Filomat 37:5 (2023), 1497–1506
https://doi.org/10.2298/FIL2305497H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A nonexistence result for a class of quasilinear Schrödinger equations with Berestycki-Lions conditions

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Abstract. In this paper, we study the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{1/2}]\frac{u}{2(1+u^2)^{1/2}} = h(u),\ \ x \in \mathbb{R}^N,$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, $V(x)$ is a potential function. Unlike $V \in C^2(\mathbb{R}^N)$, we only need to assume that $V \in C^1(\mathbb{R}^N)$. By using a change of variable, we prove the non-existence of ground state solutions with Berestycki-Lions conditions, which contain the superliner case:

$$\lim_{s\to+\infty} \frac{h(s)}{s} = +\infty$$

and asymptotically linear case:

$$\lim_{s\to+\infty} \frac{h(s)}{s} = \eta.$$

Our results extend and complement the results in related literature.

1. Introduction

This article is concerned with a class of generalized quasilinear Schrödinger equation

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{1/2}]\frac{u}{2(1+u^2)^{1/2}} = h(u),\ \ x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, $V(x)$ satisfies the following conditions:
$(\mathcal{V}_1)$ $V \in C^1(\mathbb{R}^N, \mathbb{R})$;

2020 *Mathematics Subject Classification*. Primary 35J10; Secondary 35J20
Keywords. quasilinear Schrödinger equation; nonexistence; ground state solutions; Berestycki-Lions conditions
Received: 22 October 2020; Accepted: 28 February 2021
Communicated by Marko Nedeljkov
The first author is supported by the Natural Science Foundation of Hunan Province of China(No. 2022JJ30463), and Research Foundation of Education Bureau of Hunan Province, China (No.22A0540). The second author is supported by the National Natural Science Foundation of China (Grant Nos. 11901276 and 11961045), and partly by Jiangxi Provincial Natural Science Foundation (Grant Nos. 20202BAB201001 and 20202BAB211004).
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$(\mathcal{V}_2)$ $0 < V_\infty = \lim_{|x|\to\infty} V(x) \leq V(x)$;

$(\mathcal{V}_3)$ $\nabla V(x) \cdot x \leq 0$ for all $x \in \mathbb{R}^N$, with the strict inequality holding on a subset of positive Lebesgue measure of $\mathbb{R}^N$, and the mapping $t \mapsto NV(tx) + \nabla V(tx) \cdot (tx)$ is non-increasing on $(0, \infty)$.

Standing wave of the following quasi-linear Schrödinger equation is a hot problem

$$i\partial_t z = -\Delta z + W(x)z - k(x, |z|) - \Delta l(|z|^2)l'(|z|^2)z \tag{2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. For various types of l, the quasilinear equation of the form (1) has been derived from models of several physical phenomenon. For more physical background, we can refer to [2, 8, 10, 12] and references therein.

Set $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, (2) can be reduce to the corresponding equation of elliptic type (see [7]):

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(x, u) \quad x \in \mathbb{R}^N. \tag{3}$$

If we take $g^2(u) = 1 + \frac{[(l^2(u))']^2}{2}$, then (3) turns into quasilinear elliptic equations (see [15])

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x, u), \quad x \in \mathbb{R}^N. \tag{4}$$

For (4), there are many papers (see [6, 15–17]) studying the existence of standing wave solutions. If we set

$$g^2(u) = 1 + 2u^2,$$

then (4) reduces to the following well-known quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = h(x, u).$$

Many recent studies have focused on the above quasilinear equation, see for example [3, 9, 20] and references therein.

For (1), Shen et al. [18] proved the existence of positive solutions with asymptotically linear nonlinearity. In [13], Miyagaki et al. studied the first eigenvalue for (1) and the existence of nonnegative solutions on bound domain. As far as we know, there are few paper focused on the nonexistence of ground state solutions for (1). In general, many scholars [4, 11, 14, 21] considered the nonexistence of ground state solutions with $V \in C^2(\mathbb{R}^N)$ and asymptotically linear growth as [11]. But in this paper, motivated by [5, 11, 19, 21], we consider nonexistence of ground state solutions for (1) with Berestycki-Lions conditions and we only need to assume that $V \in C^1(\mathbb{R}^N)$. So the method in [4, 11] do not work in our problem and we used some new ideas come from [5, 19, 21], which was used to deal with semilinear problems and quasilinear problems, respectively. Next, we assume $h(t) = 0$ for $t \leq 0$ and also give the following assumptions on h :

(h_1) $h \in C(\mathbb{R}, \mathbb{R})$, and there exists $C > 0$ and $p \in (2, 2^*)$ such that $|h(t)| \leq C(1 + |t|^{p-1})$;

(h_2) $h(t) = o(t)$ as $t \to 0$;

(h_3) there exists $\mathcal{L}_0 > 0$ such that $\int_0^{\mathcal{L}_0} (h(s) - V_\infty s) ds > 0$;

Remark 1.1. *Note that* (h_3'):

there exists $\beta > 0$ *such that* $h(t) \geq \beta |t|^{q-2} t$ *for* $2 < q < 2^*$

is stronger than the condition (h_3)*, which was first established by Berestyski and Lions in [1]. Indeed, in view of* (h_3')*, we infer that*

$$\frac{g(t)}{t} \geq \beta |t|^{q-2},$$

which shows that (h_4) *holds. Moreover,* (h_3) *contains the superlinear and asymptotically linear case. Indeed, the following conditions* (h_3') *and* (h_3'') *all satisfy* (h_3)*, where*

$$(h_3'') \ \lim_{s\to+\infty} \frac{h(s)}{s} = +\infty$$

and

(h_3''') $\lim_{s\to+\infty} \frac{h(s)}{s} = \eta > V_\infty$

for some $\eta \in \mathbb{R}$. So, our results extend and complement [4, 11, 14, 18, 22].

As [7], we deduce that the Euler-Lagrange functional associated with (1) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} H(u),$$

where $H(u) = \int_0^u h(s)\mathrm{d}s$. Due to the appearance of the nonlocal term $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2\mathrm{d}x$, I may be not well defined. To overcome this difficulty, a variable substitution as follows: for any $v \in H^1(\mathbb{R}^N)$, Shen and Wang [15] make a change of variable as $u = G^{-1}(v)$ and $G(u) = \int_0^u g(t)\mathrm{d}t$, and then the functional I in form can be transformed into

$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|G^{-1}(v)|^2) - \int_{\mathbb{R}^N} H(G^{-1}(v)), \quad x \in \mathbb{R}^N. \tag{5}$$

In addition, the limit energy functional of (5) is

$$\mathcal{J}^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V_\infty|G^{-1}(v)|^2) - \int_{\mathbb{R}^N} H(G^{-1}(v)), \quad x \in \mathbb{R}^N. \tag{6}$$

From our hypotheses, it is clear that $\mathcal{J}$ and $\mathcal{J}^\infty$ are well defined in $H^1(\mathbb{R}^N)$ and $\mathcal{J}, \mathcal{J}^\infty \in C^1(\mathbb{R}^N, \mathbb{R})$.

To state our result, we need define

$$\mathcal{M} = \left\{v \in H^1(\mathbb{R}^N)\backslash\{0\} : P(v) = 0\right\},$$

where

$$\begin{aligned} P(v) = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 \\ + \frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)|G^{-1}(v)|^2 - N\int_{\mathbb{R}^N} H(G^{-1}(v)). \end{aligned} \tag{7}$$

Moreover, let

$$\mathcal{M}_\infty = \{v \in H^1(\mathbb{R}^N)\backslash\{0\} : P_\infty(v) = 0\},$$

where

$$P_\infty(v) = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 - N\int_{\mathbb{R}^N} H(G^{-1}(v)). \tag{8}$$

Now, we are ready to state the main result of this paper.

Theorem 1.2. *Assume that $(\mathcal{V}_1)$-$(\mathcal{V}_3)$ and (h_1)-(h_3) are satisfied. Then $c := \inf_{\mathcal{M}} \mathcal{J}$ is not a critical level for the functional $\mathcal{J}$. In particular, the infimum c is not achieved.*

The remainder of this paper is organized as follows. In section 2, we prove Theorem 1.2.

Notations: Throughout this paper, we make use of the following notations:

- $\int_{\mathbb{R}^N} \clubsuit$ denotes $\int_{\mathbb{R}^N} \clubsuit\, \mathrm{d}x$ and C denotes the different constants;
- $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norms $\|u\|_p = \left(\int_{\mathbb{R}^N}|u|^p\right)^{\frac{1}{p}}$, where $1 \le p < \infty$;
- For any $v \in H^1(\mathbb{R}^N)\backslash\{0\}$, $v_t(x) = v(x/t)$ for $t > 0$.
- Let $H^1(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}$ with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\right)^{\frac{1}{2}}.$$

- The weak convergence in $H^1(\mathbb{R}^N)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. At first, we present some useful lemmas and corollaries.

Lemma 2.1 ([22]) *For the function G^{-1}, the following properties hold:*

(1) $1 \leq g(G^{-1}(s)) \leq \sqrt{\frac{3}{2}}$ for all $s \in \mathbb{R}$;

(2) $|G^{-1}(s)| \leq |s|$ for all $s \in \mathbb{R}$;

(3) $\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \leq |s|$ for all $s \in \mathbb{R}$;

(4) $\frac{G^{-1}(s)s}{g(G^{-1}(s))} \leq |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;

(5) $\lim\limits_{|s|\to 0} \frac{G^{-1}(s)}{s} = 1$ and $\lim\limits_{|s|\to +\infty} \frac{G^{-1}(s)}{s} = \sqrt{\frac{2}{3}}$.

By $(\mathcal{V}_3)$, we can conclude that

$$Nt^N \left[V(x) - V(tx)\right] + \left(t^N - 1\right)\left(\nabla V(x) \cdot x\right) \geq 0. \tag{9}$$

In (9), letting $t \to +\infty$, then we have

$$NV(x) + \nabla V(x) \cdot x \geq NV_\infty. \tag{10}$$

In (10), it is easy to check that $\nabla V(x) \cdot x \to 0$ as $|x| \to +\infty$.

Lemma 2.2. *Assume that $(\mathcal{V}_1)$-$(\mathcal{V}_3)$, (h_1) and (h_2) hold. Then*

$$\mathcal{J}(v) \geq \mathcal{J}(v_t) + \frac{1-t^N}{N}P(v) + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\int_{\mathbb{R}^N} |\nabla v|^2, \ \forall\, v \in H^1(\mathbb{R}^N),\ t > 0.$$

Proof. By (5) and (9), we have

$$\begin{aligned}
&\mathcal{J}(v) - \mathcal{J}(v_t)\\
&\quad= \frac{1-t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}\left[V(x) - t^N V(tx)\right]|G^{-1}(v)|^2\\
&\qquad\quad - \left(1 - t^N\right)\int_{\mathbb{R}^N} H(G^{-1}(v))\\
&\quad= \frac{1-t^N}{N}\left\{\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}\left[NV(x) + \nabla V(x)\cdot x\right]|G^{-1}(v)|^2\right.\\
&\qquad\quad \left. - N\int_{\mathbb{R}^N} H(G^{-1}(v))\right\}\\
&\qquad + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\int_{\mathbb{R}^N} |\nabla v|^2\\
&\qquad + \frac{1}{2N}\int_{\mathbb{R}^N}\left(Nt^N\left[V(x) - V(tx)\right] + (t^N - 1)\nabla V(x)\cdot x\right)|G^{-1}(v)|^2\\
&\quad\geq \frac{1-t^N}{N}P(v) + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\int_{\mathbb{R}^N} |\nabla v|^2.
\end{aligned}$$

This completes the proof. □

In view of Lemma 2.2, we get the following corollary.

Corollary 2.3. *If $(\mathcal{V}_1)$-$(\mathcal{V}_3)$, (h_1) and (h_2) hold, then for $v \in \mathcal{M}$, $\mathcal{J}(v) = \max_{t>0}\mathcal{J}(v_t)$.*

To prove $\mathcal{M} \neq \emptyset$, we set

$$\Theta = \left\{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}\left[\frac{1}{2}V_\infty |G^{-1}(v)|^2 - H(G^{-1}(v))\right] < 0\right\}.$$

Similar to the proof in [5, 19, 21], we can get the following important lemma.

Lemma 2.4. *Assume that* $(\mathcal{V}_1)$-$(\mathcal{V}_3)$, (h_1), (h_2) *and* (h_3) *hold, then* $\Theta \neq \emptyset$ *and*

$$\left\{v \in H^1(\mathbb{R}^N)\backslash\{0\} : P_\infty(v) \leq 0 \text{ or } P(v) \leq 0\right\} \subset \Theta.$$

Proof. By the proof of Theorem 2 in [1], and (h_3), it is easy to see that $\Theta \neq \emptyset$. Next, we prove the following conclusion into the two cases:

Case 1: if $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $P_\infty(v) \leq 0$, then it follows from Lemma 2.2 that $v \in \Theta$.

Case 2: if $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $P(v) \leq 0$, then it follows from (10) that

$$\begin{aligned}
&N\int_{\mathbb{R}^N}\left[\frac{1}{2}V_\infty|G^{-1}(v)|^2 - H(G^{-1}(v))\right]\\
&= P(v) - \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 - \frac{N}{2}\int_{\mathbb{R}^N}\left[(V(x)-V_\infty) + \frac{\nabla V(x)\cdot x}{N}\right]|G^{-1}(v)|^2\\
&\leq -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 - \frac{1}{2}\int_{\mathbb{R}^N}(NV(x) - NV_\infty + \nabla V(x)\cdot x)\,|G^{-1}(v)|^2\\
&\leq -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 < 0.
\end{aligned}$$

which shows that $v \in \Theta$. This completes the proof. □

Lemma 2.5. *Assume that* $(\mathcal{V}_1)$-$(\mathcal{V}_3)$ *and* (h_1)-(h_3) *hold, then for any* $v \in \Theta$, *there exists a unique* $t_v > 0$ *such that* $v_{t_v} \in \mathcal{M}$.

Proof. Let $v \in \Theta$ be fixed. Assume that

$$\Upsilon(t) := \mathcal{J}(v_t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{t^N}{2}\int_{\mathbb{R}^N}V(tx)|G^{-1}(v)|^2 - t^N\int_{\mathbb{R}^N}H(G^{-1}(v)).$$

Set

$$\begin{aligned}
\Upsilon'(t) = {}& \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N}V(tx)|G^{-1}(v)|^2\\
&+ \frac{t^{N-1}}{2}\int_{\mathbb{R}^N}(\nabla V(tx)\cdot(tx))|G^{-1}(v)|^2 - Nt^{N-1}\int_{\mathbb{R}^N}H(G^{-1}(v)) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{N-2}{2}t^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{t^N}{2}\int_{\mathbb{R}^N}[NV(tx) + (\nabla V(tx)\cdot(tx))]\,|G^{-1}(v)|^2\\
&\qquad = Nt^N\int_{\mathbb{R}^N}H(G^{-1}(v)),
\end{aligned}$$

which implies that $P(v_t) = 0 \Leftrightarrow v_t \in \mathcal{M}$. It is easy to check that $\lim_{t\to 0}\Upsilon(t) = 0$, $\Upsilon(t) > 0$ for $t > 0$ enough small. From $(\mathcal{V}_2)$ and Lebesgue Dominated Convergence Theorem, we have

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}\left(\frac{1}{2}V(tx)|G^{-1}(v)|^2 - H(G^{-1}(v))\right) = \int_{\mathbb{R}^N}\left(\frac{1}{2}V_\infty|G^{-1}(v)|^2 - H(G^{-1}(v))\right)$$

and

$$\lim_{t\to+\infty}\int_{R^N}(\nabla V(tx)\cdot(tx))|G^{-1}(v)|^2 = 0.$$

Moreover, one has

$$\begin{aligned}\Upsilon'(t) &= t^{N-3}\Big(\frac{N-2}{2}\int_{R^N}|\nabla v|^2 + \frac{N}{2}t^2\int_{R^N}V(tx)|G^{-1}(v)|^2\\&\quad+\frac{t^2}{2}\int_{R^N}(\nabla V(tx)\cdot(tx))|G^{-1}(v)|^2 - Nt^2\int_{R^N}H(G^{-1}(v))\Big)\\&= t^{N-3}\Big[\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + Nt^2\Big(\int_{R^N}\frac{1}{2}V_\infty|G^{-1}(v)|^2 - H(G^{-1}(v)) + o_t(1)\Big)\Big]\end{aligned}$$

and thus $\Upsilon(t) < 0$ for t large. Thus $\max_{t>0}\Upsilon(t)$ is achieved at some $t_v > 0$ such that $\Upsilon'(t_v) = 0$ and $v_{t_v} \in \mathcal{M}$.

Next, we prove the uniqueness. Let $\alpha(t) = 2 - Nt^{N-2} + (N-2)t^N$. For any given $v \in \Theta$, if there exist $t_1, t_2 > 0$ such that $v_{t_1}, v_{t_2} \in \mathcal{M}$. Thus $P(v_{t_1}) = P(v_{t_2}) = 0$. Therefore, we have

$$\mathcal{J}(v_{t_1}) \geq \mathcal{J}(v_{t_2}) + \frac{t_1^N - t_2^N}{Nt_1^N}P(v_{t_1}) + \frac{\alpha(t_2/t_1)}{2N}\|\nabla v_{t_1}\|_2^2 = \mathcal{J}(v_{t_2}) + \frac{\alpha(t_2/t_1)}{2N}\|\nabla v_{t_1}\|_2^2$$

and

$$\mathcal{J}(v_{t_2}) \geq \mathcal{J}(v_{t_1}) + \frac{t_2^N - t_1^N}{Nt_2^N}P(v_{t_2}) + \frac{\alpha(t_1/t_2)}{2N}\|\nabla v_{t_2}\|_2^2 = \mathcal{J}(v_{t_1}) + \frac{\alpha(t_1/t_2)}{2N}\|\nabla v_{t_2}\|_2^2,$$

which shows that $t_1 = t_2$. Thus $t_v > 0$ is unique for $v \in \Theta$. The proof is completed. □

By Corollary 2.3, Lemma 2.4 and Lemma 2.5, we can get the following lemma.

Lemma 2.6. *Assume that* $(\mathcal{V}_1)$-$(\mathcal{V}_3)$ *and* (h_1)-(h_3) *hold, then* $\inf_{\mathcal{M}}\mathcal{J} := c = \inf_{v\in\Theta}\max_{t>0}\mathcal{J}(v_t)$. *Furthermore,* $\inf_{\mathcal{M}_\infty}\mathcal{J}^\infty := c^\infty = \inf_{v\in\Theta}\max_{t>0}\mathcal{J}^\infty(v_t)$.

Lemma 2.7. *Assume that* $(\mathcal{V}_1)$-$(\mathcal{V}_3)$ *hold. Then*

(i) *there exists* $\rho > 0$ *such that* $\|\nabla v\|_2 \geq \rho$ *for any* $v \in \mathcal{M}$;

(ii) $c = \inf_{\mathcal{M}}\mathcal{J} > 0$.

Proof. (i) For any $v \in \mathcal{M}$, we have that $P(v) = 0$. By Lemma 2.1, (h_1)-(h_2) and Sobolev embedding theorem, we get

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{NV_\infty}{2}\int_{\mathbb{R}^N}|G^{-1}(v)|^2\\&\quad\leq \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}[NV(x) + \nabla V(x)\cdot x]|G^{-1}(v)|^2\\&\quad= \int_{\mathbb{R}^N}H(G^{-1}(v)) \leq \varepsilon\int_{\mathbb{R}^N}|G^{-1}(v)|^2 + C_\varepsilon\|\nabla v\|_2^{2^*}.\end{aligned}$$

If we choose $\varepsilon = \frac{NV_\infty}{4}$, then there exists $\rho > 0$ such that $\|\nabla v\|_2 \geq \rho$ for any $v \in \mathcal{M}$.

(ii) For any $v \in \mathcal{M}$, by $(\mathcal{V}_3)$, we have

$$\begin{aligned}\mathcal{J}(v) &= \mathcal{J}(v) - \frac{1}{N}P(v)\\&= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v|^2 - \frac{1}{2N}\int_{\mathbb{R}^N}(\nabla V(x), x)|G^{-1}(v)|^2\\&\geq \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v|^2 > 0.\end{aligned} \tag{11}$$

This completes the proof. □

Lemma 2.8. *For any* $v \in \mathcal{M}_\infty$, *there exists a unique* $t \geq 1$ *such that* $v_t \in \mathcal{M}$.

Proof. For any $v \in \mathcal{M}_\infty \subset \Theta$, by Lemma 2.5, there exists a unique $t > 0$ such that $v_t \in \mathcal{M}$. Next, we show $t \geq 1$. In fact, it follows from $P_\infty(v) = 0$ that

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 - N\int_{\mathbb{R}^N}H(G^{-1}(v)) = -\frac{N}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2, \tag{12}$$

which shows that $P(v) \geq 0$. Moreover, by $v_t \in \mathcal{M}$, one has

$$\begin{aligned}\frac{N-2}{2}t^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2 - Nt^N\int_{\mathbb{R}^N}H(G^{-1}(v))\\ &= -\frac{t^N}{2}\int_{\mathbb{R}^N}[NV(tx) + (\nabla V(tx)\cdot(tx))]|G^{-1}(v)|^2\\ &\leq -\frac{Nt^N}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2.\end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$t^2 \geq \frac{\frac{1}{2^*}\int_{\mathbb{R}^N}|\nabla v|^2}{\int_{\mathbb{R}^N}H(G^{-1}(v)) - \frac{1}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2} = 1.$$

The proof is completed. □

Lemma 2.9. *For any $v \in \mathcal{M}$, there exists a unique $t \in (0, 1]$ such that $v_t \in \mathcal{M}_\infty$.*

Proof. For any $v \in \mathcal{M} \subset \Theta$, similar to Lemma 2.5, we have that there exists a unique $t > 0$ such that $v_t \in \mathcal{M}_\infty$. Next, we show $t \geq 1$. In fact, it follows from $P(v) = 0$ and (10) that

$$\begin{aligned}\frac{1}{2^*}\int_{\mathbb{R}^N}|\nabla v|^2 - \int_{\mathbb{R}^N}H(G^{-1}(v)) &= -\frac{1}{2}\int_{\mathbb{R}^N}\left[V(x) + \frac{\nabla V(x)\cdot x}{N}\right]|G^{-1}(v)|^2\\ &\leq -\frac{1}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2,\end{aligned} \tag{14}$$

which shows that $P_\infty(v) \leq 0$. Moreover, by $v_t \in \mathcal{M}_\infty$, one has

$$\frac{N-2}{2}t^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{Nt^N}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2 - Nt^N\int_{\mathbb{R}^N}H(G^{-1}(v)) = 0. \tag{15}$$

It follows from (14) and (15) that

$$t^2 = \frac{\frac{1}{2^*}\int_{\mathbb{R}^N}|\nabla v|^2}{\int_{\mathbb{R}^N}H(G^{-1}(v)) - \frac{1}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2} \leq 1.$$

The proof is completed. □

Lemma 2.10. *If $v \in \mathcal{M}_\infty$, then $v(\cdot - y) \in \mathcal{M}_\infty$, for all $y \in \mathbb{R}^N$. Moreover, there exists $\theta_y > 1$ such that $v(\frac{\cdot - y}{\theta_y}) \in \mathcal{M}$ and $\lim_{|y|\to\infty}\theta_y = 1$.*

Proof. Since $v \in \mathcal{M}_\infty$, by the translation invariance, we have that $v(\cdot - y) \in \mathcal{M}_\infty$ for all $y \in \mathbb{R}^N$. By Lemma 2.8, there exists $\theta_y \geq 1$ such that $v(\frac{\cdot - y}{\theta_y}) \in \mathcal{M}$. Suppose by contradiction that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| \to +\infty$ and $\theta_{y_n} \to \bar{\theta} > 1$ or $+\infty$ as $n \to +\infty$. Next, let

$$\mathcal{Z}(\theta_{y_n}x + y_n) := V(\theta_{y_n}x + y_n) + \frac{(\nabla V(\theta_{y_n}x + y_n)\cdot(\theta_{y_n}x + y_n))}{N}.$$

It follows that

$$\begin{aligned}H(G^{-1}(v)) - \frac{1}{2}\mathcal{Z}(\theta_{y_n}x + y_n)|G^{-1}(v)|^2 &\leq H(G^{-1}(v)) - \frac{1}{2}V_\infty|G^{-1}(v)|^2\\ &\leq C\left(|G^{-1}(v)|^2 + |G^{-1}(v)|^p\right) \in L^1(\mathbb{R}^N).\end{aligned}$$

Thus by the Lebesgue Dominated Convergence Theorem, one has

$$\lim_{|y_n|\to+\infty}\int_{\mathbb{R}^N}\Big(H(G^{-1}(v))-\frac{1}{2}\mathcal{Z}(\theta_{y_n}x+y_n)|G^{-1}(v)|^2\Big) = \int_{\mathbb{R}^N}\Big(H(G^{-1}(v))-\frac{1}{2}V_\infty|G^{-1}(v)|^2\Big).$$

Moreover, for each $y_n\in\mathbb{R}^N$, we have that $v(\frac{\cdot-y_n}{\theta_{y_n}})\in\mathcal{M}$ with $\theta_{y_n}\geq 1$. Thus we know

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2 = N\theta_{y_n}^2\int_{\mathbb{R}^N}H(G^{-1}(v))-\frac{N\theta_{y_n}^2}{2}\int_{\mathbb{R}^N}\mathcal{Z}(\theta_{y_n}x+y_n)|G^{-1}(v)|^2,$$

which implies that the right hand side of the above inequality goes to $+\infty$ or

$$N\bar{\theta}^2\int_{\mathbb{R}^N}\Big[H(G^{-1}(v))-\frac{1}{2}V_\infty|G^{-1}(v)|^2\Big],$$

however, the left right hand side is fixed on $\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2>0$. Since $v\in\mathcal{M}_\infty$ and $\bar{\theta}>1$ or $+\infty$, this is a contradiction. This completes the proof. □

Lemma 2.11. $c=c_\infty$.

Proof. Let $\vartheta\in H^1(\mathbb{R}^N)$ be the ground state solution (which is positive and radially symmetric) of the autonomous problem at infinity, $\vartheta\in\mathcal{M}_\infty$ and $c_\infty=\mathcal{J}^\infty(\vartheta)$. For any given $y\in\mathbb{R}^N$, let $\vartheta_y:=\vartheta(x-y)$. By the translation invariance of the integrals, we know that $\vartheta_y\in\mathcal{M}_\infty$ and $c_\infty=\mathcal{J}^\infty(\vartheta_y)$. From Lemma 2.8, for any $y\in\mathbb{R}^N$, there exists $\theta_y\geq 1$ such that $\hat{\vartheta}_y=\vartheta_y(\cdot/\theta_y)\in\mathcal{M}$. Thus we have

$$\begin{aligned}
|\mathcal{J}(\hat{\vartheta}_y)-c_\infty| &= |\mathcal{J}(\hat{\vartheta}_y)-\mathcal{J}^\infty(\vartheta_y)|\\
&=\Big|\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\hat{\vartheta}_y|^2-\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\vartheta_y|^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(\hat{\vartheta}_y)|^2\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(\vartheta_y)|^2-\int_{\mathbb{R}^N}H(G^{-1}(\hat{\vartheta}_y))+\int_{\mathbb{R}^N}H(G^{-1}(\vartheta_y))\Big|\\
&=\Big|\frac{1}{2}(\theta_y^{N-2}-1)\int_{\mathbb{R}^N}|\nabla\vartheta|^2+\frac{1}{2}\theta_y^N\int_{\mathbb{R}^N}V(x\theta_y+y)|G^{-1}(\vartheta)|^2\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(\vartheta)|^2+(1-\theta_y^N)\int_{\mathbb{R}^N}H(G^{-1}(\vartheta))\Big|\\
&\leq\frac{1}{2}\left|\theta_y^{N-2}-1\right|\int_{\mathbb{R}^N}|\nabla\vartheta|^2+\left|1-\theta_y^N\right|\int_{\mathbb{R}^N}H(G^{-1}(\vartheta))\\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N}\left|\theta_y^NV(x\theta_y+y)-V_\infty\right||G^{-1}(\vartheta)|^2.
\end{aligned}$$

Since $\theta_y\to 1$ as $|y|\to+\infty$, we have $\frac{1}{2}\left|\theta_y^{N-2}-1\right|\int_{\mathbb{R}^N}|\nabla\vartheta|^2\to 0$ and $\left|1-\theta_y^N\right|\int_{\mathbb{R}^N}H(G^{-1}(\vartheta))\to 0$. Moreover, by $V(x\theta_y+y)\to V_\infty$ as $|y|\to+\infty$, we get $\frac{1}{2}\int_{\mathbb{R}^N}\left|\theta_y^NV(x\theta_y+y)-V_\infty\right||G^{-1}(\vartheta)|^2\to 0$. It follows that $\lim_{|y|\to+\infty}\mathcal{J}(\hat{\vartheta}_y)=c_\infty$. Thus $c=\inf\limits_{v\in\mathcal{M}}\mathcal{J}(v)\leq c_\infty$.

Next, we only need to claim that $c=\inf\limits_{v\in\mathcal{M}}\mathcal{J}(v)\geq c_\infty$. If $v\in\mathcal{M}$ and $\theta\in(0,1]$ satisfy $v(\cdot/\theta)\in\mathcal{M}_\infty$, then it follows from $(\mathcal{V}_3)$ that

$$
\begin{aligned}
\mathcal{J}(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}V(x)|f(v)|^2 - \int_{\mathbb{R}^N}H(G^{-1}(v)) \\
&= \frac{1}{N}\left(\int_{\mathbb{R}^N}|\nabla v|^2 - \int_{\mathbb{R}^N}\frac{\nabla V(x)\cdot x}{2}|G^{-1}(v)|^2\right) \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v|^2 \\
&\geq \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla v|^2 \\
&\geq \mathcal{J}^\infty(v(x/\theta)) \geq c_\infty.
\end{aligned}
$$

Thus for any $v \in \mathcal{M}$, $\mathcal{J}(v) \geq c_\infty$. Therefore $c = \inf_{v\in\mathcal{M}}\mathcal{J}(v) \geq c_\infty$. This completes the proof. □

Proof of Theorem 1.2. Suppose by contradiction, that is, there exists $\nu \in H^1(\mathbb{R}^N)$, which is a critical point of the functional $\mathcal{J}$ at level c. In particular, that, $\nu \in \mathcal{M}$ and $\mathcal{J}(\nu) = c$. By Lemma 2.9, there exists $t \in (0, 1]$ be such that $\nu_t \in \mathcal{M}_\infty$. It follows from $(\mathcal{V}_3)$ that

$$
\begin{aligned}
c = \mathcal{J}(\nu) &= \frac{1}{N}\left(\int_{\mathbb{R}^N}|\nabla \nu|^2 - \int_{\mathbb{R}^N}\frac{\nabla V(x)\cdot x}{2}|G^{-1}(\nu)|^2\right) \\
&\geq \frac{t^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla \nu|^2 - \int_{\mathbb{R}^N}\frac{\nabla V(x)\cdot x}{2N}|G^{-1}(\nu)|^2 \\
&\geq \mathcal{J}^\infty(\nu_t) \\
&\geq c_\infty = c.
\end{aligned}
$$

Thus $t = 1$, $\nu \in \mathcal{M}_\infty$ and $\mathcal{J}^\infty(\nu) = c_\infty$. In view of Lemma 2.14 in [19], $\mathcal{J}^{\infty\prime}(\nu) = 0$. We can show that $\nu > 0$ by (h_1)-(h_2) and a standard argument. Hence, it follows from

$$
c = \mathcal{J}(\nu) = \frac{t^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla \nu|^2 - \int_{\mathbb{R}^N}\frac{\nabla V(x)\cdot x}{2N}|G^{-1}(\nu)|^2 > c_\infty = c.
$$

This is a contradiction. The proof is completed. □

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