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Global existence and boundedness of solutions in a reaction-diffusion system of Michaelis-Menten-type predator-prey model with nonlinear prey-taxis and random diffusion

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Abstract. This article deals with a 2×2 reaction-diffusion-taxis model consisting of Michaelis-Menten functional response predator-prey system. The critical section of this model is that temporal-spatial evolution of the predators' velocity depends largely on the gradient of prey. But beyond that, this system also inscribes a prey-taxis mechanism that is an immediate movement of the predator u in response to a change of the prey v (which leads to the collection of u). By using contraction mapping principle, L^p estimates and Schauder estimates of parabolic equations, we prove the global existence and uniqueness of classical solutions to this model. In addition to this, we prove the global boundedness of solutions by overcome the difficulties brought by nonlinear prey-taxis.

1. Introduction

In this article, we study the following Michaelis-Menten reaction-diffusion system of predator-prey model with prey-taxis:

$$\begin{cases} u_t - d_1\Delta u + \nabla \cdot (u\chi(u)\nabla v) = -au + \beta \dfrac{cuv}{u+bv}, & x \in \Omega, t \in (0,T), \\ v_t - d_2\Delta v = rv - \dfrac{r}{K}v^2 - \dfrac{cuv}{u+bv}, & x \in \Omega, t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq 0, & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N = 1,2,3)$ with smooth boundary $\partial\Omega \in C^{2+\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$, $0 < T \leq +\infty$, initial condition $u_0(x), v_0(x) \in C^{2+\alpha}(\overline{\Omega})$ compatible on $\partial\Omega$, the constants d_1, d_2, a, K, r, β, b, c are nonnegative and ecological, and ν is the outward directional derivative normal to $\partial\Omega$. Throughout the paper, a and r reflect the death rate of u (predator) and the intrinsic growth rates of v (prey), respectively. K stands for the carrying capacity of prey v. β denotes the conversion rate of the species. We will use

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the symbols b and c to denote the handling time taken by predator to capture and expend prey and the efficiency of searching of predator, respectively.

There is the Michaelis-Menten functional response contained in the model (1), where u and v represent the population density of two species at time t with diffusion rates d_1 and d_2 (the tendency of random walks of the species), respectively. In fact, there are many well-known reaction-diffusion models such as Keller-Segel model [5, 6], Holling-type models [7], Holling-type II models [8], Ivlev-type models [9], Lotka-Volterra-type models [3, 4, 10, 11, 24–26] and so on. The system (1) was introduced by Michaelis and Menten [22]. Recently, it is of great interests to investigate the Michaelis-Menten predator-prey system. In 2011, Baek and Lim [14] discussed the dynamics of an impulsively controlled Michaelis-Menten type predator-prey system. They obtain some conditions for the existence and stability of prey-free solutions of the system by using the Floquet theory. In 2005, The stability and bifurcation analysis for a predator-prey system with the nonlinear Michaelis-Menten type predator harvesting are taken into account by Hu and Cao [15]. On the other hand, the researchers in [16] investigates the global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system. With the rise of biological mathematics, many scientists and mathematicians apply their efforts to Partial Differential Equations (PDEs), especially in nonlinear parabolic partial differential equations [17, 18]. In addition, PDEs are supposed to be sufficient in modeling of the countless processes in all fields of science. Many phenomena in physical sciences, chemistry and biology are naturally described by PDEs, such as competition systems, chemotaxis systems, predator-prey models and so on.

Under some certain conditions, Fan and Li [2] obtained the global asymptotic stability of the unique positive constant equilibrium of the following problem, a similar model to (1), by applying constructing suitable Lyapunov functions and the monotone iteration,

$$\begin{cases} u_t - d_1\Delta u = -au + \beta\dfrac{cuv}{u+bv}, & x\in\Omega, t\in(0,T),\\ v_t - d_2\Delta v = rv - \dfrac{r}{K}v^2 - \dfrac{cuv}{u+bv}, & x\in\Omega, t\in(0,T),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T),\\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)) \geq 0, & x\in\Omega, \end{cases} \tag{2}$$

which is a Michaelis-Menten reaction-diffusion system with diffusion. It is well known that the global boundedness of solutions to (2) has been proved by [23]. However, the existence of nonlinear prey-taxis brings enormous difficulties to obtain the global boundedness of solutions to (1).

In the present work, motivated by [2] and [13], we will consider the global boundedness of classical solutions to (1) under the predator-prey-taxis mechanism with simplified conditions on $\chi(u)$, which is weaker than that supposed in [1]. The following theorems are the main results of this paper.

Theorem 1.1. *Suppose that $\chi(u)$ satisfies*

(i) $\chi(u)\in C^1([0,+\infty))$;

(ii) $\chi(u)\equiv 0$ *for* $u\geq M$, *with* $M>0$;

(iii) $|\chi'(u_1)-\chi'(u_2)|\leq L|u_1-u_2|$ *for* $u_1,u_2\in[0,+\infty)$, *with* $L>0$;

(iv) $u\chi(u)$ *and* $(u\chi(u))'$ *are bounded, and* $(u\chi(u))'$ *is Lipschitz continuous,*

and $v_0\leq K$, *then there exist a unique classical solution*

$$(u(x,t),v(x,t))\in C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))$$

of the system (1) for any given $T>0$. *In addition,*

$$u\geq 0,\quad 0\leq v\leq K. \tag{3}$$

Theorem 1.2. *Suppose that $\chi(u)$ satisfies*

(i) $\chi(u) \in C^1([0, +\infty))$;

(ii) $\chi(u) \equiv 0$ *for* $u \geq M$, *with* $M > 0$;

(iii) $|\chi'(u_1) - \chi'(u_2)| \leq L|u_1 - u_2|$ *for* $u_1, u_2 \in [0, +\infty)$, *with* $L > 0$,

and $v_0 \leq K$, *then we have that the solutions to (1) are global and uniformly bounded in time.*

The prey-taxis mechanism contained in the system means a immediate movement of the predator u in response to a change of the prey v (which lead to the collection of u). Here we assume that $\chi(u) \equiv 0$ for $u \geq M$ means that there exists a marginal value M for the cumulation of predator u, over which the prey-tactic cross-diffusion $\chi(u)$ vanishes. In addition, it is necessary for the existence of classical solutions of the system (1) to suppose that $\chi'(u)$ satisfies $|\chi'(u_1) - \chi'(u_2)| \leq L|u_1 - u_2|$ for $u_1, u_2 \in [0, +\infty)$, with $L > 0$. Refer to Remark 2.1 in [13] for a detailed explanation. Throughout this paper we also denote that $\omega(u) = u\chi(u)$, then it follows from the assumptions of Theorem 1.1 that $\omega(u)$ and $\omega'(u)$ are bounded, and $\omega'(u)$ is Lipschitz continuous.

The remainder of this article is organized as follows. In Section 2, we illustrates the proof of Theorem 1.1 which are essential to the proof of Theorem 1.2. Section 3 illustrates the proof of Theorem 1.2. In Section 4, we will discuss how to generalize our results to more general setting.

2. Existence of global solutions

In this section we illustrates the proof of Theorem 1.1 which are essential in the proofs of Theorem 1.2 due to the difficulties brought by nonlinear prey-taxis. Firstly, we need some preliminary results.

Lemma 2.1. *Let* $(u, v) \in (C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega\times(0,T)))^2$ *be a solution of (1). Then* $u \geq 0$ *and* $0 \leq v \leq K_0 = \max\left\{\max_{\overline{\Omega}} v_0(x), K\right\}$.

Proof. We consider the following system of predator

$$\begin{cases} u_t - d_1\Delta u + \omega'(u)\nabla v \cdot \nabla u + \left[\chi(u)\Delta v + a - \beta\dfrac{cv}{u + bv}\right]u = 0, & x \in \Omega, t \in (0, T), \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T), \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{4}$$

Obviously, $\underline{u} \equiv 0$ is a sub-solutions to system (4). Therefore, using the maximum principle, we can obtain that $u \geq 0$. By the same way, $v \geq 0$ is also obtained.

In addition, we also study the following system of prey

$$\begin{cases} v_t - d_1\Delta v = rv - \dfrac{r}{K}v^2 - \dfrac{cuv}{u + bv} \leq rv - \dfrac{r}{K}v^2, & x \in \Omega, t \in (0, T), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T), \\ v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{5}$$

$\overline{v}(t)$ stands for a solution of the model

$$\begin{cases} \dfrac{d\overline{v}(t)}{dt} = r\overline{v}(t) - \dfrac{r}{K}\overline{v}^2(t), \\ \overline{v}(0) = \max\limits_{\Omega} v_0(x) \leq K. \end{cases} \tag{6}$$

Easily, $\overline{v}(t)$ $(0 \leq \overline{v}(t) \leq K)$ is a sup-solution to model (5). Therefore, using the maximum principle,

$$v(x, t) \leq \overline{v}(t) \leq K. \tag{7}$$

The proof is complete. □

Now, we need to establish a priori estimate of u.

Lemma 2.2. *Assume that $(u, v) \in C^{2,1}(\Omega \times (0, T))$ is a solution of (1), then there holds $\|u\|_{L^{p+1}(\Omega\times(0,T))} \leq C$ for any $p > 1$.*

Proof. Multiplying $u_t - d_1\Delta u + \nabla \cdot (\chi(u)u\nabla v) = \left(-a + \beta\frac{cv}{u+bv}\right)u$ by u^p, integrating over $\Omega \times (0, t)$, using the no-flux boundary condition $\frac{\partial u}{\partial \nu} = 0$, and noting $u \geq 0$, $0 \leq v \leq K_0$ and $0 \leq \frac{cv}{u+bv} \leq \frac{c}{b}$, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{d}{dt}u^{p+1}dt - d_1\int_0^t \int_\Omega \Delta u \cdot u^p dt \\
=&\int_\Omega u^{p+1}(t) - \int_\Omega u^{p+1}(0) + (p+1)pd_1\int_0^t \int_\Omega u^{p-1}|\nabla u|^2 dt \\
=&-\int_0^t \int_\Omega \nabla \cdot (\chi(u)u\nabla v)\cdot u^p dt + \int_0^t \int_\Omega \left(-a + \beta\frac{cv}{u+bv}\right)u^{p+1}dt \\
\leq&(p+1)p\int_0^t \int_\Omega u^p\chi(u)\nabla u \cdot \nabla v dt + \frac{\beta c}{b}\int_0^t \int_\Omega u^{p+1}dt.
\end{aligned} \tag{8}$$

Applying Young's inequality and the assumption of $\chi(u)$ yields

$$\begin{aligned}
\left|\int_0^t \int_\Omega u^p\chi(u)\nabla u \cdot \nabla v dt\right| \leq& \left|\int_0^t \int_\Omega u^p\chi(u)|\nabla u \cdot \nabla v| dt\right| \\
=&\left|\int_0^t \int_\Omega u^{\frac{p+1}{2}\cdot\frac{p-1}{2}}\chi(u)|\nabla u \cdot \nabla v| dt\right| \\
\leq&M^{\frac{p+1}{2}}\left|\int_0^t \int_\Omega u^{\frac{p-1}{2}}\chi(u)|\nabla u \cdot \nabla v| dt\right| \\
\leq&M^{\frac{p+1}{2}}\max_{0\leq u\leq M}\chi(u)\int_0^t \int_\Omega \left|u^{\frac{p-1}{2}}\nabla u \cdot \nabla v\right| dt \\
=&M^{\frac{p+1}{2}}\max_{0\leq u\leq M}\chi(u)\int_0^t \int_\Omega \left|u^{\frac{p-1}{2}}\nabla u\right| \cdot |\nabla v|\, dt \\
\leq&\varepsilon\int_0^t \int_\Omega u^{p-1}|\nabla u|^2 dt + \frac{C_0}{2\varepsilon}\int_0^t \int_\Omega |\nabla v|^2 dt
\end{aligned} \tag{9}$$

for any sufficiently small $\varepsilon > 0$.

Multiplying $v_t - d_2\Delta v = \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v$ by v, integrating over $\Omega \times (0, t)$, applying the no-flux boundary condition $\frac{\partial v}{\partial \nu} = 0$, and noting $u \geq 0$, $v \geq 0$ and $0 \leq \left(r - \frac{r}{K}v\right)v \leq rv$, we obtain

$$\begin{aligned}
\int_0^t \int_\Omega \frac{d}{dt}v^2 dt - d_2\int_0^t \int_\Omega \Delta v \cdot v dt =& \int_\Omega v^2(t) - \int_\Omega v^2(0) + 2d_2\int_0^t \int_\Omega |\nabla v|^2 dt \\
=&\int_0^t \int_\Omega \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v^2 dt \\
=&\int_0^t \int_\Omega \left(r - \frac{r}{K}v\right)v^2 dt \\
\leq& r\int_0^t \int_\Omega v^2 dt.
\end{aligned}$$

According to $0 \leq v \leq K_0$, we obtain

$$\int_0^t \int_\Omega |\nabla v|^2 dt \leq C. \tag{10}$$

Based on (8), (9) and (10), we have

$$\int_\Omega u^{p+1}(t) + (p+1)p(d_1-\varepsilon)\int_0^t\int_\Omega u^{p-1}|\nabla u|^2 dt \le C + C_0\int_0^t\int_\Omega u^{p+1}dt. \tag{11}$$

Setting $0<\varepsilon<d_1$, we can conclude that

$$\int_\Omega u^{p+1}(t) \le C + C_0\int_0^t\int_\Omega u^{p+1}dt.$$

Applying Gronwall's lemma yields

$$\int_0^t\int_\Omega u^{p+1}dt \le C.$$

The proof is complete. □

Lemma 2.3. *Assume that $(u,v)\in C^{2,1}(\Omega\times(0,T))$ is a solution of (1), then there holds $\|u,v\|_{W_p^{2,1}(\Omega\times(0,T))}\le C$ for any $p>5$.*

Proof. Assume that $(u,v)\in C^{2,1}(\Omega\times(0,T))$ is a solution of (1). Note that

$$v_t - d_2\Delta v = \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v$$

can be rewritten as follows:

$$v_t - d_2\Delta v - \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v = 0, \tag{12}$$

where

$$\left\|r - \frac{r}{K}v - \frac{cu}{u+bv}\right\|_{L^p(\Omega\times(0,T))} \le C \tag{13}$$

by $0\le v\le K_0$ and $\|u\|_{L^{p+1}(\Omega\times(0,T))}\le C$. Based on (12), (13) and the parabolic L^p-estimate, we obtain

$$\|v\|_{W_p^{2,1}(\Omega\times(0,T))}\le C. \tag{14}$$

This, together with Sobolev embedding theorem, yields

$$\|\nabla v\|_{L^\infty(\Omega\times(0,T))}\le C. \tag{15}$$

Now, we consider the equation of u. It can be rewritten as in non-divergence form:

$$u_t - d_1\Delta u + \omega'(u)\cdot\nabla v = -\omega(u)\Delta v + \left(-a+\beta\frac{cv}{u+bv}\right)u. \tag{16}$$

where

$$\|\omega'(u)\nabla v\|_{L^\infty(\Omega\times(0,T))}\le C,$$

$$\left\|-\omega(u)\Delta v + \left(-a+\beta\frac{cv}{u+bv}\right)u\right\|_{L^p(\Omega\times(0,T))}\le C$$

by (14), (15), $0\le v\le K_0$ and $\|u\|_{L^{p+1}(\Omega\times(0,T))}\le C$. Using the parabolic L^p-estimate, we have

$$\|u\|_{W_p^{2,1}(\Omega\times(0,T))}\le C.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that $(u, v) \in C^{2,1}(\Omega \times (0, T))$ is a solution of (1), then there holds $\|u, v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C$.*

Proof. Applying the Sobolev embedding theorem and Lemma 2.3, yields

$$\|u, v\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{17}$$

This, together with the parabolic Schauder estimate of $v_t - d_2\Delta v = \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v$, $\partial\Omega \in C^{2+\alpha}$, $u_0(x), v_0(x) \in C^{2+\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$, $0 \le v \le K_0$ and (17), we obtain

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{18}$$

Using the same method to the equation of u, we have

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{19}$$

The proof is complete. □

Proof. [*Proof of Theorem 1.1*] The proof of the lemma is based on $\|u, v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C$. Motivated by the pioneering work of Tao [13], utilizing Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, this proves the theorem. □

3. Global boundedness of solutions to system (1)

In this section, we will prove the global boundedness of classical solutions to (1). The following lemma is the well-known classical $L^p - L^q$ estimate for the Neumann heat semigroup on bounded domains.

Lemma 3.1. *Suppose $(e^{t\Delta})_{t>0}$ is the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then the following $L^p - L^q$ estimates hold with $C_1, C_2 > 0$ only depending on Ω:*

(i) If $1 \le q \le p \le +\infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le C_1(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in L^q(\Omega)$;

(ii) If $2 \le q \le p < +\infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le C_2(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in W^{1,q}(\Omega)$.

Lemma 3.2. *Suppose that $T \in (0, \infty]$, that $\Omega \subset \mathbb{R}^n$. $n \ge 1$, is a bounded domain, and that D, f and g comply with $D \in C^1(\overline{\Omega} \times [0, T) \times [0, \infty))$ and $D \ge 0$, $f \in C^0\left((0, T); C^0(\overline{\Omega}) \bigcap C^1(\Omega)\right)$ and $g \in C^0(\Omega \times (0, T))$ with $f \cdot v \le 0$ on $\partial\Omega \times (0, T)$. Moreover, assume that $D(x, t, s) \ge \delta s^{m-1}$, $f \in L^\infty((0, T); L^{q_1}(\Omega))$ and $g \in L^\infty((0, T); L^{q_2}(\Omega))$ for all $x \in \Omega$, $t \in (0, T)$, $\delta > 0$ and $s \ge s_0$ and for some $\delta > 0$, $m \in \mathbb{R}$ and $s_0 \ge 1$, and some $q_1 > n + 2$ and $q_2 > \frac{n+2}{2}$. Then if $u \in C^0(\overline{\Omega} \times [0, T)) \bigcap C^{2,1}(\overline{\Omega} \times [0, T))$ is a nonnegative function satisfying*

$$\begin{cases} u_t \le \nabla \cdot (D(x, t, u)\nabla u) + \nabla \cdot f(x, t) + g(x, t), & x \in \Omega, t \in (0, T), \\ \partial_\nu u(x, t) \le 0, & x \in \partial\Omega, t \in (0, T), \end{cases}$$

and if $u \in L^\infty((0, T); L^{p_0}(\Omega))$ is valid for some $p_0 \ge 1$ fulfilling

$$p_0 > 1 - m \cdot \frac{(n + 1)q_1 - (n + 2)}{q_1 - (n + 2)}$$

and

$$p_0 > 1 - \frac{m}{1 - \frac{nq_2}{(n+2)(q_2-1)}}$$

as well as

$$p_0 > \frac{n(1-m)}{2},$$

then there exists $C > 0$*, only depending on* $m, \delta, \Omega, \|f\|_{L^\infty((0,T);L^{q_1}(\Omega))}, \|g\|_{L^\infty((0,T);L^{q_2}(\Omega))}, \|u\|_{L^\infty((0,T);L^{q_0}(\Omega))}$ *and* $\|u(0)\|_{L^\infty(\Omega)}$*, such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C$$

for all $t \in (0,T)$.

Refer to the proof of Lemma A.1 in [12] for the details.

Proof. [*Proof of Theorem 1.2*] The proof consists of four parts.

Part 1: *Boundedness of* $\|u\|_{L^1(\Omega)}$.

Integrate the sum of the first equation and the β times of the second equation in (1) on Ω by parts,

$$\frac{d}{dt}\int_\Omega u + \frac{d}{dt}\int_\Omega \beta v = -a\int_\Omega u + r\beta\int_\Omega v - \frac{r\beta}{K}\int_\Omega v^2. \tag{20}$$

Employing Young's inequality, we have

$$2r\beta\int_\Omega v \le \frac{r\beta}{K}\int_\Omega v^2 + Kr\beta|\Omega|.$$

Setting the last inequality into (20), we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u + \frac{d}{dt}\int_\Omega \beta v &= -a\int_\Omega u + r\beta\int_\Omega v - \frac{r\beta}{K}\int_\Omega v^2 \\ &\le -a\int_\Omega u - r\beta\int_\Omega v + Kr\beta|\Omega|. \end{aligned} \tag{21}$$

Define

$$y_1(t) = \int_\Omega u + \int_\Omega \beta v, t > 0.$$

Then

$$y_1'(t) + k_1 y_1(t) \le k_2$$

for all $t > 0$ by (21) with $k_1 = \min\{a, r\}$ and $k_2 = Kr\delta|\Omega|$. This ensures

$$y_1(t) \le C_1 = \max\left\{y_1(0), \frac{k_2}{k_1}\right\}$$

for all $t > 0$ by the comparison principle of ordinary differential equations.

Part 2: *Boundedness of* $\|u\|_{L^p(\Omega)}$ *with* $p > 2$.

Multiply the equation of u in (1) by u^{p-1} and integrate on Ω by parts, then we have

$$\int_\Omega u_t \cdot u^{p-1} - \int_\Omega d_1\Delta u \cdot u^{p-1} + \int_\Omega \nabla\cdot(\chi(u)u\nabla v)\cdot u^{p-1} = \int_\Omega \left(-a + \beta\frac{cv}{u+bv}\right)u^p.$$

We first observe an important inequality

$$(p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v \le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2.$$

By simplifying the problem, we recall a reduced form of the last inequality

$$\chi(u)u^{p-1}\nabla u\cdot\nabla v \le \frac{d_1}{2}u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi(u)^2u^p|\nabla v|^2.$$

Applying Young's inequality with ε $\left(ab \le \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-\frac{q}{p}}}{q}b^q\right)$ and setting $p = q = 2$, $\varepsilon = d_1$, $a = u^{\frac{p-2}{2}}\nabla u$ and $b = \chi(u)u^{\frac{p}{2}}\nabla v$, we obtain

$$\begin{aligned}
&\chi(u)u^{p-1}\nabla u\cdot\nabla v\\
=&\chi(u)u^{\frac{p-2}{2}+\frac{p}{2}}\nabla u\cdot\nabla v\\
=&(u^{\frac{p-2}{2}}\nabla u)\cdot(\chi(u)u^{\frac{p}{2}}\nabla v)\\
\le&\frac{d_1}{2}u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi(u)^2u^p|\nabla v|^2.
\end{aligned}$$

Multiply the inequality by $(p-1)$ and integrate on Ω by parts yielding

$$(p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v \le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2.$$

According to

$$\int_\Omega u_t\cdot u^{p-1} = \frac{1}{p}\int_\Omega pu^{p-1}\cdot u_t = \frac{1}{p}\int_\Omega \frac{d}{dt}u^p = \frac{1}{p}\frac{d}{dt}\int_\Omega u^p,$$

$$\int_\Omega d_1\cdot\nabla\cdot(\nabla u\cdot u^{p-1}) = d_1\int_\Omega \Delta u\cdot u^{p-1} + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = 0$$

and

$$\int_\Omega \nabla\cdot(\chi(u)u\nabla v)\cdot u^{p-1} + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v = 0,$$

we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2\\
=&\int_\Omega\left(-a+\beta\frac{cv}{u+bv}\right)u^p + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v\\
\le&\int_\Omega\left(-a+\beta\frac{cv}{u+bv}\right)u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2.
\end{aligned}$$

Consequently, together with $\chi(u) \le M_1$ due to $\chi(u) \in C^1$ and $\chi(u) \equiv 0$ for $u \ge M$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\\
\le&\int_\Omega\left(-a+\beta\frac{cv}{u+bv}\right)u^p + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2\\
\le&\int_\Omega\left(-a+\beta\frac{cv}{u+bv}\right)u^p + \frac{(p-1)M_1^2M^p}{2d_1}\int_\Omega |\nabla v|^2.
\end{aligned}\tag{22}$$

Multiply the equation of v in (1) by $-\Delta v$, and integrate on Ω by parts to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2d_2\int_\Omega |\Delta v|^2 =& 2r\int_\Omega |\nabla v|^2 - 4\frac{r}{K}\int_\Omega v|\nabla v|^2 + 2c\int_\Omega \frac{uv}{u+bv}\Delta v\\ \le& 2r\int_\Omega |\nabla v|^2 + \frac{2cv}{u+bv}\int_\Omega u\Delta v\\ \le& 2r\int_\Omega |\nabla v|^2 + \frac{2cK_0}{u+bK_0}\int_\Omega u\Delta v\\ \le& 2r\int_\Omega |\nabla v|^2 + \frac{2c}{b}\int_\Omega u\Delta v.\end{aligned}$$

Employing Young's inequality, we have

$$\frac{2c}{b}\int_\Omega u|\Delta v| \le \frac{\varepsilon}{2}\int_\Omega |\Delta v|^2 + \frac{2c^2}{\varepsilon b^2}\int_\Omega u^2.$$

Setting $\varepsilon = 2d_2$, we can obtain that

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + d_2\int_\Omega |\Delta v|^2 \le 2r\int_\Omega |\nabla v|^2 + \frac{c^2}{d_2 b^2}\int_\Omega u^2. \tag{23}$$

According to

$$\begin{aligned}& d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2\\ =& d_1(p-1)\int_\Omega u^{\frac{p-2}{2}\cdot 2}|\nabla u|^2\\ =& \frac{4d_1(p-1)}{p^2}\left[\int_\Omega \left(\frac{p}{2}\right)^2 u^{(\frac{p}{2}-1)\cdot 2}|\nabla u|^2\right]\\ =& \frac{4d_1(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\end{aligned}$$

for $p > 2$, we know from (22) and (23) by Young's inequality that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 + \frac{2d_1(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + d_2\int_\Omega |\Delta v|^2\\ \le& \int_\Omega \left(-a+\beta\frac{cv}{u+bv}\right)u^p + \frac{(p-1)\chi(u)^2 u_m^p}{2d_1}\int_\Omega |\nabla v|^2 + 2r\int_\Omega |\nabla v|^2 + \frac{c^2}{d_2b^2}\int_\Omega u^2\\ =& \int_\Omega \left(-a+\beta\frac{cv}{u+bv}\right)u^p + \left(\frac{(p-1)\chi(u)^2 u_m^p}{2d_1} + 2r\right)\int_\Omega |\nabla v|^2 + \frac{c^2}{d_2b^2}\int_\Omega u^2\\ \le& \left(-a+\frac{\beta c}{b}+1\right)\int_\Omega u^p + \left(\frac{(p-1)M_1^2 u_m^p}{2d_1} + 2r\right)\int_\Omega |\nabla v|^2 + k_3\end{aligned} \tag{24}$$

with $k_3 = \frac{c^2M^2|\Omega|}{d_2b^2} > 0$.

For $\int_\Omega |\nabla v|^2$, applying the Sobolev interpolation inequality

$$||D^j v||_{p,\Omega} \le \varepsilon ||D^k v||_{p,\Omega} + C||v||_{p,\Omega},$$

setting $j = 1$, $k = 2$, $p = 2$, and integrating on Ω by parts, it's easy to see that

$$\begin{aligned}&\int_\Omega |\nabla v|^2 \le \varepsilon_1\int_\Omega |\Delta v|^2 + k_4\int_\Omega |v|^2 \le \varepsilon_1\int_\Omega |\Delta v|^2 + k_4K_0^2|\Omega|\\ =& \varepsilon_1\int_\Omega |\Delta v|^2 + k_5\end{aligned} \tag{25}$$

for any ε_1, k_4 and $k_5 = k_4 K_0^2|\Omega| > 0$ depending on ε_1.

For $\int_\Omega u^p$, by the Gagliardo-Nirenberg inequality with $u \geq 0$, we obtain

$$\begin{aligned}\int_\Omega u^p = \int_\Omega |u^{\frac{p}{2}}|^2 \leq & k_6 \left(\|\nabla u^{\frac{p}{2}}\|_2^{\frac{2Np-2N}{Np-N+2}} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{\frac{4}{Np-N+2}} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \right) \\ = & k_6 \left(\|\nabla u^{\frac{p}{2}}\|_2^{\frac{2Np-2N}{Np-N+2}} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2-\frac{2Np-2N}{Np-N+2}} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \right) \\ = & k_6 \left(\|\nabla u^{\frac{p}{2}}\|_2^{2\theta} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \right)\end{aligned} \tag{26}$$

with $k_6 > 0$ and $0 < \theta = \frac{Np-N}{Np-N+2} < 1$. Applying Young's inequality yields

$$\|\nabla u^{\frac{p}{2}}\|_2^{2\theta} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} \leq \epsilon\theta \|\nabla u^{\frac{p}{2}}\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2$$

with $\epsilon > 0$. Setting the last estimate into (26), we see that

$$\begin{aligned}\int_\Omega u^p = \int_\Omega |u^{\frac{p}{2}}|^2 \leq & k_6 \left(\|\nabla u^{\frac{p}{2}}\|_2^{2\theta} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \right) \\ \leq & k_6 \left(\epsilon\theta \|\nabla u^{\frac{p}{2}}\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \right) \\ = & k_6\epsilon\theta \|\nabla u^{\frac{p}{2}}\|_2^2 + k_6\epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 + k_6\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \\ = & k_6\epsilon\theta \|\nabla u^{\frac{p}{2}}\|_2^2 + k_6\left[\epsilon^{\frac{\theta}{\theta-1}}(1-\theta) + 1\right]\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \\ = & \varepsilon_2 \|\nabla u^{\frac{p}{2}}\|_2^2 + k_7\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 \\ = & \varepsilon_2 \|\nabla u^{\frac{p}{2}}\|_2^2 + k_7\|u\|_1^p\end{aligned}$$

for any $\varepsilon_2 = k_6\epsilon\theta > 0$, with $k_7 = k_6\left[\epsilon^{\frac{\theta}{\theta-1}}(1-\theta) + 1\right] > 0$ depending on ε_2. Because of $\|u\|_1 \leq A_1$ by Part 1, we know that

$$\int_\Omega u^p \leq \varepsilon_2 \left\|\nabla u^{\frac{p}{2}}\right\|_2^2 + k_7 A_1^p = \varepsilon_2 \left\|\nabla u^{\frac{p}{2}}\right\|_2^2 + k_8 \tag{27}$$

with $k_8 = k_7 A_1^p > 0$.

Now, we need to consider the value of ε_1 and ε_2. Fix them with

$$\left(\frac{(p-1)M_1^2 M^p}{2d_1} + 2r\right)\varepsilon_1 = \frac{d_2}{2}$$

and

$$\left(\frac{\beta c}{b} + 1\right)\varepsilon_2 = \frac{2d_1(p-1)}{p^2}.$$

We have from (24), (25) and (27) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 + \left(\frac{\beta c}{b} + 1\right)\varepsilon_2 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + 2\left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)\varepsilon_1 \int_\Omega |\Delta v|^2 \\ \leq & \left(-a + \frac{\beta c}{b} + 1\right)\int_\Omega u^p + \left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)\int_\Omega |\nabla v|^2 + k_3 \\ \leq & -a\int_\Omega u^p + \left(\frac{\beta c}{b} + 1\right)\varepsilon_2 \left\|\nabla u^{\frac{p}{2}}\right\|_2^2 + \left(\frac{\beta c}{b} + 1\right)k_8 + \left(\frac{(p-1)M_1^2 M^p}{2d_1} + 2r\right)\varepsilon_1 \int_\Omega |\Delta v|^2 \\ & + \left(\frac{(p-1)M_1^2 M^p}{2d_1} + 2r\right)k_5 + k_3.\end{aligned}$$

Obviously,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 \\
\leq& -a\int_\Omega u^p - \left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)\varepsilon_1\int_\Omega |\Delta v|^2 \\
&+ \left[\left(\frac{\beta c}{b}+1\right)k_8 + \left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)k_5 + k_3\right] \\
=& -a\int_\Omega u^p - \left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)\left(\varepsilon_1\int_\Omega |\Delta v|^2 + k_5\right) \\
&+ \left[\left(\frac{\beta c}{b}+1\right)k_8 + 2\left(\frac{(p-1)\chi(u)^2 M^p}{2d_1} + 2r\right)k_5 + k_3\right] \\
\leq& -a\int_\Omega u^p - \left(\frac{(p-1)M_1^2 M^p}{2d_1} + 2r\right)\int_\Omega |\nabla v|^2 + k_9
\end{aligned}$$

with $k_9 > 0$. Therefore, define the function

$$y_2(t) = \frac{1}{p}\int_\Omega u^p + \int_\Omega |\nabla v|^2, \quad t > 0$$

satisfies $y_2'(t) + k_{10}y_2(t) \leq k_9$ for all $t > 0$ with $k_{10} = \min\left\{\frac{(p-1)M_1^2 M^p}{2d_1} + 2r, ap\right\}$. This also ensures

$$y_2(t) \leq C_2 = \max\left\{y_2(0), \frac{k_9}{k_{10}}\right\}$$

for all $t > 0$ by the comparison principle of ordinary differential equations.

Part 3: *Boundedness of* $\| \nabla v \|_{L^\infty(\Omega)}$.

We can define $f(u,v) = \left(r - \frac{r}{K}v - \frac{cu}{u+bv}\right)v$. It follows from Part 2 and Lemma 2.3 that there is $C_3 > 0$ such that

$$\sup_{t>0}\left\|f(u,v)\right\|_{L^p(\Omega)} \leq C_3 < +\infty.$$

By the variation-of-constants formula for v, we have

$$v(\cdot,t) = e^{d_2 t\Delta}v_0 + \int_0^t e^{d_2(t-s)\Delta}f(u(s),v(s))ds, \quad t > 0.$$

Because of Lemma 3.1, we can draw a conclusion that

$$\begin{aligned}
\|\nabla v\|_{L^p(\Omega)} =& \left\|\nabla e^{d_2 t\Delta}v_0 + \int_0^t \nabla e^{d_2(t-s)\Delta}f(u(s),v(s))ds\right\|_{L^p(\Omega)} \\
\leq& \left\|\nabla e^{d_2 t\Delta}v_0\right\|_{L^p(\Omega)} + \left\|\int_0^t \nabla e^{d_2(t-s)\Delta}f(u(s),v(s))ds\right\|_{L^p(\Omega)} \\
\leq& C_2(1 + d_2 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 d_2 t}\|\nabla v_0\|_{L^p(\Omega)} + \int_0^t \left\|\nabla e^{d_2(t-s)\Delta}f(u(s),v(s))\right\|_{L^p(\Omega)}ds \\
\leq& 2C_2 e^{-\lambda_1' t}\|\nabla v_0\|_{L^p(\Omega)} + C_1\int_0^t (1 + d_2^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}})e^{-\lambda_1'(t-s)}\|f(u(s),v(s))\|_{L^p(\Omega)}ds \\
\leq& 2C_2 e^{-\lambda_1' t}\|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3\int_0^t (1 + d_2^{-\frac{1}{2}}s^{-\frac{1}{2}})e^{-\lambda_1' s}ds \\
\leq& 2C_2\|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3\left(\frac{1}{\lambda_1'} + d_2^{-\frac{1}{2}}\left(2 + \frac{1}{\lambda_1'}\right)\right)
\end{aligned}$$

for all $t > 0$. Therefore, $\|\nabla v\|_{L^p(\Omega)}$ is global bounded. We can apply

$$\frac{d\bar{v}(t)}{dt} = r\bar{v}(t) - \frac{r}{K}\bar{v}(t)^2$$

and the Moser iteration to obtain the boundedness of $\|\nabla v\|_{L^\infty(\Omega)}$, since $\|u\|_p$ for any $p > N$ is bounded.

Part 4: *Global boundedness.*

Based on Part 2, Part 3 and Lemma A.1 in [12], the global boundedness of solutions can be proved by using of the standard Moser iterative technique. The proof is complete. □

Let us mention two important Remarks of our main results.

Remark 3.3. *It is not difficult to find the global boundedness of solutions is an obvious result to the corresponding predator-prey model [19, 20] without nonlinear prey-taxis. The existence of prey-taxis in (1) makes stupendous difficulty to obtain the global boundedness, and even the global existence of solutions. On the other hand, the nonlinear prey-taxis term $\nabla \cdot (\chi(u)u\nabla v)$ contained in the system is supposed that $\chi(u) \equiv 0$ whenever $u \geq M$, where the maximal density M acts as a switch to repulsion at high densities of the predator population, very similar to the volume-filling effect or prevention of overcrowding for chemotaxis [21]. Therefore, the global boundedness of solutions established by Theorem 1.1 should be reasonable and natural.*

Remark 3.4. *To investigate the qualitative behavior of the class of reaction-diffusion equations, in which the global bounded argument is incorporated together with the prey-taxis term $\nabla \cdot (\chi(u)u\nabla v)$, a standard technique have been applied. According to the boundedness of $\|u\|_{L^1(\Omega)}$, $\|u\|_{L^p(\Omega)}$ with $p > 2$ and $\|\nabla v\|_{L^p(\Omega)}$ with $p > 2$, using the standard Moser's iterative technique of parabolic partial differential equations, we obtain a sufficient condition to verify whether the unique nonnegative solution of (1) is global bounded.*

4. Generalization and future works

The method we propose in this paper can be applied to many interesting reaction-diffusion systems with nonlinear prey-taxis. The existence of solution is an important problem to be considered. For instance, the famous predator-prey model with Lotka-Volterra functional response and continuous diffusive functions

$$\begin{cases} u_t - d_1(x,t)\Delta u + \nabla \cdot (u\chi(u)\nabla v) = -au + \beta \dfrac{cuv}{m+bv}, & x \in \Omega, t \in (0,T), \\ v_t - d_2(x,t)\Delta v = rv - \dfrac{r}{K}v^2 - \dfrac{cuv}{m+bv}, & x \in \Omega, t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq 0, & x \in \Omega, \end{cases}$$

is an interesting model worth of investigation.

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