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Magnetic Frenet curves on para-Sasakian manifolds

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In memory of Professor Simeon Zamkovoy (1977-2020)

Abstract. The study of magnetic curves, seen as solutions of Lorentz equation, has been done mainly in 3-dimensional case, motivated by theoretical physics. Then it was extended in higher dimensions, as for instance in Kählerian or Sasakian frame. This paper deals for the first time in literature with magnetic Frenet curves in higher dimensional paracontact context. Several classifications are provided here for different types of magnetic curves on para-Sasakian manifolds. Some relations between magnetic Frenet curves and Lorenz force are obtained on these spaces and examples of magnetic curves associated to paracontact magnetic fields are constructed. Some explicit equations of the paracontact magnetic curves on the classical para-Sasakian manifold $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ are given at the end.

1. Introduction

The notion of magnetic field (see [22]) was first studied in physics, but now it is of interest for both physics and mathematics. From mathematical point of view, a magnetic field on a (semi-)Riemannian manifold (M, g) of arbitrary dimension is defined as a closed 2-form F, and it gives rise to a (1,1)-tensor field ϕ, which is called the Lorentz force associated to F (see [30]). In the particular case of the 3-dimensional oriented Riemannian manifolds, the 2-forms can be identified with vector fields by using the Hodge star operator and the volume form. Moreover, from physical point of view, a static magnetic field on the Euclidian space E^3 is a divergence-free vector field, since it can be identified with a closed 2-form, by considering the orientation of E^3 (see [4]).

The trajectory around which a charged particle spirals under the action of a magnetic field is called magnetic curve (or magnetic trajectory) associated to the magnetic field. If on a (semi-)Riemannian manifold (M, g) a magnetic field F (which induces the Lorentz force ϕ) acts on a particle of charge q, then any corresponding magnetic curve γ satisfies the Lorentz equation $\nabla_{\gamma'}\gamma' = q\phi(\gamma')$, where ∇ is the Levi-Civita connection of g.

In the theory of surfaces, any nonzero constant multiple of the area form is a magnetic field, whose associated magnetic curves are studied by Sunada [30] on compact Riemann surfaces of genus ≥ 2 and Comtet [17] on the hyperbolic plane $\mathbb{H}^2$.

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Moreover, starting from the theory of Riemann surfaces, Adachi studied in [2], [3], [4], Kähler magnetic curves in non-flat complex space forms, by using the fact that any nonzero constant multiple of the fundamental 2-form plays the role of magnetic field, called Kähler magnetic field. In [23], Kalinin characterized the Kähler magnetic curves on complex space forms, using the fact that this type of manifolds is H-projectively flat. He showed that in this case the Lorentz equation reduces to a second order differential equation.

An important class of magnetic fields is given by the Killing vector fields, since their divergence vanish. The magnetic trajectories associated to a Killing magnetic field are called Killing magnetic curves, which in the 3-dimensional context may be seen as Kirchhoff elastic rods (see [6]), and also as solitons of the localized induction equation (see [7]). For the characterization of Killing magnetic curves in several 3-dimensional spaces see e.g. [6], [14], [16], [18], [19] and the references therein.

The study of magnetic curves is particularly interesting in odd dimensional Riemannian context, especially on almost contact metric manifolds, which are the odd dimensional analogous of almost Hermitian manifolds, whose fundamental 2-form is not always closed. It was natural to study magnetic curves in the context of contact metric manifolds and in particular on Sasakian manifolds, where the fundamental 2−form is exact, hence closed (see [5], [12], [13], [20]). In this case, the Reeb vector field (a particular unitary Killing vector field), defines the so-called contact magnetic field. The Sasakian manifolds are the odd dimensional analogous of Kähler manifolds, but the magnetic curves corresponding to contact magnetic fields on Sasakian manifolds of arbitrary odd dimension are not only circles, like in the Kähler context, but also geodesics obtained as integral curves of the Reeb vector field, Legendre φ-curves seen as 1-dimensional integral submanifolds of the contact distribution, and $\varphi-$helices of osculating order 3 (see [20]). The magnetic trajectories associated to contact magnetic fields could be studied on cosymplectic manifolds of arbitrary odd dimension, since the fundamental 2-form is closed. Such a study was done in [21], where a classification result, similar to that from the Sasakian case, was obtained.

A notable correspondent of contact geometry (see e.g. [9], [10], [11]) is paracontact geometry, where a huge number of papers was published. In dimension 3, several characterizations of magnetic curves were obtained on quasi-para-Sasakian manifolds and in particular on para-Sasakian manifolds (see [14] and [16]). In their paper, [1], Abbassi and Amri exposed several interesting results on the unit tangent bundle, to show the importance of magnetic trajectories in the paracontact context.

The purpose of this paper is to fill a gap in the literature of magnetic fields theory, by classifying the Frenet magnetic curves associated to paracontact magnetic fields on para-Sasakian manifolds of arbitrary odd dimension. In Section 3, we classify in a unitary way both space-like magnetic curves (with space-like or time-like acceleration) and time-like magnetic curves, corresponding to paracontact magnetic fields on para-Sasakian manifolds of arbitrary odd dimension. The magnetic curves obtained in our classification are geodesics given as integral curves of the Reeb vector field, non-geodesic φ-circles of constant paracontact (hyperbolic) angle, Legendre $\varphi-$curves, and (hyperbolic) $\varphi-$helices of osculating order 3. Different from the magnetic curves on Sasakian manifolds, which have constant contact angle, the magnetic space-like Frenet curves with space-like acceleration and the magnetic time-like Frenet curves on para-Sasakian manifolds have constant paracontact hyperbolic angle. Subsequently, different from the Sasakian case, the non-geodesic magnetic curves of the mentioned types on para-Sasakian manifolds can be φ-circles of constant paracontact hyperbolic angle and hyperbolic $\varphi-$helices of osculating order 3. In the last section, we give the explicit equations of the paracontact magnetic curves on the para-Sasakian manifold $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$.

2. Preliminaries

Firstly we recall some notions concerning the paracontact geometry, the magnetic and Frenet curves.

2.1. *Paracontact manifolds*

The notion of almost paracontact structure on a differentiable manifold of arbitrary dimension was introduced by Sato, as follows:

Definition 2.1. *([28]) A triple (φ, ξ, η) is called an almost paracontact structure on a differentiable manifold M, if φ is a $(1,1)-$tensor field, ξ is a vector field and η is a 1-form satisfying:*

$$\varphi^2 = \mathrm{I} - \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

In this case, (M, φ, ξ, η) is called an almost paracontact manifold. If, moreover, $d\eta(\xi, -) = 0$, then ξ is called the Reeb vector field.

In the sequel we recall some consequences of the above definition, given for example in [8].

Remark 2.2. *On a paracontact manifold (M, φ, ξ, η) the following items hold good:*
(a) $\varphi\xi = \eta \circ \varphi = 0$;
(b) The dimension m of the manifold may be either odd or even and $\operatorname{rank}\varphi = m - 1$;
(c) $\operatorname{Ker}\eta = \operatorname{Im}\varphi$, $\operatorname{Ker}\varphi = \operatorname{span}\{\xi\}$;
(d) The restriction of φ to the paracontact distribution $\operatorname{Ker}\eta$ is a product structure, whose eigen distributions corresponding to the eigenvalues 1 and -1 may have different ranks.

When an almost paracontact manifold carries a (semi-)Riemannian metric, then two relations of metric compatibility and anti-compatibility with the underlying structures are known in literature. Namely, the notion of almost paracontact Riemannian manifold was introduced by Sato in [29] (requiring the Riemannian metric compatibility with the almost paracontact structure), while the notion of paracontact metric manifold was given by Kaneyuki, Kozai, Williams in [24], [25] (involving a semi-Riemannian metric anti-compatible with the almost paracontact structure). Both above mentioned notions were generalized in [8]. Throughout the paper we deal with the following new notion (a particular case of that given in [8]), which we introduce as follows.

Definition 2.3. *An almost paracontact manifold (M, φ, ξ, η) is called an ε-*almost paracontact metric manifold *if it carries a semi-Riemannian metric g, related with the underlying structure by the following metricity relation:*

$$g(\varphi X, \varphi Y) = \varepsilon\eta(X)\eta(Y) - g(X, Y), \ \forall X, Y \in \mathfrak{X}(M), \tag{2}$$

where ε is 1 or -1, according as ξ is space-like or time-like, respectively.

Remark 2.4. *Accordingly to [8], a ε-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ has the following properties:*
(i) The manifold M has odd dimension $m = 2n + 1$;
(ii) The restriction φ to the distribution $\mathcal{D} = \operatorname{Ker}\eta$ is a paracomplex structure (i.e., the eigen distributions D^+ and D^- of the product structure $\varphi/_{\operatorname{Ker}\eta}$ have equal dimensions and $\mathcal{D} = D^+ \oplus D^-$);
(iii) The metric g restricted to $\mathcal{D}$ is a semi-Riemannian structure of neutral signature (n, n), and therefore, the signature of g on M is either $(n, n+1)$ or $(n+1, n)$ according as ξ is time-like (i.e. $\varepsilon = -1$) or ξ is space-like (i.e. $\varepsilon = 1$), respectively;
(iv) (φ, g) restricted to the distribution $\mathcal{D}$ is a para-Hermitian structure (see the survey [15] for this notion).
Moreover, for this type of manifolds one has
(v) $\eta(X) = \varepsilon g(X, \xi)$, $\forall X \in \mathfrak{X}(M)$.

Definition 2.5. The fundamental 2-form *of the ε-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is the 2-form Ω on M, defined by*

$$\Omega(X, Y) = g(X, \varphi Y), \forall X, Y \in \mathfrak{X}(M). \tag{3}$$

A ε-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be normal if the relation

$$N(X, Y) = 2d\eta(X, Y)\xi$$

holds for all $X, Y \in \mathfrak{X}(M)$, where N is the Nijenhuis tensor field given by

$$N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \ \forall X, Y \in \mathfrak{X}(M).$$

Recall that the exterior differential of η is expressed as

$$d\eta(X, Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X, Y])], \ \forall X, Y \in \mathfrak{X}(M).$$

Definition 2.6. *A ε-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is called a ε-paracontact metric manifold if $\Omega = d\eta$. If, moreover, the ε-paracontact metric manifold is normal, then it is called a ε-para-Sasakian manifold. A ε-paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is called ε-K-paracontact manifold if the Reeb vector field is Killing.*

Remark 2.7. *If in the above definition we consider that the Reeb vector field is space-like, i.e. $\varepsilon = 1$, we obtain the almost paracontact metric manifolds, which were classified by S. Zamkovoy and G. Nakova [31] in twelve classes (see also [27]).*

Remark 2.8. *A ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is characterized by the relation:*

$$(\nabla_X \varphi)Y = \varepsilon\eta(Y)X - g(X, Y)\xi, \ \forall X, Y \in \mathfrak{X}(M), \tag{4}$$

where ∇ is the Levi-Civita connection of g. This relation yields

$$\nabla_X \xi = -\varepsilon\varphi X, \ \forall X \in \mathfrak{X}(M). \tag{5}$$

Remark 2.9. *From* (5) *and the relation $g(\varphi X, Y) = -g(X, \varphi Y)$ it follows that on a ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the Reeb vector field ξ is a Killing vector field, i.e. the ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is a K-paracontact manifold. The converse is not true in general, but a 3-dimensional paracontact metric manifold is K-paracontact if and only if it is ε-para-Sasakian.*

We introduce now the notion of ε-φ-basis on a ε-almost paracontact metric manifold, similar to that of φ-basis on an almost paracontact metric manifold, for which we quote e.g. [14, p. 425].

Definition 2.10. *Let $(M, \varphi, \xi, \eta, g)$ be a ε-almost paracontact metric manifold and let U_p be a neighbourhood of a point $p \in M$. A pseudo-orthonormal basis $\{X_i, \varphi X_i, \xi\}$, with $i \in \{1, \dots, n\}$, in U_p (where $\{X_i, \xi\}_{i\in\{1,\dots,n\}}$ are space-like vector fields and $\{\varphi X_i\}_{i\in\{1,\dots,n\}}$ are time-like vector fields, or vice-versa) is called a ε-φ-*basis.

Remark 2.11. *Any ε-almost paracontact metric manifold admits at least locally a ε-φ-basis.*

2.2. Frenet Curves on semi-Riemannian manifolds

Extending the Frenet equations for the Minkowski space [26, p. 35] to a semi-Riemannian manifold (M, g), with g of signature $(n + 1, n)$, we define the Frenet curve of osculating order $r \geq 1$ as follows:

Definition 2.12. *Let $\gamma : I \to M$ be a pseudo-arc length parametrized space-like or time-like curve in a semi-Riemannian manifold (M, g), with the metric g of signature $(n + 1, n)$. We say that γ is a Frenet curve of osculating order $r \geq 1$ if there exist some pseudo-orthonormal vector fields $\{\dot\gamma, \nu_1, \dots, \nu_{r-1}\}$ along γ, such that*

$$\begin{aligned}
&\nabla_{\dot\gamma}\dot\gamma = \varepsilon_1\kappa_1\nu_1,\\
&\nabla_{\dot\gamma}\nu_1 = -\varepsilon_0\kappa_1\dot\gamma + \varepsilon_2\kappa_2\nu_2,\\
&\nabla_{\dot\gamma}\nu_j = -\varepsilon_{j-1}\kappa_j\nu_{j-1} + \varepsilon_{j+1}\kappa_{j+1}\nu_{j+1}, \ \text{ for } j \in \{2, r-2\},\\
&\nabla_{\dot\gamma}\nu_{r-1} = -\varepsilon_{r-2}\kappa_{r-1}\nu_{r-2},
\end{aligned} \tag{6}$$

where $g(\dot\gamma, \dot\gamma) = \varepsilon_0 = \pm 1$, $g(\nu_i, \nu_i) = \varepsilon_i = \pm 1$, and κ_i is a positive C^∞ function of the pseudo-arc length parameter s, called the i–th curvature of γ, for each $i \in \{1, \dots, r-1\}$.

In particular, a Frenet curve of osculating order 2 *with constant curvature κ_1 is a* circle.

Further, a Frenet curve of osculating order r, such that all curvatures $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ are constant is called a helix of order r.

Remark 2.13. *The relation between the causalities of $\dot\gamma$, ν_1 and ν_2 is given by $\varepsilon_2 = -\varepsilon_0\varepsilon_1$ (see [26, p. 35]).*

Example 2.14. *A* geodesic *in a semi-Riemannian manifold (M, g) is a Frenet curve of osculating order* 1.

Remark 2.15. *The curvatures of a Frenet curve γ of osculating order $r \geq 2$ in a semi-Riemannian manifold (M, g), with the metric g of signature $(n+1, n)$, have the expressions*

$$\kappa_1 = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \nu_1) = -g(\dot{\gamma}, \nabla_{\dot{\gamma}}\nu_1) > 0,$$

$$\kappa_i = g(\nabla_{\dot{\gamma}}\nu_{i-1}, \nu_i) = -g(\nu_{i-1}, \nabla_{\dot{\gamma}}\nu_i) > 0, \ i \in \{2, \ldots, r-2\}.$$

Definition 2.16. *Let γ be a Frenet curve of osculating order r on an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$.*
*When $r \geq 2$, we say γ to be a φ-*curve *in M if* $\mathrm{span}\{\dot{\gamma}, \nu_1, \ldots, \nu_{r-1}\}$ *is φ-invariant.*
*A φ-*curve *γ in M is said to be Legendre φ-*curve *if $\eta(\dot{\gamma}) = 0$.*
*A φ-curve of osculating order 2 with constant curvature κ_1 is called a φ-*circle.
*A φ-curve of osculating order $r > 2$ with all curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ constant is called a φ-*helix of order *r.*
*The functions $\tau_{ij} = g(\nu_i, \varphi\nu_j)$ for $0 \leq i < j \leq r-1$, where $\nu_0 = \dot{\gamma}$, are called the φ-*torsions *of a φ-curve γ.*

Remark 2.17. *In [20], a φ-curve of osculating order 2 was defined by the $\varphi-$invariance of* $\mathrm{span}\{\dot{\gamma}, \nu_1, \xi\}$*, but since $\varphi\xi = 0$ and φ^2 has the form* (1)*, the $\varphi-$invariant space reduces to* $\mathrm{span}\{\dot{\gamma}, \nu_1\}$*, as in Definition 2.16.*

2.3. Magnetic curves on semi-Riemannian manifolds

Let (M, g) be a semi-Riemannian manifold, on which a particle of charge q moves under the action of a magnetic field. In this case the magnetic field is identified with a closed 2-form F_q, to whom one can associate a (1,1)-tensor field ϕ_q, called the *Lorentz force*, by the relation:

$$F_q(X, Y) = g(\phi_q X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{7}$$

A *magnetic curve (or trajectory)* associated to F_q is any curve γ on M, which satisfies the *Lorentz equation* (or *Newton equation*):

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \phi_q(\dot{\gamma}), \tag{8}$$

where ∇ is the Levi-Civita connection of the metric g.

When the particle moves under the action of gravity only, i.e. the magnetic field vanishes, the relation (8) reduces to the equation of geodesics under pseudo-arc length parametrization, namely $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

A *uniform* magnetic field is a magnetic field F_q for which F_q is parallel with respect to ∇, i.e. $\nabla F_q = 0$.

An important property of the magnetic trajectories is that their speed is constant. A magnetic trajectory $\gamma(s)$ is called *normal magnetic curve* if it is parametrized by the pseudo-arc length, i.e. $g(\dot{\gamma}, \dot{\gamma}) = \pm 1$.

On a 3-dimensional semi-Riemannian manifold (M^3, g), where any closed 2-form may be identified with a divergence-free vector field via the Hodge star operator and the volume form dv_g, one can define a magnetic field F_V associated to a divergence-free vector field V on M^3, by:

$$F_V(X, Y) = dv_g(V, X, Y), \quad \forall X, Y \in \mathfrak{X}(M^3). \tag{9}$$

On the other hand, the *cross product* $\times$ on $(\mathfrak{X}(M^3), g)$ can be defined by:

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M^3). \tag{10}$$

Next, from the relations (7), (9) and (10), it follows that the Lorentz force associated to F_V can be expressed as

$$\phi(X) = V \times X, \quad \forall X \in \mathfrak{X}(M^3),$$

and the Lorentz equation (8) takes the form

$$\nabla_{\dot{\gamma}}\dot{\gamma} = V \times \dot{\gamma}.$$

If, in particular, the divergence-free vector field V is Killing, then F_V is called *Killing magnetic field*, and its trajectories are called *Killing magnetic curves*.

3. Magnetic curves on Para-Sasakian manifolds

Consider a ε-paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ and its fundamental 2-form Ω. Due to the skew-symmetry of φ with respect to g, (3) becomes

$$\Omega(X, Y) = -g(\varphi X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

and then using (7) we can define the *ε-paracontact magnetic field* with *strength* $q \in \mathbb{R}^*$ on M by

$$F_q(X, Y) = q\Omega(X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{11}$$

From (3), (7) and (11) we can express the *Lorentz force* ϕ_q associated to the ε-paracontact magnetic field F_q by the relation

$$\phi_q = -q\varphi.$$

Then, the Lorentz equation (8) takes the form

$$\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma, \tag{12}$$

where γ is a smooth curve parametrized by pseudo-arc length on M, called a *ε-paracontact normal magnetic curve of magnetism q* or a *trajectory* of the magnetic field F_q.

Definition 3.1. *Let γ be a curve on an almost paracontact manifold $(M, \varphi, \xi, \eta, g)$. The angle between $\dot\gamma$ and ξ is called* the paracontact angle of the curve *γ.*

Lemma 3.2. *Let γ be a trajectory of a ε-paracontact magnetic field F_q of magnetism $q \in \mathbb{R} \setminus \{0\}$ on a ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$. The curve γ has constant speed and the paracontact angle is constant or, equivalently, $\eta(\dot\gamma) = \eta_0 \in \mathbb{R}$.*

Proof: By considering a ε-paracontact magnetic curve on a ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ and by using relation (12), we obtain

$$\frac{d}{ds}g(\dot\gamma, \dot\gamma) = 2g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = -2qg(\varphi\dot\gamma, \dot\gamma) = 0,$$

i.e. $\dot\gamma$ has constant length.

By taking into account Definition 2.1 and relation (12), we have the following equalities:

$$\frac{d}{ds}g(\xi, \dot\gamma) = g(\nabla_{\dot\gamma}\xi, \dot\gamma) + g(\xi, \nabla_{\dot\gamma}\dot\gamma) = -\varepsilon g(\varphi\dot\gamma, \dot\gamma) - q\eta(\varphi\dot\gamma) = 0,$$

which yield $g(\xi, \dot\gamma) = const.$, i.e. $\eta(\dot\gamma) = const.$ $\square$

Proposition 3.3. *Let γ be a Frenet curve of osculating order r on a ε-para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$. If γ is a non-geodesic normal magnetic curve associated to a ε-paracontact magnetic field F_q of magnetism $q \in \mathbb{R} \setminus \{0\}$ on M, then:*

- *i) The curve γ has osculating order $r \leq 3$.*
- *ii) The Lorentz force satisfies:*

$$\phi_q\dot\gamma = \varepsilon_1\kappa_1\nu_1,$$
$$\phi_q\nu_1 = \varepsilon_1(1 - \varepsilon_0\eta_0^2)\frac{q^2}{\kappa_1}\dot\gamma + \varepsilon_2\eta_0 q\,\mathrm{sgn}(q\eta_0 - \varepsilon_0)\nu_2,$$
$$\phi_q\nu_2 = -\varepsilon_1 q\eta_0\,\mathrm{sgn}(q\eta_0 - \varepsilon_0)\nu_1,$$

where sgn *denotes the signum function.*

- *iii) The Reeb vector field ξ is space-like (i.e. the manifold is para-Sasakian), given by:*

$$\xi = \varepsilon_0\eta_0\dot{\gamma} + \varepsilon_0 \operatorname{sgn}(q\eta_0 - \varepsilon_0)\frac{\kappa_1}{q}\nu_2. \tag{13}$$

- *iv) One has $\varepsilon_1(\eta_0^2 - \varepsilon_0) > 0$ and the first curvature of γ is*

$$\kappa_1 = |q|\sqrt{\varepsilon_1(\eta_0^2 - \varepsilon_0)}.$$

In particular,

  - *If $r = 2$, then $\eta_0 = \frac{\varepsilon_0}{q}$, and $\kappa_1 = \sqrt{\varepsilon_1(1 - \varepsilon_0 q^2)}$.*
  - *If $r = 3$, then the second curvature of γ is $\kappa_2 = |q\eta_0 - \varepsilon_0|$.*

Proof: From the Lorentz equation (12) and the first Frenet formula in (6) we obtain

$$\varepsilon_1\kappa_1\nu_1 = -q\varphi\dot{\gamma}, \tag{14}$$

and then, by using the metricity condition (2) and Lemma 3.2, it follows that

$$\varepsilon_1\kappa_1^2 = q^2(\varepsilon\eta_0^2 - \varepsilon_0), \tag{15}$$

hence one has $\varepsilon_1(\varepsilon\eta_0^2 - \varepsilon_0) > 0$, and

$$\kappa_1 = |q|\sqrt{\varepsilon_1(\varepsilon\eta_0^2 - \varepsilon_0)}. \tag{16}$$

From (14), we have that the action of the Lorentz force on the tangent vector to γ is

$$\phi_q\dot{\gamma} = \varepsilon_1\kappa_1\nu_1,$$

which yields

$$\eta(\nu_1) = 0. \tag{17}$$

Next, applying to (14) the covariant derivative with respect to $\dot{\gamma}$, using the relation (4), the Frenet equations (6), Definition 2.1, and the expression (16) of κ_1, we obtain the following identity:

$$\varepsilon_2\kappa_1\kappa_2\nu_2 = \varepsilon_1\varepsilon q(q\eta_0 - \varepsilon_0)(\varepsilon_0\eta_0\dot{\gamma} - \varepsilon\xi). \tag{18}$$

When γ has osculating order 2, i.e. $\kappa_2 = 0$, the relation (18) yields two cases: $\eta_0 = \frac{\varepsilon_0}{q}$ or $\dot{\gamma} = \frac{\varepsilon}{\varepsilon_0\eta_0}\xi$.

In the second case, γ is an integral curve of ξ, which implies that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, i. e. γ is a geodesic, which must be excluded from the classification.

In the case when $\eta_0 = \frac{\varepsilon_0}{q}$, the relation (15) yields $\varepsilon_1(\varepsilon - \varepsilon_0 q^2) > 0$ and then from (16) we obtain

$$\kappa_1 = \sqrt{\varepsilon_1(\varepsilon - \varepsilon_0 q^2)}.$$

When γ has osculating order $r > 2$, i.e. $\kappa_2 \neq 0$, by taking the norm of the two hand sides of equation (18), a simple calculation yields $\kappa_2^2 = \varepsilon(q\eta_0 - \varepsilon_0)^2$. It follows, on one hand, that $\varepsilon = 1$, i.e. ξ is space-like and, on the other hand, that

$$\kappa_2 = |q\eta_0 - \varepsilon_0|.$$

Next, from (18) we get that ξ has the expression (13) from the statement.

Since we proved that the signature of ξ is 1, we obtain from (13) that

$$\eta_0^2\varepsilon_0 + \frac{\kappa_1^2}{q^2}\varepsilon_2 = 1,$$

hence

$$\kappa_1^2 = \varepsilon_2 q^2(1-\eta_0^2\varepsilon_0),$$

and replacing ε_2 by $-\varepsilon_0\varepsilon_1$, we can write

$$\kappa_1^2 = \varepsilon_1 q^2(\eta_0^2-\varepsilon_0).$$

Applying φ to (13) and using (14), we obtain that

$$\varphi\nu_2 = \varepsilon_1\eta_0 \operatorname{sgn}(q\eta_0-\varepsilon_0)\nu_1, \tag{19}$$

which shows that the action of the Lorentz force on ν_2 is the one given by the theorem.

Next, since $\kappa_1 \neq 0$, by applying φ to (14) and replacing ξ from (13), we obtain

$$\varphi\nu_1 = \varepsilon_1\frac{q}{\kappa_1}(\varepsilon_0\eta_0^2-1)\dot\gamma - \varepsilon_2\eta_0 \operatorname{sgn}(q\eta_0-\varepsilon_0)\nu_2, \tag{20}$$

i.e. the action of ϕ_q on ν_1 is that given in the statement.

If we take the covariant derivative with respect to $\dot\gamma$, then (20) yields

$$\nabla_{\dot\gamma}\nu_2 = [q(\varepsilon_0\eta_0^2-1)\nu_1 - (\nabla_{\dot\gamma}\varphi)\nu_1 - \varphi(\nabla_{\dot\gamma}\nu_1)]\frac{\varepsilon_2 \operatorname{sgn}(q\eta_0-\varepsilon_0)}{\eta_0}. \tag{21}$$

From (2), (17) and $g(\dot\gamma,\nu_1)=0$ we have that $\nabla_{\dot\gamma}(\varphi\nu_1)=0$. Moreover, by using the second Frenet formula in (6), (14), (19), and the fact that $\varepsilon_2=-\varepsilon_0\varepsilon_1$, (21) becomes

$$\nabla_{\dot\gamma}\nu_2 = -\varepsilon_1\kappa_2\nu_1.$$

Comparing the above expression with the one given by the third Frenet equation from (6), we obtain $\kappa_3=0$, i.e. the Frenet magnetic curve has osculting order 3. □

Remark 3.4. *Concerning the statement i) of Proposition 3.3, the curve γ is a helix, as we commented in Example 2.14.*

In the geometry of manifolds, there is a concern to determine the shape, by using curvature.

In the sequel, we use the curvatures of a Frenet curve to determine the shape of normal magnetic curves in the context of para-Sasakian manifolds. Thus, we now provide a classification result for the normal paracontact magnetic curves, in the mentioned background.

Theorem 3.5. *Let γ be a Frenet curve and let F_q be the paracontact magnetic field of magnetism $q \neq 0$ on a para-Sasakian manifold (M,φ,ξ,η,g). Then γ is a normal magnetic curve on M associated to F_q, if and only if it is one of the following:*

i) *a geodesic obtained as an integral curve of ξ;*

ii) *a non-geodesic φ-circle of constant paracontact angle such that $\eta_0=\frac{\varepsilon_0}{q}$ and $0<\varepsilon_1(1-\varepsilon_0q^2)=\kappa_1^2$;*

iii) *a φ-(hyperbolic) helix of osculating order 3, with space-like axis ξ, and with curvatures satisfying $0<q^2\varepsilon_1(\eta_0^2-\varepsilon_0)=\kappa_1^2$ and $\kappa_2=|q\eta_0-\varepsilon_0|$, where $\eta_0=\eta(\dot\gamma)=const\in\mathbb{R}$.*

Proof: Let us consider a normal magnetic curve γ associated to F_q.

If γ is a geodesic on $(M, \varphi, \xi, \eta, g)$, we have on one hand that $\nabla_{\dot\gamma}\dot\gamma = \phi_q\dot\gamma$, where ∇ is the Levi-Civita connection of g and on the other hand that $\nabla_{\dot\gamma}\dot\gamma = 0$, hence $\phi_q\dot\gamma = 0$. Since $\phi_q\dot\gamma = -q\varphi\dot\gamma$, and on a para-Sasakian manifold $\ker\varphi = \mathrm{span}\{\xi\}$, it follows that $\dot\gamma \in \mathrm{span}\{\xi\}$. Then, the properties of $\dot\gamma$ and ξ to be unitary, yield $\dot\gamma = \pm\xi$, i.e. γ is an integral curve of ξ up to a change of the sign of the parameter, which yields the item i).

On the other hand, if γ is not a geodesic, then by i) of Proposition 3.3, γ is a circle ($r = 2$) or a hyperbolic helix ($r = 3$). By ii) of Proposition 3.3, for $r = 2$ (respectively $r = 3$), $\{\dot\gamma, \nu_1\}$ (respectively $\{\dot\gamma, \nu_1, \nu_2\}$) is φ-invariant, i.e. γ is a φ-circle (respectively a φ-helix). Next, iv) Proposition 3.3 yields the items ii) and iii).

To prove the converse part of the statement we shall show that each curve described at items i), ii) and iii) is a paracontact magnetic curve on the manifold.

Firstly, on a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ we take a curve from item i), i.e. a geodesic γ obtained as an integral curve of ξ, that is $\nabla_{\dot\gamma}\dot\gamma = 0$ and $\dot\gamma = \xi$. Since on the para-Sasakian manifold we have $\varphi\xi = 0$, by using the two previous equalities, we obtain $\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma$, i.e. γ is a paracontact magnetic curve on $(M, \varphi, \xi, \eta, g)$.

Next, we consider a curve from item iii), i.e. a φ-(hyperbolic) helix γ of osculating order 3, of space-like axis ξ and of curvatures $\kappa_1 = |q|\sqrt{\varepsilon_1(\eta_0^2 - \varepsilon_0)}$, $\kappa_2 = |q\eta_0 - \varepsilon_0|$, where $\eta_0 = \eta(\dot\gamma) = const. \in \mathbb{R}$. Since $\eta(\dot\gamma) = g(\xi, \dot\gamma)$, it follows that $\frac{d}{ds}g(\xi, \dot\gamma) = 0$, or equivalently $g(\nabla_{\dot\gamma}\xi, \dot\gamma) + g(\xi, \nabla_{\dot\gamma}\dot\gamma) = 0$. By using (5) and the first Frenet equation from (6), we obtain $\varepsilon_1\kappa_1\eta(\nu_1) = 0$, i.e. $\eta(\nu_1) = g(\xi, \nu_1) = 0$, hence $\xi \in \mathrm{span}\{\dot\gamma, \nu_2\}$, which can be written as

$$\xi = \varepsilon_0\eta_0\dot\gamma + \varepsilon_2\eta(\nu_2)\nu_2. \tag{22}$$

Since ξ is space-like and unitary, i.e. $g(\xi, \xi) = 1$, and $\varepsilon_2 = -\varepsilon_0\varepsilon_1$, it follows that $|\eta(\nu_2)| = \sqrt{\varepsilon_1(\eta_0^2 - \varepsilon_0)}$, and taking into account the expression of κ_1, we have:

$$|\eta(\nu_2)| = \frac{\kappa_1}{|q|}.$$

Applying the covariant derivative with respect to $\dot\gamma$ in the expression (22) of ξ and then using (5), followed by the Frenet formulas (6) for osculating order 3, we obtain:

$$\varphi\dot\gamma = (\varepsilon_1\varepsilon_2\eta(\nu_2)\kappa_2 - \varepsilon_0\varepsilon_1\eta_0\kappa_1)\,\nu_1,$$

and it follows that

$$g(\varphi\dot\gamma, \varphi\dot\gamma) = (\varepsilon_1\varepsilon_2\eta(\nu_2)\kappa_2 - \varepsilon_0\varepsilon_1\eta_0\kappa_1)^2\,\varepsilon_1. \tag{23}$$

The relation (23) replaced into the metricity condition (2), yields:

$$\varepsilon_1\kappa_2^2\eta(\nu_2)^2 - 2\varepsilon_0\varepsilon_1\varepsilon_2\eta_0\kappa_1\kappa_2\eta(\nu_2) + (\varepsilon_1\kappa_1^2 - 1)\eta_0^2 + \varepsilon_0 = 0. \tag{24}$$

By solving the equation (24) with respect to $\eta(\nu_2)$ and then replacing κ_2 from the hypothesis, we obtain two solutions:

$$\eta(\nu_2) = \frac{\kappa_1}{|q|}\varepsilon_1\frac{-|q|\eta_0 + \varepsilon_0\,\mathrm{sgn}\,q}{|q\eta_0 - \varepsilon_0|} = -\frac{\kappa_1}{q}\varepsilon_1\,\mathrm{sgn}(q\eta_0 - \varepsilon_0)$$

and

$$\eta(\nu_2) = \frac{\kappa_1}{|q|}\varepsilon_1\frac{-|q|\eta_0 - \varepsilon_0\,\mathrm{sgn}\,q}{|q\eta_0 - \varepsilon_0|}.$$

By substituting the first above expression of $\eta(\nu_2)$ into (22), we obtain the Reeb vector field, given by:

$$\xi = \varepsilon_0\eta_0\dot\gamma - \varepsilon_1\varepsilon_2\frac{\kappa_1}{q}\,\mathrm{sgn}(q\eta_0 - \varepsilon_0)\nu_2.$$

Then, applying $\nabla_{\dot{\gamma}}$ and using the Frenet formulas from (6) with $r = 3$, we have

$$\nabla_{\dot{\gamma}}\xi = [\varepsilon_0\varepsilon_1\eta_0\kappa_1 + \varepsilon_2\frac{\kappa_1}{q}\operatorname{sgn}(q\eta_0 - \varepsilon_0)\kappa_2]\nu_1.$$

Next from (5) and the expressions of κ_2 in the hypothesis, it follows that

$$-\varphi\dot{\gamma} = \varepsilon_1\frac{\kappa_1}{q}\nu_1, \text{ i.e. } \phi_q\dot{\gamma} = \varepsilon_1\kappa_1\nu_1.$$

Finally, from the first Frenet formula we conclude that γ is a magnetic curve associated to F_q.

For the second expression of $\eta(\nu_2)$ from above, by following the same directions as for the previous case, we obtain a change of sign in the expression of $\phi_q\dot{\gamma}$, hence γ is a magnetic curve corresponding to $-F_q$. Since the magnetism q can be either positive or negative, it follows that the magnetic curves corresponding to F_q coincide with those associated to $-F_q$. Hence γ from item iii) is a magnetic curve corresponding to F_q.

In particular, the curves exposed in ii) are also paracontact magnetic curves. □

Remark 3.6. *When the paracontact magnetic curve γ falls in case* i) of Theorem 3.5, then γ must be space-like.

In the sequel, by considering each causality of the Frenet curve γ, from Theorem 3.5 and Remark 3.6, we obtain, as consequences, the following characterizations of the paracontact magnetic curves.

Corollary 3.7. *On a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, a space-like Frenet curve γ with space-like acceleration is a paracontact normal magnetic curve if and only if γ is one of the following:*

a) *a geodesic obtained as an integral curve of ξ;*

b) *a non-geodesic φ-circle of constant paracontact hyperbolic angle $\theta = \pm\operatorname{argcosh}\left(\frac{1}{|q|}\right)$ with $\kappa_1 = \sqrt{1-q^2}$, when the magnetism is $q \in (-1, 1) \setminus \{0\}$;*

c) *a φ-hyperbolic helix of osculating order* 3, *with axis ξ, and with curvatures $\kappa_1 = |q \sinh\theta|$, $\kappa_2 = |1 - q\widetilde{\cosh}\,\theta|$, where θ is the constant paracontact hyperbolic angle and $\widetilde{\cosh}\,\theta = \pm\cosh\theta$.*

Proof: Consider a Frenet paracontact normal magnetic curve γ on M, with $\varepsilon_0 = \varepsilon_1 = 1$.

Item a) is already proved by Theorem 3.5 i).

If in Theorem 3.5 iii) we take $\varepsilon_0 = \varepsilon_1 = 1$, we obtain $\eta_0^2 > 1$, because $\kappa_1 > 0$. Subsequently, there exists $\theta \in \mathbb{R}$ such that $\eta_0 = \pm\cosh\theta$, which will be denoted in the sequel by $\widetilde{\cosh}\,\theta$. In this case it follows that $\kappa_1 = |q|\sqrt{\cosh^2\theta - 1} = |q\sinh\theta|$, $\kappa_2 = |1 - q\widetilde{\cosh}\,\theta|$, and thus item c) is proved.

When the osculating order is $r = 2$, by using Theorem 3.5 ii) we obtain that $\eta_0 = \frac{1}{q}$ and $\kappa_1^2 = 1 - q^2 > 0$, i.e. $|q| < 1$ and $\eta_0^2 > 1$. Thus we have again that there exists $\theta \in \mathbb{R}$ such that $\eta_0 = \widetilde{\cosh}\,\theta = \frac{1}{q}$. Since $\cosh\theta > 1$ it follows that $\cosh\theta = \frac{1}{|q|}$, or equivalently, $\theta = \pm\operatorname{argcosh}\left(\frac{1}{|q|}\right)$. Thus the statement b) follows, which completes the proof.

To prove the converse, we take into account that any curve γ from items a), b), c), is a curve described in Theorem 3.5 at item i), ii), iii), respectively, and then γ is a paracontact magnetic curve on the manifold. □

Corollary 3.8. *There do not exist space-like Legendre $\varphi-$curves with space-like acceleration, which are paracontact normal magnetic curves in a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$.*

Proof: If we suppose $\eta(\dot{\gamma}) = 0$, it follows that $\cosh\theta = 0$, but since $\cosh\theta \geq 1$ for every $\theta \in \mathbb{R}$, which contradicts the assumption, and hence we conclude the statement. □

Corollary 3.9. *On a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, a space-like Frenet curve γ with time-like acceleration is a paracontact normal magnetic curve if and only if γ is one of the following curves:*

a) *a geodesic obtained as an integral curve of* ξ*;*

b) *a non-geodesic* φ*-circle of constant paracontact angle* $\theta = \arccos\frac{1}{q} \in [0,\pi]$*, with* $\kappa_1 = \sqrt{q^2-1}$*, when* $|q| \geq 1$*;*

c) *a Legendre* φ*-curve in* M*, with* $\kappa_1 = |q|$*,* $\kappa_2 = 1$*, i.e. a* $1-$*dimensional integral submanifold of the paracontact distribution. Moreover, in this case* $\xi = -\operatorname{sgn}(q)\nu_2$*;*

d) *a* φ*-helix of osculating order* 3*, with axis* ξ*, and of curvatures* $\kappa_1 = |q|\sin\theta$*,* $\kappa_2 = |1-q\cos\theta|$*, where* $\theta \in [0,\pi]$ *is the constant paracontact angle.*

Proof: Let γ be a Frenet curve with $\varepsilon_0 = -\varepsilon_1 = 1$ on M. Item a) is already proved by Theorem 3.5 i).

Taking $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$, we obtain from Theorem 3.5 iii) that $\eta_0^2 < 1$, i.e. there exists $\theta \in [0,\pi]$ such that $\eta_0 = \cos\theta$ and then $\kappa_1 = |q|\sqrt{1-\cos^2\theta} = |q|\sin\theta$, $\kappa_2 = |1-q\cos\theta|$, and thus any curve from item d) is a paracontact magnetic curve.

For $r = 2$, we obtain from Theorem 3.5 ii) that $\eta_0 = \frac{1}{q}$ and $\kappa_1^2 = q^2 - 1 > 0$, i.e. $|q| > 1$ and $\eta_0^2 < 1$, hence there exists $\theta \in [0,\pi]$ such that $\eta_0 = \cos\theta = \frac{1}{q}$, or equivalently, $\theta = \arccos\frac{1}{q}$. Thus, any curve described at b) is a paracontact magnetic curve.

To prove item c), we assume that $\eta(\dot\gamma) = 0$, which yields $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2}$. Then, by substituting this particular value of θ, the expressions of κ_1 and κ_2 from item d) reduce to $\kappa_1 = |q|$ and $\kappa_2 = 1$. Then, from (13) it follows that $\xi = -\operatorname{sgn}(q)\nu_2$, hence d) is proved.

The converse follows by the fact that any curve γ described at a), b), d), is a curve from items i), ii), iii) of Theorem 3.5, respectively, and any curve from c) is obtained by particularizing d) for $\theta = \frac{\pi}{2}$. □

Corollary 3.10. *On a para-Sasakian manifold* (M,φ,ξ,η,g)*, a time-like Frenet curve* γ *is a paracontact normal magnetic curve if and only if one of the following instances holds good:*

a) γ *is a non-geodesic* φ*-circle of constant paracontact hyperbolic angle* $\theta = \operatorname{argsh}\left(-\frac{1}{q}\right)$*, and with* $\kappa_1 = \sqrt{q^2+1}$*;*

b) γ *is a Legendre* φ*-curve on* M*, with* $\kappa_1 = |q|$*,* $\kappa_2 = 1$*, i.e. a* $1-$*dimensional integral submanifold of the paracontact distribution. In this case* $\xi = \operatorname{sgn}(q)\nu_2$*;*

c) γ *is a* φ*-hyperbolic helix of osculating order* 3*, with axis* ξ*, and of curvatures* $\kappa_1 = |q|\cosh\theta$*,* $\kappa_2 = |1+q\sinh\theta|$*, where* θ *is the constant paracontact hyperbolic angle and* $\eta_0 = \sinh\theta$*.*

Proof: By taking a time-like Frenet curve γ on a para-Sasakian manifold (M,φ,ξ,η,g), from the first Frenet equation, (12) and (2) we have that

$$g(\nu_1,\nu_1) = \frac{1}{\kappa_1^2}g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = \frac{q^2}{\kappa_1^2}g(\varphi\dot\gamma,\varphi\dot\gamma) = \frac{q^2}{\kappa_1^2}(\eta_0^2-\varepsilon_0) = \frac{q^2}{\kappa_1^2}(\eta_0^2+1) > 0.$$

Since the first normal to a time-like curve is a space-like vector, we shall take a Frenet curve γ on M, such that $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$.

Taking into account Remark 3.6, we conclude that a time-like Frenet magnetic curve can fall either in case ii) or iii), Theorem 3.5.

Then, by taking a non-geodesic Frenet curve with $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$, we obtain from Theorem 3.5 iii), that η_0 is a real constant, hence there exists $\theta \in \mathbb{R}$ such that $\eta_0 = \sinh\theta$. In this case the curvatures become $\kappa_1 = |q|\sqrt{1+\eta_0^2} = |q|\cosh\theta$ and $\kappa_2 = |1+q\sinh\theta|$, and the proof of item c) is concluded.

If the magnetic curve γ is a Legendre curve, i.e. $\eta(\dot\gamma) = 0$, it follows that $\sinh\theta = 0$, or equivalently $\theta = 0$. Then, the above expressions of κ_1 and κ_2 reduce to $\kappa_1 = |q|$ and $\kappa_2 = 1$. Moreover, the expression (13) of ξ becomes $\xi = \operatorname{sgn}(q)\nu_2$. Thus, item b) is proved.

When $r = 2$, by using Theorem 3.5 ii), we obtain that $\eta_0 = -\frac{1}{q}$ and $\kappa_1^2 = 1 + q^2$, for all $q \neq 0$, hence there exists $\theta \in \mathbb{R}\setminus\{0\}$ such that $\eta_0 = \sinh\theta = -\frac{1}{q}$, or equivalently $\theta = \operatorname{argsh}\left(-\frac{1}{q}\right)$.

Conversely, since any curve γ described at a) and c), is a curve from items ii) and iii) of Theorem 3.5, respectively, and any curve from b) is obtained from c) by taking $\theta = 0$, it follows by Theorem 3.5, that γ is a normal paracontact magnetic curve on the manifold. □

Proposition 3.11. *A φ-helix γ of order 3 on a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ has constant paracontact angle if and only if $\tau_{02} = 0$. In this case we have the following properties:*

i) All $\varphi-$torsions are constant and $\tau_{01} \neq 0$;

ii) The following relations hold good:

$$g(\nu_2, \xi) = \varepsilon_0\varepsilon_1(\kappa_2\tau_{01} - \kappa_1\tau_{12}) \text{ and } \tau_{12}^2 - \varepsilon_1\tau_{01}^2 = \varepsilon_0; \tag{25}$$

iii) γ is a paracontact magnetic curve of magnetism $q = -\frac{\kappa_1}{\tau_{01}}$, having constant paracontact angle given by $g(\xi, \dot{\gamma}) = \varepsilon_0 \frac{\tau_{01}\tau_{12}}{\kappa_1\tau_{12} - \kappa_2\tau_{01}}$;

iv) If $\tau_{12} = 0$, then γ is space-like (respectively time-like) Legendre $\varphi-$curve given in Corollary 3.9 c) (or respectively in Corollary 3.10 b)).

Proof: The Frenet equations for a $\varphi-$helix γ of osculating order 3 on a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ are

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma} &= \varepsilon_1\kappa_1\nu_1, \\ \nabla_{\dot{\gamma}}\nu_1 &= -\varepsilon_0\kappa_1\dot{\gamma} + \varepsilon_2\kappa_2\nu_2, \\ \nabla_{\dot{\gamma}}\nu_2 &= -\varepsilon_1\kappa_2\nu_1. \end{aligned} \tag{26}$$

The derivatives of the $\varphi-$torsions of γ with respect to the pseudo-arc length parameter s are

$$\begin{aligned} \frac{d}{ds}\tau_{01} &= \frac{d}{ds}g(\dot{\gamma}, \varphi\nu_1) = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\nu_1) + g(\dot{\gamma}, (\nabla_{\dot{\gamma}}\varphi)\nu_1) + g(\dot{\gamma}, \varphi\nabla_{\dot{\gamma}}\nu_1), \\ \frac{d}{ds}\tau_{02} &= \frac{d}{ds}g(\dot{\gamma}, \varphi\nu_2) = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\nu_2) + g(\dot{\gamma}, (\nabla_{\dot{\gamma}}\varphi)\nu_2) + g(\dot{\gamma}, \varphi\nabla_{\dot{\gamma}}\nu_2), \\ \frac{d}{ds}\tau_{12} &= \frac{d}{ds}g(\nu_1, \varphi\nu_2) = g(\nabla_{\dot{\gamma}}\nu_1, \varphi\nu_2) + g(\nu_1, (\nabla_{\dot{\gamma}}\varphi)\nu_2) + g(\nu_1, \varphi\nabla_{\dot{\gamma}}\nu_2). \end{aligned}$$

By using (26) and the metricity condition (4), the three above derivatives reduce respectively to

$$\frac{d}{ds}\tau_{01} = \varepsilon_0\eta(\nu_1) + \varepsilon_2\kappa_2\tau_{02}, \tag{27}$$

$$\frac{d}{ds}\tau_{02} = \varepsilon_1\kappa_1\tau_{12} + \varepsilon_0\eta(\nu_2) - \varepsilon_1\kappa_2\tau_{01}, \tag{28}$$

$$\frac{d}{ds}\tau_{12} = -\varepsilon_0\kappa_1\tau_{02}. \tag{29}$$

From (26) and (5) we obtain the following relations:

$$\frac{d}{ds}g(\dot{\gamma}, \xi) = \varepsilon_1\kappa_1 g(\nu_1, \xi), \tag{30}$$

$$\frac{d}{ds}g(\nu_1, \xi) = -\varepsilon_0\kappa_1 g(\dot{\gamma}, \xi) + \varepsilon_2\kappa_2 g(\nu_2, \xi) + \tau_{01},$$

$$\frac{d}{ds}g(\nu_2, \xi) = -\varepsilon_1\kappa_2 g(\nu_1, \xi) + \tau_{02}.$$

With respect to the Frenet frame, the Reeb vector field ξ decomposes as follows:

$$\xi = \varepsilon_0 g(\dot{\gamma}, \xi)\dot{\gamma} + \varepsilon_1 g(\nu_1, \xi)\nu_1 + \varepsilon_2 g(\nu_2, \xi)\nu_2,$$

and since $\varphi\xi = 0$ we have that

$$\varphi\xi = \varepsilon_0 g(\dot{\gamma}, \xi)\varphi\dot{\gamma} + \varepsilon_1 g(\nu_1, \xi)\varphi\nu_1 + \varepsilon_2 g(\nu_2, \xi)\varphi\nu_2 = 0.$$

Then, from the above relation and $g(\dot{\gamma}, \varphi\xi) = g(\nu_1, \varphi\xi) = g(\nu_2, \varphi\xi) = 0$, we obtain the following equalities:

$$\varepsilon_1 g(\nu_1, \xi)\tau_{01} + \varepsilon_2 g(\nu_2, \xi)\tau_{02} = 0, \tag{31}$$

$$-\varepsilon_0 g(\dot{\gamma}, \xi)\tau_{01} + \varepsilon_2 g(\nu_2, \xi)\tau_{12} = 0, \tag{32}$$

$$-\varepsilon_0 g(\dot{\gamma}, \xi)\tau_{02} - \varepsilon_1 g(\nu_1, \xi)\tau_{12} = 0. \tag{33}$$

We suppose that γ has constant paracontact angle θ and the curvatures $\kappa_1,\ \kappa_2$ are nonzero constants. It follows that $g(\dot{\gamma}, \xi) = constant$, and from (30) we have that $g(\nu_1, \xi) = 0$, which together with (33) yield either i) $\tau_{02} = 0$, or ii) $g(\dot{\gamma}, \xi) = 0$. In Case i) the relation (27) (respectively (29)) leads to $\tau_{01} = constant$ (respectively to $\tau_{12} = constant$). In Case ii), since $g(\dot{\gamma}, \xi) = g(\nu_1, \xi) = 0$, it follows that ξ is collinear to ν_2, hence $g(\nu_2, \xi) \neq 0$. Then from (31) (respectively (32)) one obtains $\tau_{02} = 0$ (respectively $\tau_{12} = 0$), i.e. ii) is a particular case of i).

Conversely, if $\tau_{02} = 0$, the relations (31) and (33) lead respectively to

$$g(\nu_1, \xi)\tau_{01} = 0,\ g(\nu_1, \xi)\tau_{12} = 0,$$

from which we have either Case I) $g(\nu_1, \xi) \neq 0$ and $\tau_{01} = \tau_{12} = 0$, or Case II) $g(\nu_1, \xi) = 0$. Since in Case I) we obtain that all $\varphi-$torsions vanish, which is a contradiction, we conclude that Case II) is the only possible case. Consequently, $g(\nu_1, \xi) = 0$, and then from (30) it follows that $\frac{d}{ds}g(\dot{\gamma}, \xi) = 0$, i.e. the paracontact angle θ is constant.

Next, if the paracontact angle is constant, or equivalently $\tau_{02} = 0$, the relation (28) leads to

$$g(\nu_2, \xi) = \varepsilon_0\varepsilon_1(\kappa_2\tau_{01} - \kappa_1\tau_{12}), \tag{34}$$

i.e. the first relation in (25) is proved.

In this case, as proven before, $g(\nu_1, \xi) = 0$, hence the expression of ξ reduces to

$$\xi = \varepsilon_0 g(\dot{\gamma}, \xi)\dot{\gamma} + \varepsilon_2 g(\nu_2, \xi)\nu_2,$$

and then

$$\varphi\xi = \varepsilon_0 g(\dot{\gamma}, \xi)\varphi\dot{\gamma} + \varepsilon_2 g(\nu_2, \xi)\varphi\nu_2 = 0. \tag{35}$$

Moreover, since $\varphi\dot{\gamma}$ and ν_1 are both orthogonal to $\dot{\gamma}$, it follows that $\varphi\dot{\gamma}$ is collinear to ν_1 and one has

$$\varphi\dot{\gamma} = -\tau_{01}\nu_1, \tag{36}$$

hence $\tau_{01} \neq 0$. Similarly, we can write

$$\varphi\nu_2 = \tau_{12}\nu_1 \tag{37}$$

and then from (35) and (34) we obtain

$$g(\dot{\gamma}, \xi) = \varepsilon_0(\kappa_1\tau_{12} - \kappa_2\tau_{01})\tau_{12}/\tau_{01}. \tag{38}$$

By using (36), (37), and the metricity relation (2) we obtain

$$g(\dot{\gamma}, \xi)^2 = \varepsilon_0 + \varepsilon_1\tau_{01}^2, \tag{39}$$

$$g(\xi, \nu_2)^2 = \varepsilon_2 + \varepsilon_1\tau_{12}^2, \tag{40}$$

$$g(\varphi\dot{\gamma}, \varphi\nu_2) = -\varepsilon_1\tau_{01}\tau_{12} = g(\dot{\gamma}, \xi)g(\xi, \nu_2).$$

Then, from (40), (34) and (38) we have the following relation involving the $\varphi-$torsions and the curvatures of γ:

$$(\kappa_2\tau_{01} - \kappa_1\tau_{12})^2\tau_{12}/\tau_{01} = \tau_{01}\tau_{12}. \tag{41}$$

Moreover, (38) and (39) yield

$$(\kappa_2\tau_{01} - \kappa_1\tau_{12})^2\tau_{12}^2/(\tau_{01}^2) = \varepsilon_0 + \varepsilon_1\tau_{01}^2. \tag{42}$$

By replacing (41) into (42) we obtain

$$\tau_{12}^2 = \varepsilon_0 + \varepsilon_1\tau_{01}^2, \tag{43}$$

and thus the second relation in (25) is proved.

If $\tau_{12} \neq 0$, since $\tau_{01} \neq 0$, then (41) and (38) become respectively

$$\frac{\kappa_1\tau_{12} - \kappa_2\tau_{01}}{\tau_{01}} = \frac{\tau_{01}}{\kappa_1\tau_{12} - \kappa_2\tau_{01}},$$

and

$$g(\xi, \dot{\gamma}) = \varepsilon_0\frac{\tau_{01}\tau_{12}}{\kappa_1\tau_{12} - \kappa_2\tau_{01}},$$

which prove the first relation in (25).

When $\tau_{12} = 0$, (43) yields $\tau_{01}^2 = -\varepsilon_0\varepsilon_1$, and then from (39) it follows that $g(\dot{\gamma}, \xi) = 0$, i.e. γ is a Legendre $\varphi-$helix and (40) becomes $g(\xi, \nu_2)^2 = \varepsilon_2$. This relation holds only when $\varepsilon_2 = 1$, more precisely when $\varepsilon_0 = -\varepsilon_1$. Moreover, the relations $\tau_{12} = 0$, (34) and (40) yield $\varepsilon_2 = \kappa_2^2\tau_{01}^2$, and since $\tau_{01}^2 = \varepsilon_2$, it follows that $\kappa_2 = 1$, and the last part of the proposition is proved. □

Remark 3.12. *The space-like and time-like Frenet magnetic curves on para-Sasakian manifolds of dimension $2n + 1$, characterized in the corollaries 3.7, 3.9, 3.10, yield (for n=1) the helices obtained in [14] on $\beta-$para-Sasakian manifolds with $\beta = -1$.*

4. Examples of magnetic curves associated to paracontact magnetic fields

We construct first a para-Sasaki structure (φ, ξ, η, g) on $\mathbb{R}^{2n+1}$ and then we provide the magnetic curves corresponding to the paracontact magnetic field on $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$, which for $n = 1$ can be particularized to those obtained in [14] on the hyperbolical Heisenberg group H_h^3.

We denote by $(x^i, y^i, z)_{i\in\{1,\ldots,n\}}$ the canonical coordinates of $\mathbb{R}^{2n+1}$, on which we consider a vector field

$$\xi = 2\frac{\partial}{\partial z},$$

and a 1-form η, defined by

$$\eta = \frac{1}{2}dz + \sum_{h=1}^{n}(x^h dy^h - y^h dx^h).$$

As the distribution $\operatorname{Ker}\eta$ is parallelizable, let $\{X_i, Y_i\}_{i\in\{1,\ldots,n\}}$ be a basis of vector fields, which gives a parallelization of $\operatorname{Ker}\eta$. If we take a $(1, 1)-$tensor field φ on $\mathbb{R}^{2n+1}$, such that

$$\varphi X_i = Y_i, \quad \varphi Y_i = X_i, \quad \varphi\xi = 0, \quad i \in \{1, \ldots, n\}, \tag{44}$$

then the structure (φ, ξ, η) turns out to be an almost paracontact structure on $\mathbb{R}^{2n+1}$.

From now on, we take, in particular,

$$X_i = \frac{\partial}{\partial x^i} + 2y^i\frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y^i} - 2x^i\frac{\partial}{\partial z}, \ i \in \{1, \ldots, n\}, \tag{45}$$

which have the Lie brackets

$$[X_i, X_j] = [Y_i, Y_j] = 0, \quad [X_i, Y_j] = -2\delta_{ij}\xi, \quad [X_i, \xi] = [Y_i, \xi] = 0, \quad i, j \in \{1, \dots, n\},$$

where δ_{ij} is the Kronecker delta.

The structure (φ, ξ, η, g) on $\mathbb{R}^{2n+1}$ is almost paracontact metric, where

$$g = \eta \otimes \eta + \sum_{h=1}^{n} \left((dx^h)^2 - (dy^h)^2\right). \tag{46}$$

Moreover, since $d\eta = g(\cdot, \varphi)$, i.e. $d\eta = \Omega$, it follows that this structure is a paracontact metric one on $\mathbb{R}^{2n+1}$.

By appling the Koszul formula to the metric g, given by (46), we obtain the components of the Levi-Civita connection with respect to the φ−basis $\{X_i, Y_i, \xi\}_{i\in\{1,\dots,n\}}$:

$$\begin{aligned} \nabla_{X_i} X_j = \nabla_{Y_i} Y_j = \nabla_\xi \xi = 0, \quad \nabla_{Y_i} X_j = -\nabla_{X_i} Y_j = \delta_{ij}\xi \\ \nabla_{X_i}\xi = \nabla_\xi X_i = -Y_i, \quad \nabla_{Y_i}\xi = \nabla_\xi Y_i = -X_i, \quad i, j \in \{1, \dots, n\}. \end{aligned} \tag{47}$$

The relations (44)-(47) imply (4), which based on Proposition 3.3 iii) shows that $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is a para-Sasakian manifold.

On this para-Sasakian manifold $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$, we classify in the sequel the magnetic curves corresponding to the paracontact magnetic field F_q, and we give their explicit parameterizations.

Theorem 4.1. *A smooth curve γ on the para-Sasakian manifold $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$, parameterized by the pseudo-arc length s, is a normal magnetic curve associated to the paracontact magnetic field F_q if and only if γ is one of the following curves:*

a) a straight line parameterized as

$$\gamma(s) = \left(c_1^i s + x_0^i,\ c_2^i s + y_0^i,\ 2\left(\eta_0 + \sum_{h=1}^{n}(c_1^h y_0^h - c_2^h x_0^h)\right)s + z_0\right)_{i\in\{1,\dots,n\}},$$

where c_1^i, $c_2^i \in \mathbb{R}$ and $x^i(0) = x_0^i \in \mathbb{R}$, $y^i(0) = y_0^i \in \mathbb{R}$, and $g(\xi, \dot\gamma) = \eta_0 \in \mathbb{R}$.

b) a φ-helix with axis ξ parameterized as

$$\begin{aligned} \gamma(s) = \Bigg(& \frac{\alpha_1^i}{2\eta_0 - q}\sinh[(2\eta_0 - q)s] + \frac{\alpha_2^i}{2\eta_0 - q}\cosh[(2\eta_0 - q)s] + x_0^i, \\ & \frac{\alpha_1^i}{2\eta_0 - q}\cosh[(2\eta_0 - q)s] + \frac{\alpha_2^i}{2\eta_0 - q}\sinh[(2\eta_0 - q)s] + y_0^i, \\ & 2\left(\eta_0 + \frac{\sum_{h=1}^{n}\left((\alpha_1^h)^2 - (\alpha_2^h)^2\right)}{2\eta_0 - q}\right)s + \\ & + \frac{2}{2\eta_0 - q}\sum_{h=1}^{n}(\alpha_1^h y_0^h - \alpha_2^h x_0^h)\sinh[(2\eta_0 - q)s] + \\ & + \frac{2}{2\eta_0 - q}\sum_{h=1}^{n}(\alpha_2^h y_0^h - \alpha_1^h x_0^h)\cosh[(2\eta_0 - q)s] + z_0\Bigg)_{i\in\{1,\dots,n\}}, \end{aligned} \tag{48}$$

where α_1^i, $\alpha_2^i \in \mathbb{R}$ and $x^i(0) = x_0^i \in \mathbb{R}$, $y^i(0) = y_0^i$, $z(0) = z_0 \in \mathbb{R}$ and $g(\xi, \dot\gamma) = \eta_0 \in \mathbb{R}$.

c) a φ-circle of constant paracontact angle, whose equation is (48)*, in which $\eta_0 = \frac{\varepsilon_0}{q}$.*

Proof: Let γ be a curve parameterized by pseudo-arc length, given by

$$\gamma(s) = (x^i(s), y^i(s), z(s))_{i\in\{1,\dots,n\}}.$$

Let us write the speed $\dot\gamma$ with respect to the φ-basis $\{X_i, Y_i, \xi\}_{i\in\{1,\dots,n\}}$ in the form

$$\dot\gamma(s) = \alpha^i(s)X_i + \beta^i(s)Y_i + c(s)\xi,\ i \in \{1, \dots, n\}, \tag{49}$$

where $\alpha^i(s)$, $\beta^i(s)$ and $c(s) = \eta(\dot\gamma(s))$ are smooth functions.

On the other hand, from $\dot\gamma = \dot x^i \frac{\partial}{\partial x^i} + \dot y^i \frac{\partial}{\partial y^i} + \dot z \frac{\partial}{\partial z}$ and (45), we obtain

$$\dot\gamma = \dot x^i X_i + \dot y^i Y_i + \left(2\sum_{i=1}^n x^i \dot y^i - 2\sum_{i=1}^n \dot x^i y^i + \dot z\right)\frac{\partial}{\partial z},$$

which, compared with (49), yields

$$\dot x^i = \alpha^i,\ \dot y^i = \beta^i,\ \dot z = 2\Big(c - \sum_{i=1}^n x^i \dot y^i + \sum_{i=1}^n \dot x^i y^i\Big). \tag{50}$$

The curve γ is a magnetic curve associated to the paracontact magnetic field $F_q = q\Omega = -qg(\varphi X, Y)$ if it satisfies the Lorentz equation (12). By replacing (49) and using (47), then (12) becomes

$$(\dot\alpha^i - 2c\beta^i)X_i + (\dot\beta^i - 2c\alpha^i)Y_i + \dot c\xi = -q\beta^i X_i - q\alpha^i Y_i,$$

which is equivalent to the following ordinary differential equations system:

$$\begin{cases} \dot\alpha^i = (2c - q)\beta^i \\ \dot\beta^i = (2c - q)\alpha^i \\ \dot c = 0. \end{cases} \tag{51}$$

Since the third above equation has the solution $c(s) = const \in \mathbb{R}$, i.e. $\eta(\dot\gamma) = const \in \mathbb{R}$, we use the notation $c(s) = \eta_0 \in \mathbb{R}$.

Now we distinguish two cases which lead to the equations in the statement.

Case I) When $q = 2\eta_0$, the system (51) has the solution

$$\alpha^i(s) = c_1^i = const \in \mathbb{R},\ \beta^i(s) = c_2^i = const \in \mathbb{R},\ c = \eta_0 \in \mathbb{R}. \tag{52}$$

Next, from (50) and (52), we obtain that

$$x^i = c_1^i s + x_0^i, \quad y^i = c_2^i s + y_0^i, \quad z = 2\left(\eta_0 + \sum_{h=1}^n (c_1^h y_0^h - c_2^h x_0^h)\right)s + z_0,$$

i.e. in this case the magnetic curve is a straight line with the equation from *a*).

Since $\varepsilon_0 = g(\dot\gamma, \dot\gamma) = \eta_0^2 + \sum_{h=1}^n \left((c_1^h)^2 - (c_2^h)^2\right)$ and the curve is parametrized by pseudo-arc length, then the line is either space-like (respectively time-like) according as $\varepsilon_0 = 1$ (respectively $\varepsilon_0 = -1$), or light-like when $\eta_0^2 = \sum_{h=1}^n \left((c_2^h)^2 - (c_1^h)^2\right)$.

Case II) When $q \neq 2\eta_0$, the system (51) is equivalent to

$$\begin{cases} \ddot\alpha^i(s) = (2\eta_0 - q)^2\alpha^i \\ \dot\beta^i(s) = (2\eta_0 - q)\alpha^i \\ c(s) = \eta_0 = const \in \mathbb{R}, \end{cases}$$

which has the solution

$$\begin{cases} \alpha^i(s) = \alpha_1^i \cosh[(2\eta_0 - q)s] + \alpha_2^i \sinh[(2\eta_0 - q)s], \\ \beta^i(s) = \alpha_1^i \sinh[(2\eta_0 - q)s] + \alpha_2^i \cosh[(2\eta_0 - q)s], \\ c(s) = \eta_0 = const \in \mathbb{R}, \end{cases} \tag{53}$$

where $\alpha_1^i,\ \alpha_2^i \in \mathbb{R}$ and $x^i(0) = x_0^i \in \mathbb{R},\ y^i(0) = y_0^i, \in \mathbb{R}$, for all $i \in \{1, \dots, n\}$ and $g(\xi, \dot{\gamma}) = \eta_0 \in \mathbb{R}$.

By replacing (53) into (50) and integrating, we obtain the following equations for the coordinates of the points of the magnetic curve:

$$\begin{cases} x^i(s) = \frac{\alpha_1^i}{2\eta_0 - q} \sinh[(2\eta_0 - q)s] + \frac{\alpha_2^i}{2\eta_0 - q} \cosh[(2\eta_0 - q)s] + x_0^i, \\ y^i(s) = \frac{\alpha_1^i}{2\eta_0 - q} \cosh[(2\eta_0 - q)s] + \frac{\alpha_2^i}{2\eta_0 - q} \sinh[(2\eta_0 - q)s] + y_0^i, \\ z(s) = 2\left(\eta_0 + \frac{\sum\limits_{h=1}^{n}\left((\alpha_1^h)^2 - (\alpha_2^h)^2\right)}{2\eta_0 - q}\right)s \\ \qquad + \frac{2}{2\eta_0 - q} \sum\limits_{h=1}^{n} (\alpha_1^h y_0^h - \alpha_2^h x_0^h) \sinh[(2\eta_0 - q)s] \\ \qquad + \frac{2}{2\eta_0 - q} \sum\limits_{h=1}^{n} (\alpha_2^h y_0^h - \alpha_1^h x_0^h) \cosh[(2\eta_0 - q)s] + z_0, \end{cases} \tag{54}$$

where $\alpha_1^i,\ \alpha_2^i \in \mathbb{R}$ and $x^i(0) = x_0^i \in \mathbb{R},\ y^i(0) = y_0^i,\ z(0) = z_0 \in \mathbb{R}$, for all $i \in \{1, \dots, n\}$.

Subsequently, in Case II) the magnetic curve is the helix given at item *b*) of the Theorem. Since the curve is parametrized by pseudo-arc length and $\varepsilon_0 = g(\dot{\gamma}, \dot{\gamma}) = \eta_0^2 + \sum_{h=1}^n \left((\alpha_1^h)^2 - (\alpha_2^h)^2\right)$, then the helix is either space-like (respectively time-like) accordingly as $\varepsilon_0 = 1$ (respectively $\varepsilon_0 = -1$), or light-like when $\eta_0^2 = \sum_{h=1}^n \left((\alpha_2^h)^2 - (\alpha_1^h)^2\right)$.

If in (54) we take $\eta_0 = \frac{\varepsilon_0}{q}$, we obtain, according to Theorem 3.5 ii), that the magnetic curve is that given at item c).

Conversely, one can easily verify that the curves from items *a*), *b*) and *c*) of the Theorem satisfy the Lorentz equation (12), where φ is the (1,1) tensor field on $\mathbb{R}^{2n+1}$ given by (44). □

Remark 4.2. *The normal paracontact magnetic curves on the para-Sasakian manifold* $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$*, described in Theorem 4.1, at items a), c), b), are respectively of the types i), ii), iii) of Theorem 3.5. The curvatures of these three classes of curves have the constant values mentioned in Theorem 3.5 at each item.*

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