h,1}f)du = \frac{1}{2}\int_0^1 \varphi_r(u)\left(\int_0^{u+1} C_{hv}f\,dv + \int_0^{1-u} C_{hv}dv\right)du,$$

where, interchanging the order of integration, we obtain

$$2I_1 = \int_0^1\left(\int_0^1 \varphi_r(u)du\right)C_{hv}f\,dv + \int_1^2\left(\int_{v-1}^1 \varphi_r(u)du\right)C_{hv}f\,dv + \int_0^1\left(\int_0^{1-v} \varphi_r(u)du\right)C_{hv}f\,dv.$$

For $1 \le u \le r$ we have by (2)

$$C_{hu}(S_{h,1}f) = \frac{1}{2}\int_{u-1}^{u+1} C_{hv}f\,dv,$$

hence, for the second integral of (6) for $r \ge 3$ we can write, after interchanging the order of integration,

$$\begin{aligned} I_2 :&= \int_1^r \varphi_r(u)C_{hu}(S_{h,1}f)du = \frac{1}{2}\int_1^r \varphi_r(u)\int_{u-1}^{u+1} C_{hv}f\,dv \\ &= \frac{1}{2}\int_0^2\left(\int_1^{v+1} \varphi_r(u)du\right)C_{hv}f\,dv + \frac{1}{2}\int_2^{r-1}\left(\int_{v-1}^{v+1} \varphi_r(u)du\right)C_{hv}f\,dv \\ &\quad + \frac{1}{2}\int_{r-1}^{r+1}\left(\int_{v-1}^{r} \varphi_r(u)du\right)C_{hv}f\,dv. \end{aligned}$$

Both integrals, I_1 and I_2, used together in (6), give

$$\begin{aligned} S_{h,r+1}f &= \frac{1}{2}\int_0^1\left(\int_0^{1-v} \varphi_r(u)du + \int_0^{1+v} \varphi_r(u)du\right)C_{hv}dv + \frac{1}{2}\int_1^{r-1}\left(\int_{v-1}^{v+1} \varphi_r(u)du\right)C_{hv}dv \\ &\quad + \frac{1}{2}\int_{r-1}^{r+1}\left(\int_{v-1}^{r} \varphi_r(u)du\right)C_{hv}dv. \end{aligned}$$

■

Corollary 2.5. *The kernel function φ_{r+1} has for $r \ge 2$ continuous derivatives*

1. $\varphi'_{r+1}(v) = \frac{1}{2}(\varphi_r(1+v) - \varphi_r(1-v))$ *on the interval* $[0,1]$,
2. $\varphi'_{r+1}(v) = \frac{1}{2}(\varphi_r(v+1) - \varphi_r(v-1))$ *on the interval* $[1, r-1]$,
3. $\varphi'_{r+1}(v) = -\frac{1}{2}\varphi_r(v-1)$ *on the interval* $[r-1, r+1]$,

and on the breakpoints $\varphi'_{r+1}(0) = 0$, $\varphi'_{r+1}(1) = \frac{1}{2}(\varphi_r(2) - \varphi_r(0))$, $\varphi'_{r+1}(r-1) = -\frac{1}{2}\varphi_r(r-2)$ *and* $\varphi'_{r+1}(r+1) = 0$.

Examples.

a. By Definition 2.1 we have $\varphi_1(u) = 1$ for $0 \le u \le 1$.

b. Using equality (4) we get $\varphi_2(u) = 1 - \frac{u}{2}$ for $0 \le u \le 2$.

c. Let us calculate $\varphi_3(u)$ for $0 \le u \le 1$ by Lemma 2.4,

$$\varphi_3(u) = \frac{1}{2}\int_0^{1-u}\left(1-\frac{v}{2}\right)dv + \frac{1}{2}\int_0^{1+u}\left(1-\frac{v}{2}\right)dv = \frac{1}{4}(3-u^2).$$

Next we calculate $\varphi_3(u)$ for $1 \le u \le 3$,

$$\varphi_3(u) = \frac{1}{2}\int_{u-1}^{2}\left(1-\frac{v}{2}\right)dv = \frac{1}{8}(u-3)^2.$$

Proposition 2.6. *Let $S_{h,r}f = \int_0^r \varphi_r(u)C_{hu}f du$, as in Lemma 2.4. Then the following statements hold*

a) $\varphi_r(u) \ge 0, 0 \le u \le r$,

b) $\varphi_r \in C_{[0,r]}$,

c) $\varphi_r(r) = 0$, *for* $r \ge 2$,

d) $\int_0^r \varphi_r(u)du = 1$.

Proof. It follows directly from Lemma 2.4. ■

Remark 2.7.

An anonymous referee asked: Can we consider the Steklov averages of C-regularized cosine operator functions in a similar way? The **C-regularized cosine operator functions** are defined (see [6]), using an operator $C \in [X]$ and replacing the conditions $(i), (ii)$ in Definition 1.1 by conditions

(i) $C_0 = C$(the operator, given above),

(ii) $C_{h_1} \cdot C_{h_2} = \frac{1}{2}(C_{h_1+h_2} + C_{|h_1-h_2|})C$.

Let us denote the family of the C-regularized cosine operator functions by $\{C_h^C\}_{h\ge 0}$ and let us define the Steklov averages as in Definition 2.1, i.e. $S_{h,1}^C f := \int_0^1 C_{ht}^C f dt$ for every $h > 0$ and $S_{h,r}^C f := S_{h,1}^C(S_{h,r-1}^C f)$ for $r = 2, 3, ...$ Then

$$S_{h,r}^C f = \int_0^r \varphi_r(u)C_{hu}^C(C^{r-1}f)du,$$

where φ_r is the same as in Proposition 2.6. Proof is a small change from the previous one.

Remark 2.8. *If*

$$C_h f(x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right),\ h \ge 0,$$

then the representation of $S_{h,r}$ in Proposition 2.6 with a) - d) corresponds to the representation [8], Theorem 2.1 and Remark 2.2,

$$S_{h,r}f(x) = L_{r,h}f(x) = \frac{1}{(2rh)^r}\int_{x-rh}^{x+rh}\cdots\int_{x-rh}^{x+rh} f\left(\frac{t_1+...+t_r}{r}\right)dt_1...dt_r.$$

Moreover, as in [8], let us denote by B_{n-1}^{nh} the B-spline function of degree $n-1$ associated to the $n+1$ points $-nh < -(n-2)h < -(n-4)h < ... < (n-2)h < nh$. Then, by [8], formula (2.7), we can prove that the kernel function φ_r in Lemma 2.4 is related to the B-spline by

$$\varphi_r(t) = 2B_{r-1}^r(t), \quad 0 \le t \le r.$$

3. Steklov averages and the infinitesimal generator

We need further facts about the strongly continuous operator cosine function C_h, $h \geq 0$ (see [7]). The family $\{C_h\}$ is called the strongly continuous, if $\lim_{h\to 0+} \|C_h f - f\| = 0$.

Theorem A. *If $C_t : X \to X$ is a strongly continuous operator cosine function, then there are constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|C_t\|_{[X]} \leq Me^{\omega t}$$

for all $t \geq 0$.

The concept of the infinitesimal generator $A : D(A) \to X$ of a strongly continuos operator cosine function is needed.

Definition B. *The **infinitesimal generator** $A : D(A) \to X$ is defined by*

$$D(A) := \left\{ f \in X \,\middle|\, t \to C_t f \text{ is 2-times differentiable in } 0 \right\},$$

$$Af := C_0'' f \text{ for } f \in D(A).$$

Theorem C. The following statements are true:
a) The generator A is a densely defined on X and closed linear operator.
b) For all $t \geq 0$ is true $C_t D(A) \subset D(A)$ and $C_t(Af) = A(C_t f)$ for all $f \in D(A)$.
c) For $f \in D(A)$ and $t \geq 0$
(i) $\int_0^t (t-u) C_u f du \in D(A)$
(ii) $C_t f - f = \int_0^t (t-u) C_u (Af) du = A\left(\int_0^t (t-u) C_u f du\right)$.

The equation (ii) can be generalized using the Steklov averages.

Proposition 3.1. *Let $A : D(A) \to X$ be the generator of $C_t : X \to X$ and $S_{h,r} : X \to X$ be the Steklov averages of order $r \in \mathbb{N}$. Then for $A^r := A(A^{r-1})$ $(r = 2, 3, ...)$, $A^1 = A$ there hold $S_{h,2r} f \in D(A^r)$ and*

$$A^r(S_{h,2r} f) = \frac{1}{2^r h^{2^r}} (C_{2h} - I)^r f \tag{7}$$

for any $f \in X$ and $h > 0$.

Proof. The case $r = 1$ is the statement of Theorem C. Indeed, by (4)

$$S_{h,2} f = \frac{1}{2} \int_0^2 (2-v) C_{hv} f dv = \frac{1}{2h^2} \int_0^{2h} (2h-u) C_u f du$$

which gives by Theorem C (ii)

$$A(S_{h,2} f) = \frac{1}{2h^2} (C_{2h} f - f).$$

The general equation (7) follows by induction. ∎

Remark 3.2. *For the C-regularized cosine operator functions $\{C_h^C\}_{h\geq 0}$ there is defined the (sub)generator A, see [6], Definition 1.2.1, but that definition is not sufficient to prove a counterpart of Proposition 3.1. As proof, all statements of Theorem C must be valid in case $f \in D(A)$, moreover, we must assume that all operator pairs (A, C_h^C), (A, C), (C, C_h^C) are commuting. If all these conditions are met, then*

$$A^r(S_{h,2r}^C f) = \frac{1}{2^r h^{2^r}} (C_{2h}^C - C)^r (C^r f) \quad (f \in X).$$

To continue, we recall the trigonometric identity ([16], formula 1.320)

$$(\cos x - 1)^l = \frac{1}{2^{l-1}} {\sum_{v=0}^{l}}{}' (-1)^{l-v} \binom{2l}{l-v} \cos vx, \tag{8}$$

where $\sum'$ here and in the following means that the term with $v = 0$ is halved.

We prove a similar identity for the cosine operator function. So, we hope to show that the following proposition holds.

Proposition 3.3. *It holds true for $l \in \mathbb{N}$ that*

$$(C_h - I)^l = \frac{1}{2^{l-1}} {\sum_{v=0}^{l}}{}' (-1)^{l-v} \binom{2l}{l-v} C_{vh}. \tag{9}$$

Proof. For $l = 1$ (9) is true. Suppose that (9) is true for $l = n$ and consider

$$(C_h - I)^{n+1} = \frac{1}{2^{n-1}} {\sum_{v=0}^{n}}{}' (-1)^{n-v} \binom{2n}{n-v} C_{vh}(C_h - I).$$

Denote here the right-hand side expression by

$$I_{n+1} := \frac{1}{2^n}(-1)^n \binom{2n}{n}(C_h - I) + \frac{1}{2^{n-1}} \sum_{v=1}^{n} (-1)^{n-v} \binom{2n}{n-v} C_{vh}(C_h - I), \tag{10}$$

and let us denote

$$\begin{aligned} S_1 : \; &= \sum_{v=1}^{n} (-1)^{n-v} \binom{2n}{n-v} \left(\frac{1}{2}[C_{(v+1)h} + C_{(v-1)h}] - C_{vh} \right) \\ &= \frac{1}{2} \sum_{v=1}^{n} (-1)^{n-v} \binom{2n}{n-v} \left([C_{(v+1)h} - C_{vh}] - [C_{vh} - C_{(v-1)h}] \right). \end{aligned}$$

We continue, denoting $U_{v+1} := C_{(v+1)h} - C_{vh}$, and therefore, using a combinatorial identity

$$\binom{m}{p+1} + \binom{m}{p} = \binom{m+1}{p+1},$$

we have

$$\begin{aligned} S_1 &= \frac{1}{2} \left(\sum_{v=2}^{n+1} (-1)^{n-v+1} \binom{2n}{n-v+1} U_v - \sum_{v=1}^{n} (-1)^{n-v} \binom{2n}{n-v} U_v \right) \\ &= \frac{1}{2} \left((-1)^n \binom{2n}{n-1} U_1 + \sum_{v=2}^{n+1} (-1)^{n-v+1} \binom{2n+1}{n+1-v} U_v \right). \end{aligned}$$

For

$$S_2 := \sum_{v=2}^{n+1} (-1)^{n-v+1} \binom{2n+1}{n+1-v} U_v \quad (U_v := C_{vh} - C_{(v-1)h})$$

we obtain

$$S_2 = (-1)^n \binom{2n+1}{n-1} C_h + \sum_{v=2}^{n+1} (-1)^{n-v+1} \binom{2n+2}{n+1-v} C_{vh}.$$

Inserting S_2 into S_1, then S_1 into (10), we have

$$\begin{aligned} I_{n+1} &= \frac{(-1)^n}{2^n}\binom{2n}{n}(C_h - I) + \frac{1}{2^n}\left[(-1)^n\binom{2n}{n-1}(C_h - I) + (-1)^n\binom{2n+1}{n-1}C_h\right] \\ &+ \frac{1}{2^n}\sum_{v=2}^{n+1}(-1)^{n+1-v}\binom{2n+2}{n+1-v}C_{vh} \\ &= \frac{(-1)^{n+1}}{2^n}\binom{2n+1}{n}I + \frac{1}{2^n}\sum_{v=1}^{n+1}(-1)^{n+1-v}\binom{2n+2}{n+1-v}C_{vh}. \end{aligned}$$

Since $\frac{1}{2}\binom{2n+2}{n+1} = \binom{2n+1}{n}$, we obtain the equation

$$I_{n+1} = \frac{1}{2^n}\sum_{v=0}^{n+1}{}'(-1)^{n-v+1}\binom{2n+2}{n+1-v}C_{vh},$$

which gives (9) for $l = n + 1$. ∎

Remark 3.4. *For the C-regularized cosine operator functions $\{C_h^C\}_{h\geq 0}$ the equality (9) is valid in the form*

$$(C_h^C - C)^l = \frac{1}{2^{l-1}}\left(\sum_{v=0}^{l}{}'(-1)^{l-v}\binom{2l}{l-v}C_{vh}^C\right)(C^{l-1}),$$

if C_h^C and C commute, i.e. $C_h^C(Cf) = C(C_h^C f)$ for each $f \in X,\ h > 0$.

4. Steklov averages and the approximation problems

In this part of the article we need the following notion (compare [1], Section 91).

Definition 4.1. *The **modulus of continuity** of order $k \in \mathbb{N}$ of $f \in X$ is defined for $\delta \geq 0$ via the cosine operator function in Definition 1.1 by*

$$\omega_k(f, \delta) := \sup_{0\leq h\leq \delta} \|(C_h - I)^k f\|.$$

Motivated by the trigonometric approximation [14], let us define

$$Q_{h,l}f := \frac{2}{\binom{2l}{l}}\sum_{v=1}^{l}(-1)^{v+1}\binom{2l}{l-v}S_{vh,2l}f. \tag{11}$$

Proposition 4.2. *Let $l \in \mathbb{N}$, $h > 0$. Then for any $f \in X$ we have $Q_{h,l}f \in D(A^l)$ and*
1) $\left\|f - Q_{h,l}f\right\| \leq \frac{2^l}{\binom{2l}{l}}\omega_l(f, 2lh)$,
2) $\left\|A^l(Q_{h,l}f)\right\| \leq \frac{1}{2^{l-1}\binom{2l}{l}h^{2l}}\sum_{v=1}^{l}\binom{2l}{l-v}v^{-2l}\omega_l(f, 2lh)$.

Proof. 1) By Proposition 2.6 we get

$$Q_{h,l}f = \frac{2}{\binom{2l}{l}}\sum_{v=1}^{l}(-1)^{v+1}\binom{2l}{l-v}\int_0^{2l}\varphi_{2l}(u)C_{vhu}f\,du = \frac{2}{\binom{2l}{l}}\int_0^{2l}\varphi_{2l}(u)\sum_{v=1}^{l}(-1)^{v+1}\binom{2l}{l-v}C_{vhu}f\,du.$$

By (8), taking $x = 0$, we have

$$\sum_{v=1}^{l}(-1)^{v+1}\binom{2l}{l-v} = \frac{1}{2}\binom{2l}{l}. \tag{12}$$

By (12) and Proposition 2.6, d) we have

$$f = \frac{2}{\binom{2l}{l}} \int_0^{2l} \varphi_{2l}(u) \sum_{v=1}^{l} (-1)^{v+1} \binom{2l}{l-v} f du,$$

hence

$$Q_{h,l}f - f = \frac{2}{\binom{2l}{l}} \int_0^{2l} \varphi_{2l}(u) \sum_{v=1}^{l} (-1)^{v+1} \binom{2l}{l-v} (C_{vhu} - I) f du. \tag{13}$$

By (9)

$$(-1)^l 2^{l-1} (C_h - I)^l = \frac{1}{2}\binom{2l}{l} I + \sum_{v=1}^{l} (-1)^v \binom{2l}{l-v} C_{vh}$$

and using (12) we obtain

$$(-1)^l 2^{l-1} (C_h - I)^l = \sum_{v=1}^{l} (-1)^v \binom{2l}{l-v} (C_{vh} - I).$$

We use in (13) the equality above, which gives

$$f - Q_{h,l}f = \frac{(-1)^l 2^l}{\binom{2l}{l}} \int_0^{2l} \varphi_{2l}(u) (C_{hu} - I)^l f du. \tag{14}$$

By properties a), d) in Proposition 2.6 and by definition of the modulus of continuity we obtain

$$\left\| f - Q_{h,l}f \right\| \leq \frac{2^l}{\binom{2l}{l}} \int_0^{2l} \varphi_{2l}(u) \omega_l(f, hu) du \leq \frac{2^l}{\binom{2l}{l}} \omega_l(f, 2lh).$$

2) According to Proposition 3.1 and equation (11) we get

$$A^l(Q_{h,l}f) = \frac{1}{2^{l-1}\binom{2l}{l} h^{2l}} \sum_{v=1}^{l} (-1)^{v+1} \binom{2l}{l-v} v^{-2l} (C_{2vh} - I)^l f,$$

which proves the statement 2). ■

Corollary 4.3. *The following evaluations hold for any $f \in X, h > 0$*
1) $\left\| f - S_{h,2}f \right\| \leq \omega(f, 2h)$;
2) $\left\| A(S_{h,2}f) \right\| \leq \frac{1}{2h^2} \omega(f, 2h)$.

Proof. It follows from Proposition 4.2 if we take $l = 1$. ■

Remark 4.4. *For the case of*

$$C_h f(x) = \frac{1}{2} \left(f(x+h) + f(x-h) \right), \ h \geq 0,$$

the infinitesimal generator of Definition B equals to the second derivative, i.e.

$$Af(x) = f''(x)$$

and the modulus of continuity of Definition 4.1 is related to the ordinary modulus of continuity of order two $\omega^*(f, \delta)$ (see [1], Section 91) by

$$\omega(f, \delta) := \omega_1(f, \delta) = \frac{1}{2} \omega^*(f, 2\delta).$$

Therefore, the statement of Corollary 4.3 precisely, including constants, coincides with a classical result in [1], Section 95.

To simplify the notation, let us denote the constants in Proposition 4.2 as follows:

$$q_1(l) := \frac{2^l}{\binom{2l}{l}}, \quad q_2(l) := \frac{1}{2^{l-1}\binom{2l}{l}} \sum_{v=1}^{l} \binom{2l}{l-v} v^{-2l} \quad (l \in \mathbb{N}). \tag{15}$$

Let us make some remarks concerning the constants $q_1(l), q_2(l)$. First, from equation (8) for $x = \pi$ it follows

$$\sum_{v=1}^{l} \binom{2l}{l-v} = 2^{2l-1} - \frac{1}{2}\binom{2l}{l}.$$

Hence,

$$q_2(l) \leq \frac{1}{2^{l-1}\binom{2l}{l}} \sum_{v=1}^{l} \binom{2l}{l-v} = \frac{2^l}{\binom{2l}{l}} - \frac{1}{2^l} \leq q_1(l).$$

Moreover, by the Stirling formula, $q_1(l) \approx \frac{\sqrt{\pi l}}{2^l}$, and

$$q_2(l) \leq q_1(l) \leq \frac{\sqrt{l}}{2^{l-1}}, \tag{16}$$

i.e. the constants in (15) are quite rapidly decreasing (comment: $q_1(l+1) = \frac{l+1}{2l+1} q_1(l)$, $q_1(1) = 1$, $q_1(2) = \frac{2}{3}$, $q_1(3) = \frac{2}{5}$, $q_1(4) = \frac{8}{35}, \ldots$).

The next theorem shows a general way to get direct or Jackson-type inequalities with constants in an explicit form for certain approximation methods.

Theorem 4.5. *Suppose a semi-norm $P : X \to (0, \infty) = \mathbb{R}^+$ is given such that there exist finite numbers*

$$M_0 := \sup_{f \in X} \frac{P(f)}{\|f\|}, \tag{17}$$

$$M_r := \sup_{f \in D(A^r)} \frac{P(f)}{\|A^r f\|} \quad (r \in \mathbb{N}). \tag{18}$$

Then for any $f \in X$, $h > 0$, $r \in \mathbb{N}$

$$P(f) \leq (M_0 q_1(r) + M_r q_2(r) h^{-2r}) \omega_r(f, 2rh), \tag{19}$$

where the constants $M_0, M_r, q_1(r), q_2(r)$ are defined by (17), (18) and (15), respectively.

Proof. According to assumptions (17) and (18) we write

$$P(f) \leq P(f - Q_{h,r} f) + P(Q_{h,r} f) \leq M_0 \|f - Q_{h,r} f\| + M_r \|A^r(Q_{h,r} f)\|.$$

The statement (19) follows now from Proposition 4.2. ∎

Remark 4.6. *The approach given above for the functional spaces was used in [14], [15] and in papers cited therein.*

Beside the modulus of continuity, another quantity, used in approximation problems, is the (Peetre) K-functional (see, e.g. [5], Ch. 6).

Definition 4.7. *(compare in [5], Ch. 6, equation (1.11)) Let be given the infinitesimal generator $A : D(A) \to X$ of a strongly continuous operator cosine function. The **K-functional** of an element $f \in X$ is defined for $t \geq 0$ via the formula*

$$K(f, t) := K(f, t; X, D(A^r)) := \inf_{g \in D(A^r)} (\|f - g\| + t\|A^r g\|), \quad t \geq 0.$$

In Approximation Theory, it is an important subject to compare the modulus of continuity and the K-functional (see, e.g. [5], Ch. 6). It appears that the Steklov avarages are good tools for that comparison.

We begin with some properties of the modulus of continuity in Definition 4.1, which are adaptations of the well-known properties of the ordinary modulus of continuity (see, e.g. [1], [5], [13])

Proposition 4.8. *The modulus of continuity $\omega_k(f,\delta)$ $(\omega(f,\delta) := \omega_1(f,\delta))$ in Definition 4.1 has the following properties:*

(i) $\omega_k(f, m\delta) \le m^k(1 + (m-1)T)^k\omega_k(f,\delta)$, $m \in \mathbb{N}$;

(ii) $\omega_k(f,\lambda\delta) \le ([\lambda]+1)^k(1+[\lambda]T)^k\omega_k(f,\delta)$, $\lambda > 0$, ($[\lambda] \le \lambda$ *is the entire part of* $\lambda \in \mathbb{R}$);

(iii) $\omega_k(f,\delta) \le (1+T)^{k-l}\omega_l(f,\delta)$, $k \ge l$ *and* $k, l = 0, 1, 2, \dots$;

(iv) for $f \in D(A^k)$, $k = 0, 1, ...,$ *there holds*

$$\omega_{k+l}(f,h) \le \left(\frac{Th^2}{2}\right)^k \omega_l(A^k f, h) \quad (l = 0, 1, \dots).$$

Proof. The inequalities (i)–(iii) can be proved in a very similar way as the classical ones. For (iv), first we prove that for $f \in D(A^k)$, $k = 0, 1, ...$ there holds

$$\left\|(C_t - I)^{k+l} f\right\| \le \left(\frac{Tt^2}{2}\right)^k \left\|(C_t - I)^l A^k f\right\| \quad (l = 0, 1, ...).$$

Then (iv) follows by taking supremum. Let us fix $l = 0, 1, ...$ and consider induction by k. For $k = 0$ the statement is obvious. Consider

$$(C_t - I)^{k+1+l} f = (C_t - I)(C_t - I)^{k+l} f = \int_0^t (t-u) C_u A (C_t - I)^{k+l} f du,$$

which is valid by Theorem C, (ii). Since $\|C_u\|_{[X]} \le T$ and A and C_t are commutative we get

$$\left\|(C_t - I)^{k+1+l} f\right\| \le \int_0^t (t-u)\, \|C_u\|_{[X]} \left\|(C_t - I)^{k+l} A f\right\| du \le \frac{Tt^2}{2} \left\|(C_t - I)^{k+l} A f\right\|.$$

Therefore, by assumption of the induction

$$\left\|(C_t - I)^{k+1+l} f\right\| \le \frac{Tt^2}{2} \left(\frac{Tt^2}{2}\right)^k \left\|(C_t - I)^l A^{k+1} f\right\|.$$

■

The comparison theorem between the modulus of continuity and the K-functional reads as follows.

Theorem 4.9. *For any* $f \in X$, $t > 0$, $r \in \mathbb{N}$ *there hold the inequalities*

$$c_1(r)\omega_r(f,t) \le K(f, t^{2r}) \le c_2(r)\omega_r(f,t),$$

where the constants $c_2(r) \ge c_1(r) > 0$ *are independent on* $f \in X$ *and* $t > 0$.

Proof. 1) For the left-hand side inequality let us take $g \in D(A^r)$. By the properties of the modulus of continuity we have

$$\begin{aligned} \omega_r(f,t) &\le \omega_r(f-g,t) + \omega_r(g,t) \le (T+1)^r\|f-g\| + (T/2)^r t^{2r}\|A^r g\| \\ &\le \max((T+1)^r, (T/2)^r) K(f, t^{2r}). \end{aligned}$$

2) For the right-hand side inequality let us notice that for any $g \in D(A^r)$ we have

$$K(f, t^{2r}) \leq \|f - g\| + t^{2r}\|A^r g\|, \quad t \geq 0.$$

By Proposition 4.2 $Q_{t,r}f \in D(A^r)$, hence

$$K(f, t^{2r}) \leq \|f - Q_{t,r}f\| + t^{2r}\|A^r(Q_{t,r}f)\|.$$

Again, by Proposition 4.2 using notations in (15) we have

$$K(f, t^{2r}) \leq q_1(r)\omega_r(f, 2rt) + q_2(r)t^{2r}t^{-2r}\omega_r(f, 2rt) = (q_1(r) + q_2(r))\omega_r(f, 2rt).$$

On the one hand, by inequality (16) $q_1(r) + q_2(r) \leq \sqrt{r}/2^{r-2}$. On the other hand, by the property of the modulus of continuity

$$\omega_r(f, 2rt) \leq (2r)^r(1 + (2r - 1)T)^r\omega_r(f, t).$$

Therefore, we get

$$K(f, t^{2r}) \leq 4r^{r+1/2}(1 + (2r - 1)T)^r\omega_r(f, t).$$

■

Remark 4.10. *The constant $c_2(r)$ in Theorem 4.9 is somehow smaller than that in [5], Ch. 6, Theorem 2.4, or in [3], Theorem 1. Indeed, in these cases $T = 1$ and for the constant $c_2(r)$ we get the estimate $c_2(r) \leq 4r^{r+1/2}(1+(2r-1)T)^r = 2^{r+2}r^{2r+1/2}$, whereas in [3], Theorem 1, $c_2(r) \leq 2^{5r}r^{2r} + 2^r$.*

5. A Jackson-type theorem with Chebyshev cosine operator function

Below, X stands for one of the Banach spaces $C_{[-1,1]}$ or $X = L^p_w, (1 \leq p < \infty)$, defined as in Remark 2.3. For these spaces a suitable cosine operator function is defined by (in following notations the letter C stands for Chebyshev)

$$(C^C_h f)(x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1 - x^2}\sin h) + f(x\cos h - \sqrt{1 - x^2}\sin h)\right\}. \tag{20}$$

Our purpose is to study the **best algebraic approximation** of $f \in X$ by polynomials of degree n in X,

$$E_n(f) := \inf_{p\in P_n} \|f - p\|,$$

for which we will apply Theorem 4.5, i.e. we consider the semi-norm $P(f) = E_n(f)$. Obviously, since $E_n(f) \leq \|f\|$, in Theorem 4.5 the constant $M_0 = 1$.

For the constant M_r we need further results from [3]. The main tool to prove statements in [3] is the Chebyshev translation operator (see [3], formula (1.2))

$$(\tau_h f)(x) := \frac{1}{2}\left\{f(xh + \sqrt{(1 - x^2)(1 - h^2)}) + f(xh - \sqrt{(1 - x^2)(1 - h^2)})\right\}, \quad x, h \in [-1, 1],$$

which is related with the cosine operator function (20) via equality

$$(C^C_h f)(x) = (\tau_{\cos h}f)(x), \quad h \in [0, \pi].$$

Obviously, $\tau_h : X \to X$ is equibounded, i.e. $\|\tau_h f\| \leq \|f\|$, moreover, strongly continuous (see [3], Lemma 2), i.e. $\lim_{h\to 1-} \|\tau_h f - f\| = 0$. Hence C^C_h satisfies $\lim_{h\to 0+} \|C^C_h f - f\| = 0$, i.e. strongly continuous and by Definition B we may calculate the generator

$$(Af)(x) = (1 - x^2)f''(x) - xf(x), \quad x \in [-1, 1].$$

Analogously to [3], formula (1.3), we define the (Chebyshev) strong derivative as the function $g \in X$ for which

$$\lim_{h\to 0+} \|\frac{C_h^C f - f}{1-\cos h} - g\| = 0,$$

and write $D_C^1 f = g$. The higher derivatives are defined iteratively for $r = 1, 2, \dots$ by $D_C^{r+1} f = D_C^1(D_C^r f)$ with $D_C^0 f = f$. The (Sobolev) class W_X^r is the set of $f \in X$ for which $D_C^r f$ exists.

Denote by $X_{2\pi}$ the space of 2π-periodic functions as the counterpart of space X in the beginning of this subsection. Then a quite obvious statement holds.

Proposition 5.1. *The function $f \in X$ has the Chebyshev derivative $D_C^r f \in X (r = 0, 1, 2, \dots)$ iff $f \circ \cos \in X_{2\pi}$ has the ordinary derivative of order $2r$. Moreover,*

$$(D_C^r f)(\cos x) = (-1)^r (D_{2\pi}^{2r}(f \circ \cos))(x).$$

Calculating

$$(D_{2\pi}''(f \circ \cos))(x) = \frac{d^2}{dx^2} f(\cos x) = \sin^2 x f''(\cos x) - \cos x f'(\cos x),$$

we see that

$$(D_C^1 f)(u) = -(Af)(u).$$

Hence, the Sobolev class W_X^1 coincides with the domain $D(A)$ of the generator A, and, in general, $W_X^r = D(A^r)$.

Now we are able to reformulate, in our notations, an important statement from [3].

Theorem 5.2. *([3], Proposition 3.a)) If $f \in D(A^r), r \in \mathbb{N}$, then there exists a constant $C_r > 0$, being independent of f and n, such that*

$$E_n(f) \le C_r n^{-2r} \|A^r f\|.$$

Remark 5.3. *In Trigonometric Approximation there is a deep statement (sometimes called as Theorem of Akhiezer-Krein-Favard), analogous to the previous theorem, but with the exact constant $K_r < \frac{\pi}{2}$, called as the Favard numbers (see, e.g. [5], Ch. 5, §5, and Ch. 7, Theorem 4.3). Unfortunately, here we do not know any idea to have a reasonable estimate for the constant C_r. At least, the proof of Proposition 3 of [3] overestimates that constant very much.*

Anyway, to apply Theorem 4.5, we have now by previous theorem that $M_r = C_r n^{-2r}$, and if take $h = 1/n$, then we get for any $f \in X, r \in \mathbb{N}$ a Jackson-type estimate

$$E_n(f) \le (q_1(r) + C_r q_2(r))\omega_r(f, \frac{2r}{n}),$$

or by the estimate (16) in a more compact form as

$$E_n(f) \le \frac{\sqrt{r}}{2^{r-1}}(1 + C_r)\omega_r(f, \frac{2r}{n}).$$

This estimate is not new by the order, compare [3], the new part of that estimate is the constant $\frac{\sqrt{r}}{2^{r-1}}(1 + C_r)$ only.

The Chebyshev cosine operator function is useful for the Chebyshev-Fourier series. One might ask how about other classical orthogonal polynomials?

6. A Jackson-type theorem using Legendre polynomials

In this Section, as above, X stands for one of the Banach spaces $C_{[-1,1]}$ or $X = L^p_w$, $(1 \leq p < \infty)$. Here we may present a general approach due to S. Z. Rafal'son [9], [10]. Let the system of algebraic polynomials $\{P_k\}_{k=0}^\infty$, defined on $[-1,1]$, be orthonormal with respect to the weight $w(x) > 0$. Moreover, suppose $\|P_k\|_C = P_k(1)$. The Fourier coefficients of $f \in X$ via the system $\{P_k\}_{k=0}^\infty$ is denoted by $c_k(f)$. The generalized translation operator $\tau_t : X \to X, t \in [0,\pi]$ is defined by properties:

a) for any $f \in X$ and $t \in [0,\pi]$ there holds $\|\tau_t f\| \leq \|f\|$,

b) for any $f \in X$ and $t \in [0,\pi]$ the Fourier coefficients of $\tau_t f$ satisfy the equation

$$c_k(\tau_t f) = c_k(f) P_k(\cos t)/P_k(1), \quad k = 0,1,2,\ldots$$

In this framework S. Z. Rafal'son [9] announced (without proof) an Akhiezer-Krein-Favard-type theorem for the Legendre polynomials (then $w(x) = 1$). Unfortunately, in case of Legendre polynomials, the operator $\tau_t : X \to X$ does not form the cosine operator function. Indeed, by the definition of $\tau_t : X \to X$, comparing the Fourier coefficients, for the cosine operator function we should have

$$P_k(\cos s) P_k(\cos t) = \frac{P_k(1)}{2}(P_k(\cos(s+t)) + P_k(\cos(s-t))), \quad s,t \in [0,\pi], \quad k = 0,1,2,\ldots \tag{21}$$

The equation (21) is valid for the orthonormal system of the Chebyshev polynomials $\{T_0(x) = \sqrt{\frac{2}{\pi}}, T_k(x) = \frac{2}{\sqrt{\pi}}\cos(k \arccos x), k \in N\}$. For the Chebyshev system the generalized translation operator $\tau_h : X \to X$ coincides with the cosine operator function (20) (see [9], Section 3).

S. Z. Rafal'son [9] considers more precisely the system of Legendre polynomials, consisting of polynomials

$$P_k(x) = \frac{(-1)^k}{2^k k!}\frac{d^k}{dx^k}(1-x^2)^k \quad (x \in [-1,1]; k = 0,1,2,\ldots).$$

In particular, $P_2(x) = (3x^2-1)/2$, which does not satisfy the equation (21), hence, the corresponding generalized translation operator ([9], Section 3; letter L in the notation below stands for Legendre)

$$(\tau_t^L f)(x) := \frac{1}{\pi}\int_{-1}^1 f(x\cos t + u\sqrt{1-x^2}\sin t)\frac{du}{\sqrt{1-u^2}}, \tag{22}$$

does not form the cosine operator function. Nevertheless, the given generalized translation operator is very useful in approximation theory ([12]). Therefore, we cite here by S. Z. Rafal'son [9] an Akhiezer-Krein-Favard-type theorem with the (almost) exact constant therein.

For $f \in C_{[-1,1]}$, for which there exists $f^{(2r)} \in C_{[-1,1]}$, we denote

$$(D^r f)(t) := \frac{d^r}{dt^r}\left((1-t^2)^r f^{(r)}(t)\right).$$

For $r = 1$ this is, up to a constant factor, as the strong (Legendre-) derivative $D_L^r f$ of $f \in X$ in [12], Subsection 3, Definition 1, namely $D_L^1 f = -\frac{1}{2}D^1 f$. Unfortunately, for the higher derivatives there are no simple relations between these derivatives, for example, $D_L^2 f = \frac{1}{4}(D^2 f - 2D^1 f)$.

Theorem 6.1. *([9], Subsection 6, Theorem 6) If $f^{(2r)} \in C_{[-1,1]}$, $r \in \mathbb{N}$, then for any $n \geq r-1$*

$$E_n(f)_C \leq M_{r,n} E_n(D^r f)_C,$$

where

$$M_{r,n} := \frac{4(n+2)\Gamma(r+1/2)}{\sqrt{\pi}\Gamma(r)}\sum_{\nu=0}^\infty \frac{((2\nu+1)(n+2)-r-1)!}{((2\nu+1)(n+2)+r)!}.$$

Moreover, $c_1(r)M_{r,n} \leq n^{-2r} \leq c_2(r)M_{r,n}$ with certain constants $0 < c_1(r) \leq c_2(r)$, not depending on n.

Since we are interested in constants in Jackson-type inequalities, we shall look for estimates of the constants $M_{r,n}$.

Proposition 6.2. *For $n \geq r-1$ we have*

$$\frac{4\Gamma(r+1/2)}{\sqrt{\pi}\Gamma(r)(n+2)^{2r}}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^{2r+1}} \leq M_{r,n} \leq \frac{4\Gamma(r+1/2)(r+1)^{2r}}{\sqrt{\pi}\Gamma(r)(2r+1)^{r}(n+2)^{2r}}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^{2r+1}} \quad (r \geq 2),$$

in particular, $\frac{2\zeta(3)}{(n+2)^2} \leq M_{1,n} \leq \frac{7\zeta(3)}{3(n+2)^2}$, where $\zeta(z)$ is the Riemann zeta function.

Proof. Denote $\ell := (2\nu+1)(n+2)$, then we may rewrite the sum in the expression of $M_{r,n}$ as

$$S_{r,n} := \frac{1}{(n+2)^{2r+1}}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^{2r+1}}\frac{1}{(1-\frac{r^2}{\ell^2})(1-\frac{(r-1)^2}{\ell^2})\dots(1-\frac{1^2}{\ell^2})} \geq \frac{1}{(n+2)^{2r+1}}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^{2r+1}}.$$

Since $\ell \geq n+2 \geq r+1$, for $j = 1, 2, \dots r$ we estimate

$$\frac{1}{1-\frac{j^2}{\ell^2}} \leq \frac{1}{1-\frac{j^2}{(n+2)^2}} = 1 + \frac{j^2}{(n+2)^2 - j^2} \leq 1 + \frac{r^2}{2n+3} \leq 1 + \frac{r^2}{2r+1} = \frac{(r+1)^2}{2r+1}$$

for $r \geq 2$. If $j = r = 1$, then $\frac{1}{1-1/\ell^2} \leq \frac{4}{3}$. ∎

To prove a Jackson-type theorem, e.g. Theorem 4.5, it is not ultimately necessary to work in the cosine operator framework. It is important to have in hand a good approximation operator such that the statements of Proposition 4.2 are true. In the Legendre case the suitable operator is introduced and studied in [4] and [12]. In [12], Sec. 3, there had been introduced the (Legendre-) integral or Steklov means $S_h f, f \in X, h \in (-1,1)$. They are defined, using the kernel ($\log^{-1}(z) \equiv 1/\log(z)$)

$$\kappa(x;h) := \log\frac{(1+x)(1-h)}{(1-x)(1+h)}\log^{-1}(\frac{2}{1+h}), \quad -1 < h \leq x < 1,$$

and $\kappa(x;h) := 0$ otherwise. It is important to mention that this kernel is non-negative for $x, h \in [-1,1]$ and

$$\frac{1}{2}\int_{-1}^{1}\kappa(u;h)du = 1.$$

Then the Steklov means are defined via the integral

$$(S_h f)(x) := \frac{1}{2}\int_{-1}^{1}(\tau_u^L f)(x)\kappa(u;h)du,$$

where the translation operator $\tau_u^L : X \to X$ is defined in (22).

The counterpart of the modulus of continuity takes here the next form (compare Definition 4.1)

$$\omega^L(f,\delta) := \sup_{\delta \leq h \leq 1}\|\tau_h^L f - f\|,$$

for which $\lim_{\delta\to 1-}\omega^L(f,\delta) = 0$. The set of all $f \in X$ for which the strong (Legendre-) derivative $D_L^1 f$ exists as an element of X is denoted by W_X^1.

We reformulate (and improve a little) a statement from [4] and [12].

Proposition 6.3. *([4], Theorem 3, Part a) for 1)and [12], Corollary 2, (d) for 2), respectively) For the Steklov means $S_h f, f \in X$, for each $h \in (-1,1)$ it holds:*

1) $\left\|S_h f - f\right\| \leq \omega^L(f,h)$,

2) $S_h f \in W_X^1$ and $\left\|D_L^1(S_h f)\right\| \leq \frac{1}{1-h}\omega^L(f,h)$.

Proof. 1) By the definition of the kernel $\kappa(u;h)$ we have

$$(S_h f)(x) - f(x) = \frac{1}{2}\int_h^1 \Big((\tau_u^L f)(x) - f(x)\Big)\kappa(u;h)du,$$

hence, by the definition of the modulus of continuity

$$\|S_h f - f\| \le \frac{1}{2}\int_h^1 \sup_{h\le u\le 1}\|\tau_u^L f - f\|\kappa(u;h)du = \omega^L(f,h).$$

2) By [12], Corollary 2, (d)

$$(D_L^1(S_h f))(x) = \frac{1}{2}\log^{-1}(\frac{2}{1+h})\Big(f(x) - (\tau_h^L f)(x)\Big).$$

Since $1-h \le 2\log(\frac{2}{1+h}),\ h\in(-1,1]$, we have proved the statement 2). ■

Proposition 6.3 yields, analogously to Theorem 4.5, next

Theorem 6.4. *Suppose a semi-norm $P: X \to (0,\infty) = \mathbb{R}^+$ is given in space X such that there exist finite numbers*

$$M_0 := \sup_{f\in X}\frac{P(f)}{\|f\|}, \tag{23}$$

$$M_1 := \sup_{f\in W_X^1}\frac{P(f)}{\|D_L^1 f\|}. \tag{24}$$

Then for any $f\in X$, $0<h<1$,

$$P(f) \le (M_0 + \frac{1}{1-h}M_1)\omega^L(f,h). \tag{25}$$

As a corollary we obtain a Jackson-type inequality with a constant in explicit shape for the Legendre framework.

Corollary 6.5. *For all $f\in C_{[-1,1]}, n\in\mathbb{N}$ the algebraic best approximation satisfies the inequality*

$$E_n(f)_C \le \Big(1 + \frac{14}{3}\zeta(3)\Big)\omega^L(f, 1-\frac{1}{(n+2)^2})_C,$$

where $\zeta(3) = 1.2020569\ldots$ (Riemann dzeta function).

Proof. Firstly we take in Theorem 6.4 $P(f) = E_n(f)_C$, then we get by Rafal'son's Theorem 6.1 the constant M_1 in (24) equals to $\frac{14}{3(n+2)^2}\zeta(3)$, since $D^1 f = -2D_L^1$ and therefore $E_n(D^1 f)_C = 2E_n(D_L^1 f)_C \le 2\|D_L^1 f\|_C$. Lastly, we prove the statement using (25) with $h = 1-\frac{1}{(n+2)^2}$. ■

References

[1] N. I. Akhiezer, *Lectures in the Theory of Approximation*. Second revised and enlarged edition, Izdat. "Nauka", Moscow, 1965 (in Russian).

[2] P. L. Butzer, A. Gessinger, Ergodic theorems for semigroups and cosine operator functions at zero and infinity with rates; applications to partial differential equations. A survey. In: *Mathematical Analysis, Wavelets, and Signal Processing.*, Contemporary Mathematics, 190(1995), 67-94.

[3] P. L. Butzer and R. Stens, Chebyshev transform methods in the theory of best algebraic approximation. Abh. Math. Seminar. Univ. Hamburg, 45 (1976), 165 - 190.

[4] P. L. Butzer, R. L. Stens, M. Wehrens, Approximation by algebraic convolution integrals. In: *Approximation Theory and Functional Analysis.* J. B. Prolla (ed.), North-Holland Publ. Co., 1979, 71–120.

[5] R. A. DeVore, G. G. Lorentz, *Constructive Approximation.* Springer-Verlag, 1993.

[6] M. Kostić, Generalized Semigroups and Cosine Functions. Mathematical Institute Belgrade, 2011.

[7] D. Lutz. Strongly continuous operator cosine functions. In: *Functional Analysis. Proc., Dubrovnik, 1981*, Lect Notes in Math. 948 (1982), Eds. Butković,D. , Kaljević, H., Kurepa, S.

[8] D. Popa, I. Raça, Steklov Averages as Positive Linear Operators. Filomat 30:5 (2016), 1195–1201. DOI 10.2298/FIL1605195P

[9] S. Z. Rafal'son, Generalized shift operator in the theory of orthogonal polynomials. In: *Constructive function Theory '81*, Sofia, 1983, 150–157.

[10] S. Z. Rafal'son, Generalized shift, generalized convolution and some extremal relations in the theory of the approximation of functions. *J. Math. Sci.* 42 (1988), 1646–1651. https://doi.org/10.1007/BF01665053 Translated from *Zapiski Nauchnykh Seminarov POMI* 149 (1986), 150—157.

[11] M. Sova, Cosine operator functions. *Rozprawy matematyczne*, 49 (1966), Warszawa, (Dissertation, 49 pp.).

[12] R. L. Stens, M. Wehrens, Legendre transform methods and best algebraic approximation. *Commentationes mathematicae* 21 (1979), no 2, 351-380. DOI:10.14708/cm.v21i2.5996

[13] A. F. Timan, Theory of Approximation of Functions of a Real Variable, Dover Publications, New York, 1994.

[14] O. L. Vinogradov, V. V. Zhuk, Estimates for functionals with a known moment sequence in terms of deviations of Steklov type means. *J. Math. Sci.*, Springer US, 178 (2) (2011) DOI: 10.1007/s10958-011-0531-3 (Translated from *Zapiski Nauchnykh Seminarov POMI*, 383 (2010), 5–32.).

[15] V. V. Zhuk, Inequalities of the type of the generalized Jackson theorem for the best approximations. *J Math Sci*, Springer US, 2013, Vol. 193, Issue 1, pp 75–88 (Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 404, 2012, pp. 135–156.).

[16] D. Zwillinger and V. Moll (eds.), Grandshteyn and Ryzhik's Table of Integrals, Series and Products. Eighth edition. Academic Press, 2014.