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Characterisations of the radical in a Banach algebra

Heinrich Raubenheimer[a], **Tshikhudo Lukoto**[b]

[a]*University of Johannesburg, South Africa*
[b]*University of Limpopo, South Africa*

Abstract. In this paper we characterise the radical in a Banach algebra $\mathcal{A}$ in terms of the perturbation ideal $\mathcal{P}(R)$ of a set R in $\mathcal{A}$.

1. Preliminaries

In Zemánek's proof of his well known characterisation of the radical in a Banach algebra [14–16], an important tool was the use of Jacobson's Density Theorem. Since Zemánek's characterisation of the radical in 1977, many authors [2–4] proved spectral characterisations of the radical by using analytic techniques, avoiding representation theory altogether. In 1971 Lebow and Schechter, [7], characterised the radical in a Banach algebra by proving that it is equal to the perturbation ideal of the group of invertible elements. Their proof is short and it is an algebraic proof. In 2008 G.R. Allan, [1], characterized the radical by using elementary arguments from complex analysis.
We are going to follow the same approach as Lebow and Schechter [7], and in section 4 characterise the radical in terms of the perturbation ideal of lower semiregularities. In section 5 we characterise the radical in terms of the perturbation ideal of upper semiregularities. Finally, in the last section of the paper, we characterise the radical in terms of the perturbation ideal of sets which are neither lower nor upper semiregularities. In sections 1, 2 and 3 we establish our notation and basic results which are relevant to our proofs.

Let $\mathcal{A}$ be a complex Banach algebra with a unit element $1_{\mathcal{A}}$ and for any $\lambda \in \mathbb{C} \setminus \{0\}$, simply write λ for $\lambda \cdot 1_{\mathcal{A}}$. We will denote by $\mathcal{A}^{-1}$ the group of all invertible elements in $\mathcal{A}$ while $\mathcal{A}_l^{-1}$ ($\mathcal{A}_r^{-1}$) represents the set of all left (right) invertible elements in $\mathcal{A}$. Note that $\mathcal{A}^{-1} = \mathcal{A}_l^{-1} \cap \mathcal{A}_r^{-1}$. The set

$$\sigma(x) = \sigma(x, \mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda - x \notin \mathcal{A}^{-1}\},$$

is the usual spectrum of $x \in \mathcal{A}$. It is well known that $\sigma(x)$ is non-empty and a compact subset of the complex plane $\mathbb{C}$. The *spectral radius of x in $\mathcal{A}$* is defined by

$$r(x) = r(x, \mathcal{A}) = \sup\{|\lambda| : \lambda \in \sigma(x, \mathcal{A})\}.$$

If $\sigma(a) = \{0\}$, then $a \in \mathcal{A}$ is called a *quasinilpotent element* and the set of all quasinilpotent elements in the Banach algebra $\mathcal{A}$ is denoted by $QN(\mathcal{A})$. If K is a subset of topological space, then ∂K denotes the boundary

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Email addresses: heinrichr@uj.ac.za (Heinrich Raubenheimer), tshikhudo.lukoto@ul.ac.za (Tshikhudo Lukoto)

of K. By *ideal* in $\mathcal{A}$ we mean a two-sided ideal. An ideal J is *proper* if $J \subsetneq \mathcal{A}$. A *maximal left (right) ideal* is a proper left (right) ideal which is not contained in any proper left (right) ideal. The *radical* of $\mathcal{A}$, denoted by $\mathrm{Rad}(\mathcal{A})$, is the intersection of all maximal ideals of $\mathcal{A}$. Hence $\mathrm{Rad}(\mathcal{A})$ is a two-sided ideal. It can be shown that

$$\mathrm{Rad}(\mathcal{A}) = \{x \in \mathcal{A} : r(xz) = 0 \text{ for all } z \in \mathcal{A}\}. \tag{1}$$

It follows from (1) that the radical is contained in the set of quasinilpotent elements.
Our next result is the famous Zemánek's characterisation of the radical in a Banach algebra. This result plays a central role in this paper.

Theorem 1.1. ([3], Theorem 5.3.1) Let $\mathcal{A}$ be a Banach algebra. Then the following properties are equivalent:

(i) $a \in \mathrm{Rad}(\mathcal{A})$,

(ii) $\sigma(a + x) = \sigma(x)$, for all $x \in \mathcal{A}$,

(iii) $r(a + x) = 0$, for all quasinilpotent elements $x \in \mathcal{A}$,

(iv) $r(a + x) = 0$, for all quasinilpotent elements x in a neighbourhood of 0 in $\mathcal{A}$,

(v) there exists $C > 0$ such that $r(x) \leq C\|x - a\|$, for all x in a neighbourhood of $a \in \mathcal{A}$.

Many of the new characterisations of the radical that we provide in Section 3 follow from the equivalence (i)⇔(ii) or the equivalence (i)⇔(iii) in Theorem 1.1.

An element $a \in \mathcal{A}$ is called a *left topological divisor of zero* if

$$\inf\{\|ax\| : x \in \mathcal{A},\ \|x\| = 1\} = 0.$$

Similarly, $a \in \mathcal{A}$ is a *right topological divisor of zero* if

$$\inf\{\|xa\| : x \in \mathcal{A},\ \|x\| = 1\} = 0.$$

We define $\exp\mathcal{A}$ as the set

$$\exp\mathcal{A} = \{e^x : x \in \mathcal{A}\},$$

and we let $\mathrm{Exp}\mathcal{A}$ be the set

$$\mathrm{Exp}\mathcal{A} = \{e^{a_1}e^{a_2}\cdots e^{a_n} : n \in \mathbb{N}, a_1, a_2, \ldots, a_n \in \mathcal{A}\}.$$

$\mathrm{Exp}\mathcal{A}$ is the component of the invertible elements containing $1_{\mathcal{A}}$ and it is an open and closed normal subgroup of $\mathcal{A}^{-1}$ generated by e^a for all $a \in \mathcal{A}$.
In the middle 1990's and the early 2000's Kordula, Mbekhta and Müller defined the notions of regularity and semiregularity in Banach algebras, [6] and [11, sections 6 and 23], to facilitate an axiomatic approach to spectral theory in Banach algebras. Since we are going to employ these notions in sections 3, 4 and 5, we define them here.

Definition 1.2. ([11], Definition 6.1) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a *regularity* if it satisfies the following conditions:

(i) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \Leftrightarrow a^n \in R$;

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R.$$

In many cases it is possible to verify the axioms of a regularity by using the following criterion:

Theorem 1.3. ([11], Theorem 6.4) Let R be a non-empty subset of a Banach algebra $\mathcal{A}$ satisfying

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R \tag{P1}$$

for all commuting elements $a, b \in \mathcal{A}$. Then R is a regularity.

One can divide the definition of a regularity into two parts:

Definition 1.4. ([11], Definition 23.1) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a *lower semiregularity* if

(i) $a \in \mathcal{A}, n \in \mathbb{N}, a^n \in R \Rightarrow a \in R$,

(ii) a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1_{\mathcal{A}}$, and $ab \in R$, then $a, b \in R$.

Remark 1.5. ([11], Remark 23.3) Let R be a non-empty subset of a Banach algebra $\mathcal{A}$ satisfying

$$a, b \in \mathcal{A}, ab = ba, ab \in R \Rightarrow a \in R \text{ and } b \in R. \tag{P1'}$$

Then clearly R is a lower semiregularity.

Definition 1.6. ([11], Definition 23.10) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called an *upper semiregularity* if

(i) $a \in R, n \in \mathbb{N} \Rightarrow a^n \in R$,

(ii) a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1_{\mathcal{A}}$, and $a, b \in R$, then $ab \in R$,

(iii) R contains a neighbourhood of the unit element $1_{\mathcal{A}}$.

Remark 1.7. A semigroup containing a neighborhood of the unit element of $\mathcal{A}$ is an upper semiregularity because it already satisfies conditions (i) and (ii) of Definition 1.6.

We can deduce that R is a regularity if and only if it is both a lower semiregularity and an upper semiregularity.

If $\mathcal{A}$ is a Banach algebra and $\mathcal{S} \subseteq \mathcal{A}$, then one can define in a natural way a spectrum relative to $\mathcal{S}$ for any $a \in \mathcal{A}$ by

$$\sigma_{\mathcal{S}}(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{S}\}.$$

If $\mathcal{S}$ is a regularity or a semiregularity, then $\sigma_{\mathcal{S}}(a)$ has interesting properties, see ([11], Theorem 6.7, Theorem 23.4 and Theorem 23.18).

2. Perturbation Classes

In this section, we mention results on perturbation of a set due to Lebow and Schechter ([7]). These results will be used in subsequent sections.

Definition 2.1. Let X be a complex Banach space, and let $\mathcal{S}$ be a subset of X. The *perturbation* of $\mathcal{S}$, denoted by $\mathcal{P}(\mathcal{S})$, is the set of all $x \in X$ such $x + s \in \mathcal{S}$ for all $s \in \mathcal{S}$, i.e.,

$$\mathcal{P}(\mathcal{S}) = \{x \in X : x + s \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}.$$

We say $\mathcal{P}(\mathcal{S})$ is the set of elements of X that perturb $\mathcal{S}$ into itself. We shall throughout assume

$$\alpha\mathcal{S} \subseteq \mathcal{S} \tag{2}$$

for each $\alpha \in \mathbb{C} \setminus \{0\}$. Although in general $\mathcal{P}(\mathcal{S})$ is not an ideal, we will call $\mathcal{P}(\mathcal{S})$ the *perturbation ideal* of $\mathcal{S}$.

Remark 2.2. Let $\mathcal{S}$ be a subset of a Banach space X. If $0 \in \mathcal{S}$, then $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$: Let $0 \in \mathcal{S}$. If $x \in \mathcal{P}(\mathcal{S})$, then $x + a \in \mathcal{S}$ for every $a \in \mathcal{S}$. Since $0 \in \mathcal{S}$, it then follows that $x \in \mathcal{S}$. Hence $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$.

Lemma 2.3. ([7], Lemma 2.1) Let X be a Banach space with $\mathcal{S} \subseteq X$. If $\mathcal{S}$ satisfies (2), then $\mathcal{P}(\mathcal{S})$ is a linear subspace of X. If, in addition, $\mathcal{S}$ is an open subset of X, then $\mathcal{P}(\mathcal{S})$ is closed.

Lemma 2.4. ([7], Lemma 2.2) Let X be a Banach space and let $\mathcal{S}_1$, $\mathcal{S}_2$ be subsets of X which satisfy (2). Assume that $\mathcal{S}_1$ is open, that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and that $\partial\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Then $\mathcal{P}(\mathcal{S}_2) \subseteq \mathcal{P}(\mathcal{S}_1)$.

Lemma 2.5. ([7], Lemma 2.3) Let $\mathcal{A}$ be a Banach algebra and $\mathcal{S} \subseteq \mathcal{A}$. If $\mathcal{A}^{-1}\mathcal{S} \subseteq \mathcal{S}$, then $\mathcal{P}(\mathcal{S})$ is a left ideal. If $\mathcal{S}\mathcal{A}^{-1} \subseteq \mathcal{S}$ then $\mathcal{P}(\mathcal{S})$ is a right ideal.

By the Lemmas above we have the following results.

Theorem 2.6. ([7], Theorem 2.4) Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{S}$ is an open subset of $\mathcal{A}$ which satisfies

$$\mathcal{A}^{-1}\mathcal{S} \subseteq \mathcal{S}, \quad \mathcal{S}\mathcal{A}^{-1} \subseteq \mathcal{S},$$

then $\mathcal{P}(\mathcal{S})$ is a closed two-sided ideal.

One of the reasons that we are interested in the perturbation ideal $\mathcal{P}(\mathcal{S})$ of a subset $\mathcal{S}$ of a Banach algebra $\mathcal{A}$ is the following:

Proposition 2.7. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{S} \subseteq \mathcal{A}$ satisfy $\alpha\mathcal{S} \subseteq \mathcal{S}$ for all $0 \neq \alpha \in \mathbb{C}$. Then $a \in \mathcal{P}(\mathcal{S})$ if and only if $\sigma_{\mathcal{S}}(x + a) = \sigma_{\mathcal{S}}(x)$ for all $x \in \mathcal{A}$.

Proof. Suppose $a \in \mathcal{P}(\mathcal{S})$ and $x \in \mathcal{A}$. If $\lambda \in \sigma_{\mathcal{S}}(x + a)$, then

$$\begin{aligned} \lambda - (x + a) &= (\lambda - x) - a \notin \mathcal{S} \\ &\Leftrightarrow \lambda - x \notin \mathcal{S} \text{ since } -a \in \mathcal{P}(\mathcal{S}), \text{ Lemma 2.3.} \\ &\Leftrightarrow \lambda \in \sigma_{\mathcal{S}}(x). \end{aligned}$$

Hence $\sigma_{\mathcal{S}}(x + a) = \sigma_{\mathcal{S}}(x)$. Conversely, suppose $\sigma_{\mathcal{S}}(x + a) = \sigma_{\mathcal{S}}(x)$ for all $x \in \mathcal{A}$. In particular, if $y \in \mathcal{S}$, then by (2), $-y = 0 - y \in \mathcal{S}$. In view of our assumption, $0 \notin \sigma_{\mathcal{S}}(y) = \sigma_{\mathcal{S}}(y + a)$. Hence, $0 - (y + a) = -(y + a) \in \mathcal{S}$. Again by (2), $y + a \in \mathcal{S}$. Since y was an arbitrary element of $\mathcal{S}$, it then follows that $a \in \mathcal{P}(\mathcal{S})$. □

Proposition 2.8. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{S} \subseteq \mathcal{A}$ satisfy $\alpha\mathcal{S} \subseteq \mathcal{S}$ for all $0 \neq \alpha \in \mathbb{C}$. Then $\sigma_{\mathcal{S}}(\alpha x) = \alpha\sigma_{\mathcal{S}}(x)$ for all $x \in \mathcal{A}$.

Proof. Let $x \in \mathcal{A}$ and $\lambda \in \sigma_{\mathcal{S}}(\alpha x)$. Then $\lambda - \alpha x \notin \mathcal{S}$. By (2), $\frac{\lambda}{\alpha} - x \notin \mathcal{S}$. This implies $\frac{\lambda}{\alpha} \in \sigma_{\mathcal{S}}(x)$ and so $\lambda \in \alpha\sigma_{\mathcal{S}}(x)$. We have shown that $\sigma_{\mathcal{S}}(\alpha x) \subseteq \alpha\sigma_{\mathcal{S}}(x)$. The inclusion $\alpha\sigma_{\mathcal{S}}(x) \subseteq \sigma_{\mathcal{S}}(\alpha x)$ follows similarly. □

Our next result gives information on the perturbation ideal of a set and its complement.

Proposition 2.9. Let $\mathcal{A}$ be a Banach algebra and suppose $R \subseteq \mathcal{A}$ and $\mathcal{A} \setminus R$ are closed under multiplication by nonzero scalars. Then

$$\mathcal{P}(R) = \mathcal{P}(\mathcal{A} \setminus R).$$

Proof. Let $a \in \mathcal{P}(R)$ and let $y \in \mathcal{A} \setminus R$. Since R is closed under scalar multiplication, it then follows by Lemma 2.3 that $\mathcal{P}(R)$ is a linear subspace and so $-a \in \mathcal{P}(R)$. If $a + y \in R$, then $-a + (a + y) = y \in R$, which is a contradiction. Hence $a + y \in \mathcal{A} \setminus R$, and since y is an arbitrary element in $\mathcal{A} \setminus R$, we get that $a \in \mathcal{P}(\mathcal{A} \setminus R)$. Hence $\mathcal{P}(R) \subseteq \mathcal{P}(\mathcal{A} \setminus R)$. The inclusion $\mathcal{P}(\mathcal{A} \setminus R) \subseteq \mathcal{P}(R)$ can be proved similarly. □

In Section 6 we will provide applications of the above Proposition.

3. Perturbation Ideals of Regularities

In this section, we are going to investigate the perturbation ideals of some well known regularities. Recall that if R is a regularity in a Banach algebra $\mathcal{A}$, then $A^{-1} \subseteq R$ ([11], Proposition 6.2 (ii)).

If $\mathcal{A}$ is a Banach algebra, then the following subsets of $\mathcal{A}$ are regularities because they satisfy condition (P1) in Theorem 1.3:

(i) $R_1 = \mathcal{A}$;

(ii) $R_2 = \mathcal{A}^{-1}$;

(iii) $R_3 = \mathcal{A}_l^{-1}$ and $R_4 = \mathcal{A}_r^{-1}$;

(iv) $R_5 = H_l$ the set of all elements in $\mathcal{A}$ that are not left topological divisors of zero and $R_6 = H_r$, the set of all elements in $\mathcal{A}$ that are not right topological divisors of zero.

If $\mathcal{A}$ is a Banach algebra and $R_1 = \mathcal{A}$, then $a + R_1 \subseteq R_1$ for all $a \in \mathcal{A}$. Hence, $\mathcal{P}(R_1) = \mathcal{A}$, i.e., $\mathcal{A}$ is the perturbation ideal of $\mathcal{A}$.

The equivalence (i)⇔(ii) in Theorem 1.1 together with Proposition 2.7 is

Theorem 3.1. ([7], Theorem 2.5) Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(\mathcal{A}^{-1}) = \mathrm{Rad}(\mathcal{A}).$$

The proof of Theorem 3.1 by Lebow and Schechter ([7]) is an algebraic proof. Their proof is short and neat. Later, Aupetit and Zemánek ([3], Theorem 5.3.1) proved a stronger result (See Proposition 2.7). However, the proof of Theorem 5.3.1 in [3] is more involved and the proof uses arguments of representation theory. It can be shown that the representation theory arguments can be replaced by arguments involving the theory of subharmonic functions.

Proposition 3.2. ([6], Proposition 1.3) Let R be a regularity in a Banach algebra $\mathcal{A}$. If $a, b \in \mathcal{A}$, $ab = ba$ and $a \in \mathcal{A}^{-1}$ then

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R.$$

Note that if R is a regularity in a Banach algebra $\mathcal{A}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, it then follows from Proposition 3.2 that $\alpha R \subseteq R$.
We are now ready to prove one of the main results in this section.

Theorem 3.3. Let $\mathcal{A}$ be a Banach algebra and R a regularity with $\partial\mathcal{A}^{-1} \cap R = \emptyset$. Then

$$\mathcal{P}(R) \subseteq \mathrm{Rad}(\mathcal{A}).$$

Proof. First we note that $\mathcal{A}^{-1}$ is an open subset of $\mathcal{A}$ with $\alpha\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}$ for all nonzero scalars α. In view of $\mathcal{A}^{-1} \subseteq R$ and $\alpha R \subseteq R$ for all nonzero scalars α, our assumption $\partial\mathcal{A}^{-1} \cap R = \emptyset$ together with Lemma 2.4 and Theorem 3.1 gives $\mathcal{P}(R) \subseteq \mathcal{P}(\mathcal{A}^{-1}) = \mathrm{Rad}(\mathcal{A})$. □

We now provide applications of Theorem 3.3.

Proposition 3.4. Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(R_i) \subseteq \mathrm{Rad}(\mathcal{A})$$

for $i = 3, 4, 5, 6$.

Proof. Since the regularities R_i ($i = 3, 4, 5, 6$) satisfy $\partial\mathcal{A}^{-1} \cap R_i = \emptyset$, see ([11], Theorem 1.14), it follows from Theorem 3.3 that $\mathcal{P}(R_i) \subseteq \mathrm{Rad}(\mathcal{A})$. □

For the regularities R_3 and R_4, we have

Theorem 3.5. ([7], Theorem 2.6) Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(R_i) = \mathrm{Rad}(\mathcal{A})$$

for $i = 3, 4$.

4. Perturbation Ideals of Lower Semiregularities

In this section we investigate perturbation ideals of lower semiregularities.

If R is a lower semiregularity in a Banach algebra $\mathcal{A}$, then it is well known that $\mathcal{A}^{-1} \subseteq R$ ([11], Lemma 23.2). If one can prove that $\partial\mathcal{A}^{-1} \cap R = \emptyset$, then one can employ Lemma 2.4 to conclude that $\mathcal{P}(R) \subseteq \mathcal{P}(\mathcal{A}^{-1}) = \mathrm{Rad}(\mathcal{A})$. This is the case for our first result: If $R = R_7 = \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$, then R is a lower semiregularity because it satisfies the (P1′) condition.

Theorem 4.1. Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \mathrm{Rad}(\mathcal{A}).$$

Proof. We first note that $\mathcal{A}^{-1}$ is open in $\mathcal{A}$ and $\mathcal{A}^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. Now we show that $\partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \emptyset$. Assume $\partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) \neq \emptyset$. So there exists $a \in \partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1})$ and hence $a \in \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. Suppose $a \in \mathcal{A}_l^{-1}$, then by ([11], Theorem 1.14 (i)), a is not a left topological divisor of zero. Similarly if $a \in \mathcal{A}_r^{-1}$, then a is not a right topological divisor of zero. Since $a \in \partial\mathcal{A}^{-1}$, by ([11], Theorem 1.14 (iv)), a is both a left and a right topological divisor of zero, which is a contradiction. So $\partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \emptyset$. Since both $\mathcal{A}^{-1}$ and $\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$ satisfy (2), it follows from Lemma 2.4 that $\mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) \subseteq \mathcal{P}(\mathcal{A}^{-1})$. To prove the inclusion $\mathcal{P}(\mathcal{A}^{-1}) \subseteq \mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1})$, let $x \in \mathcal{P}(\mathcal{A}^{-1})$ and $a \in \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. If $a \in \mathcal{A}_l^{-1}$, then by Theorem 3.5, $x + a \in \mathcal{A}_l^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. Similarly, if $a \in \mathcal{A}_r^{-1}$, then $x + a \in \mathcal{A}_r^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. If we combine our arguments, we get $\mathcal{P}(\mathcal{A}^{-1}) \subseteq \mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1})$, and hence $\mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \mathcal{P}(\mathcal{A}^{-1})$=$\mathrm{Rad}(\mathcal{A})$. □

Let $\mathcal{A}$ be a Banach algebra. Define the set

$$R = R_8 = \mathcal{A} \setminus QN(\mathcal{A}) = \{x \in \mathcal{A} : r(x) > 0\}.$$

Proposition 4.2. Let $\mathcal{A}$ be a Banach algebra and $R = \mathcal{A} \setminus QN(\mathcal{A})$. Then R is a lower semiregularity.

Proof. To prove that R is a lower semiregularity, we are going to verify condition (P1′), see Remark 1.5. Let $a, b \in \mathcal{A}$, with $ab = ba$ and suppose $ab \in R$. Then

$$0 < r(ab) \leq r(a)r(b).$$

This implies that $r(a) > 0$ and $r(b) > 0$. Hence, a, $b \in R$, i.e., R is a lower semiregularity. □

We have proved above that R_8 is a lower semiregularity. In view of ([11], Lemma 23.2) this means that $\mathcal{A}^{-1} \subseteq R_8$. However, for the semiregularity R_8 one can prove directly that $\mathcal{A}^{-1} \subseteq R_8$: If $a \in \mathcal{A}^{-1}$, then $0 \notin \sigma(a)$. Since $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$, it follows that $r(a) > 0$.

Remark 4.3. Let $\mathcal{A}$ be a Banach algebra and let $R = \mathcal{A} \setminus QN(\mathcal{A})$. Then

$$\alpha R \subseteq R \text{ for every } \alpha \in \mathbb{C} \setminus \{0\}.$$

Proof. Let $\alpha a \in \alpha R$ where $a \in R$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Now $r(\alpha a) = |\alpha| r(a) > 0$ and hence $\alpha a \in R$. Therefore $\alpha R \subseteq R$ for every $\alpha \in \mathbb{C} \setminus \{0\}$. □

A key step in the proof of our next result is the equivalence (i)⇔(iii) in Theorem 1.1.

Theorem 4.4. Let $\mathcal{A}$ be a Banach algebra and $R = \mathcal{A} \setminus QN(\mathcal{A})$. Then

$$\mathcal{P}(R) = \text{Rad}(\mathcal{A}).$$

Proof. Let $a \in QN(\mathcal{A})$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Then $r(\alpha a) = |\alpha| \cdot 0 = 0$. Hence, $QN(\mathcal{A})$ is closed under multiplication by nonzero scalars. by Remark 4.3 R is also closed under multiplication by nonzero scalars. In view of the equivalence (i)⇔(iii) in Theorem 1.1 and Proposition 2.9 we get $\mathcal{P}(R) = \mathcal{P}(QN(\mathcal{A})) = \text{Rad}(\mathcal{A})$. □

5. Perturbation Ideals of Upper Semiregularities

In this section, our focus is to characterise the radical in $\mathcal{A}$ in terms of the perturbation ideals of upper semiregularities.
Let $R = R_9 = \text{Exp}\mathcal{A}$, which is the connected component of $\mathcal{A}^{-1}$ containing $1_{\mathcal{A}}$. It is also called the principal component of $\mathcal{A}^{-1}$ ([3], Theorem 3.3.7). Since R is an open semigroup containing the identity of $\mathcal{A}$, it is an upper semiregularity, see Remark 1.7. The spectrum associated with Exp$\mathcal{A}$ is

$$\varepsilon(x) = \varepsilon(x, \mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda - x \notin \text{Exp}\mathcal{A}\}, \;\; x \in \mathcal{A}.$$

It is called the *exponential spectrum* of $x \in \mathcal{A}$. This spectrum is well known in spectral theory of Banach algebras. For basic properties of this spectrum see [5].

Remark 5.1. If $\mathcal{A}$ is a Banach algebra, then $\mathcal{A} = \text{Exp}\mathcal{A} + \text{Exp}\mathcal{A}$.

Proof. To prove the nontrivial containment, let $a \in \mathcal{A}$. Choose $\lambda \in \mathbb{C}$ with $|\lambda|$ large enough. Then $a = \lambda - (\lambda - a) = \lambda \cdot 1_{\mathcal{A}} - (\lambda - a)$. If $|\lambda|$ is large enough, then $\sigma(\lambda - a)$ does not separate 0 from infinity. Hence, by ([3], Theorem 3.3.6), $\lambda - a \in \exp\mathcal{A} \subseteq \text{Exp}\mathcal{A}$. Since the complex exponential function is onto, there exists $\alpha \in \mathbb{C}$ with $\lambda = e^{\alpha}$, and so $\lambda \cdot 1_{\mathcal{A}} = e^{\alpha \cdot 1_{\mathcal{A}}} \in \exp\mathcal{A} \subseteq \text{Exp}\mathcal{A}$. We have shown that $\mathcal{A} \subseteq \text{Exp}\mathcal{A} + \text{Exp}\mathcal{A}$. □

Theorem 5.2. Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(\text{Exp}\mathcal{A}) = \text{Rad}(\mathcal{A}).$$

Proof. Let $a \in \text{Rad}(\mathcal{A})$ and $x \in \text{Exp}\mathcal{A}$. Then $x + a = x(1_{\mathcal{A}} + x^{-1}a)$. Now by the Spectral Mapping Theorem ([11], Theorem 1.34) and the fact that the radical is contained in the set of quasinilpotent elements, it then follows that $\sigma(1_{\mathcal{A}} + x^{-1}a) = \{1\}$. This spectrum does not separate 0 from infinity. Hence by Theorem 3.3.6 in [3], $1_{\mathcal{A}} + x^{-1}a = e^{y} \in \exp\mathcal{A}$ for some $y \in \mathcal{A}$. If $x \in \text{Exp}\mathcal{A}$, then $x + a = x(1_{\mathcal{A}} + x^{-1}a) \in \text{Exp}\mathcal{A}$, and so $a \in \mathcal{P}(\text{Exp}\mathcal{A})$. Conversely, let $a \in \mathcal{P}(\text{Exp}\mathcal{A})$ and $x \in \mathcal{A}$. Now since $\mathcal{A} = \text{Exp}\mathcal{A} + \text{Exp}\mathcal{A}$, we get that

$$\begin{aligned}
1_{\mathcal{A}} - xa &= 1_{\mathcal{A}} - (x_1 + x_2)a \;\text{ where }\; x_1, x_2 \in \text{Exp}\mathcal{A}, \\
&= 1_{\mathcal{A}} - x_1 a - x_2 a \\
&= x_1(x_1^{-1} - a) - x_2 a \\
&= x_2(x_2^{-1} x_1 (x_1^{-1} - a) - a).
\end{aligned}$$

Since Exp$\mathcal{A}$ is closed under multiplication, and since $-a \in \mathcal{P}(\text{Exp}\mathcal{A})$ we get $x_2(x_2^{-1} x_1(x_1^{-1} - a) - a) \in \text{Exp}\mathcal{A} \subseteq \mathcal{A}^{-1}$.

Hence, $1_{\mathcal{A}} - xa \subseteq \mathcal{A}^{-1}$ and so $a \in \text{Rad}(\mathcal{A})$, see ([3], Theorem 3.1.3). This completes the proof. □

Let $R = \mathcal{A}^{-1} \cup \partial\mathcal{A}^{-1}$. Since $1_{\mathcal{A}} \in \mathcal{A}^{-1} \subseteq R$, it follows that R contains a neighbourhood of the identity. In view of R being closed under multiplication, we get from Remark 1.7 that R is an upper semiregularity. Note that $R = \overline{\mathcal{A}^{-1}}$ (the closure of a set is the union of the set with its boundary).

Theorem 5.3. Let $\mathcal{A}$ be a Banach algebra and let $R = \overline{\mathcal{A}^{-1}}$. Then $\mathcal{P}(R)$ is an ideal and

$$\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(R) \subseteq \overline{\mathcal{A}^{-1}}.$$

Proof. Let $x \in$Rad$(\mathcal{A})$ and let $a \in R$. If $a \in \mathcal{A}^{-1}$, then by Theorem 3.1, $x + a \in \mathcal{A}^{-1} \subseteq \mathcal{A}^{-1} \cup \partial\mathcal{A}^{-1}$. If $a \in \partial\mathcal{A}^{-1}$, then $x + a \in \partial\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1} \cup \partial\mathcal{A}^{-1}$, see Corollary 6.3. By combining these arguments, it follows that $\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(R)$. Now if $0 \neq \alpha \in \mathbb{C}$, then $\alpha\partial\mathcal{A}^{-1} \subseteq \partial\mathcal{A}^{-1}$, see the comment which follows the proof of Lemma 2.6 in [8]. Since $\alpha\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}$, it follows that $\alpha R \subseteq R$. From this we then use Lemma 2.3 to deduce that $\mathcal{P}(R)$ is a linear subspace. Since $\mathcal{A}^{-1}\mathcal{A}^{-1} \subseteq \mathcal{A}^{-1}$ and by Lemma 2.6 in [8], it becomes clear that $\mathcal{A}^{-1}R \subseteq R$ and $R\mathcal{A}^{-1} \subseteq R$. By Lemma 2.5, $\mathcal{P}(R)$ is an ideal. Since $0 \in \overline{\mathcal{A}^{-1}}$, it follows from Remark 2.2 that $\mathcal{P}(R) \subseteq \overline{\mathcal{A}^{-1}}$. □

Our next result hints that it is not true in Theorem 5.3 that $\mathrm{Rad}(\mathcal{A}) = \mathcal{P}(R)$.

Corollary 5.4. Let $\mathcal{A}$ be a Banach algebra and $R = \overline{\mathcal{A}^{-1}}$. Then

$$1_{\mathcal{A}} - ba \in R$$

for all $b \in \mathcal{A}$ and $a \in \mathcal{P}(R)$.

Proof. Let $a \in \mathcal{P}(R)$ and $b \in \mathcal{A}$. Since $1_{\mathcal{A}} - ba = 1_{\mathcal{A}} - r$ with $r \in \mathcal{P}(R)$, because $\mathcal{P}(R)$ is an ideal, we get that $1_{\mathcal{A}} - ba = 1_{\mathcal{A}} - r = s$ with $s \in R$ because $r \in \mathcal{P}(R)$. □

6. Perturbation Ideals of other Spectra

In this section we characterise the radical in terms of the perturbation ideal of sets which are neither regularities nor semiregularities.

Let $\mathcal{A}$ be a Banach algebra and $R = \mathcal{A} \setminus \partial\mathcal{A}^{-1}$. Since $\partial\mathcal{A}^{-1}$ is a closed set, R is an open set with $\mathcal{A}^{-1} \subseteq R$. In the usual way, the set R gives rise to a spectrum

$$\sigma_R(x) = \{\lambda \in C : \lambda - x \in \partial\mathcal{A}^{-1}\}$$

for all $x \in \mathcal{A}$. In the literature this spectrum is known as the *boundary spectrum*, (see [8] and [9]). Although the set R that generates the boundary spectrum σ_R is neither an upper nor a lower semiregularity (see [12], Example 1.1, 1.2 and [13]), our interest in the boundary spectrum is to determine the perturbation ideal of the set R.

Remark 6.1. Let $\mathcal{A}$ be a Banach algebra and let set $R = \mathcal{A} \setminus \partial\mathcal{A}^{-1}$. Then

$$\alpha R \subseteq R \text{ for every } \alpha \in \mathbb{C} \setminus \{0\}.$$

Proof. Let $\alpha a \in \alpha R$ where $a \in R$ and $\alpha \in \mathbb{C} \setminus \{0\}$. If a is invertible, then αa is invertible and so $\alpha a \in R$. If a is not invertible, then αa is also not invertible and it is easy to see that $\alpha a \notin \partial\mathcal{A}^{-1}$. Hence, $\alpha a \in R$. □

Our next result is one of the main theorems in this section. A key step in the proof of this result is the equivalence (i)⇔(ii) in Theorem 1.1 and Proposition 2.9.

Theorem 6.2. Let $\mathcal{A}$ be a Banach algebra and let $R = \mathcal{A} \setminus \partial\mathcal{A}^{-1}$. Then

$$\mathcal{P}(R) = \mathrm{Rad}(\mathcal{A}).$$

Proof. We first note that both $\mathcal{A}^{-1}$ and R are open sets in $\mathcal{A}$ with $A^{-1} \subseteq R$. Also, both $\mathcal{A}^{-1}$ and R are closed under scalar multiplication, see Remark 6.1. Since $\partial\mathcal{A}^{-1} \cap R = \emptyset$, it follows from Lemma 2.4 and Theorem 3.1 that $\mathcal{P}(R) \subseteq \mathcal{P}(\mathcal{A}^{-1}) = \mathrm{Rad}(\mathcal{A})$. We claim that $\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(R)$: Let $r \in \mathrm{Rad}(\mathcal{A})$ and $x \in R$. If $x \in \mathcal{A}^{-1}$, then in view of Theorem 3.1, $r + x \in \mathcal{A}^{-1} \subseteq R$. If $x \in R$ and $x \notin \mathcal{A}^{-1}$, then $r + x \notin \partial\mathcal{A}^{-1}$: If $r + x \in \partial\mathcal{A}^{-1}$, then there is a sequence (y_n) in $\mathcal{A}^{-1}$ with $y_n \to r + x$. Hence $y_n - r \to x$ and $(y_n - r)$ is a sequence in $\mathcal{A}^{-1}$, by Theorem 3.1. Hence, $x \in \partial\mathcal{A}^{-1}$ which is a contradiction. Consequently, $r + x \in R$ and so $r \in \mathcal{P}(R)$. If we combine our arguments we get $\mathcal{P}(R) = \mathrm{Rad}(\mathcal{A})$. $\square$

Let $\mathcal{A}$ be a Banach algebra and let $R = \partial\mathcal{A}^{-1}$. The set R is neither an upper nor a lower semiregularity since $1_{\mathcal{A}} \notin R$. However in view of Proposition 2.9 and Theorem 6.2 we have

Corollary 6.3. Let $\mathcal{A}$ be a Banach algebra and let $R = \partial\mathcal{A}^{-1}$. Then

$$\mathcal{P}(R) = \mathrm{Rad}(\mathcal{A}).$$

Proof. If $0 \neq \alpha \in \mathbb{C}$, then $\alpha R \subseteq R$ (see the comment which follows after the proof of Lemma 2.6 in [8]). In view of Proposition 2.9 and Theorem 6.2, our result is proved. $\square$

In our next results, we will investigate the perturbation ideals of the components of $\mathcal{A}^{-1}$ different from $\mathrm{Exp}\mathcal{A}$.

Since $\mathrm{Exp}\mathcal{A}$ is a normal subgroup of the set of invertible elements $\mathcal{A}^{-1}$, the quotient group $\mathcal{A}^{-1}/\mathrm{Exp}\mathcal{A}$ is the set of cosets

$$\mathcal{A}^{-1}/\mathrm{Exp}\mathcal{A} = \{a \cdot \mathrm{Exp}\mathcal{A} : a \in \mathcal{A}^{-1}\}.$$

All the components of $\mathcal{A}^{-1}$ can be represented by cosets $a \cdot \mathrm{Exp}\mathcal{A}$ where $a \in \mathcal{A}^{-1}$. In particular if $a \in \mathrm{Exp}\mathcal{A}$, we obtain the principal component of $\mathcal{A}^{-1}$. Let $\mathcal{A}_\gamma$ represent any other component of $\mathcal{A}^{-1}$ different from $\mathrm{Exp}\mathcal{A}$, i.e., $\mathcal{A}_\gamma = a \cdot \mathrm{Exp}\mathcal{A}$ for $a \in \mathcal{A}^{-1} \setminus \mathrm{Exp}\mathcal{A}$. For any $a \in \mathcal{A}^{-1} \setminus \mathrm{Exp}\mathcal{A}$, $\mathcal{A}_\gamma = a \cdot \mathrm{Exp}\mathcal{A}$ is an open and closed connected subset of $\mathcal{A}^{-1}$. Note that the components of $\mathcal{A}^{-1}$ are disjoint. We note that $R = \mathcal{A}_\gamma$ is neither a lower nor an upper semiregularity since $1_{\mathcal{A}} \notin \mathcal{A}_\gamma$. Since $\alpha\mathrm{Exp}\mathcal{A} \subseteq \mathrm{Exp}\mathcal{A}$ for each $\alpha \in \mathbb{C} \setminus \{0\}$, it follows easily that $\alpha\mathcal{A}_\gamma \subseteq \mathcal{A}_\gamma$ for all $\alpha \in \mathbb{C} \setminus \{0\}$.

Proposition 6.4. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}_\gamma$ be any component of $\mathcal{A}^{-1}$. Then

$$\mathcal{A} = \mathcal{A}_\gamma + \mathcal{A}_\gamma.$$

Proof. Let $\mathcal{A}_\gamma$ be a component of $\mathcal{A}^{-1}$ different from $\mathrm{Exp}\mathcal{A}$. Suppose $\mathcal{A}_\gamma$ takes the form $\mathcal{A}_\gamma = a \cdot \mathrm{Exp}\mathcal{A}$ with $a \in \mathcal{A}^{-1}\setminus\mathrm{Exp}\mathcal{A}$. Since $\mathcal{A}_\gamma \subseteq \mathcal{A}$ and since $\mathcal{A}$ is closed under addition, $\mathcal{A}_\gamma + \mathcal{A}_\gamma \subseteq \mathcal{A}$. Let $x \in \mathcal{A}$. Since $\varepsilon(a^{-1}x)$ is a compact subset of $\mathbb{C}$ (see [5], Theorem 1), it is possible to choose $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda \notin \varepsilon(a^{-1}x)$, i.e., $\lambda - a^{-1}x \in \mathrm{Exp}\mathcal{A}$. Then, $x = \lambda a - (\lambda a - x) = \lambda a - a(\lambda - a^{-1}x)$. From the arguments above we get $x \in \mathcal{A}_\gamma + \mathcal{A}_\gamma$. $\square$

Theorem 6.5. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}_\gamma$ be any component of $\mathcal{A}^{-1}$. Then

$$\mathcal{P}(\mathcal{A}_\gamma) = \mathrm{Rad}(\mathcal{A}).$$

Proof. If $\mathcal{A}_\gamma$ is the principal component of $\mathcal{A}^{-1}$, then we have proved in Theorem 5.2 that $\mathcal{P}(\mathcal{A}_\gamma) = \mathrm{Rad}(\mathcal{A})$. Let $r \in \mathrm{Rad}(\mathcal{A})$ and $az \in \mathcal{A}_\gamma = a \cdot \mathrm{Exp}\mathcal{A}$ with $z \in \mathrm{Exp}\mathcal{A}$ and $a \in \mathcal{A}^{-1}\setminus\mathrm{Exp}\mathcal{A}$. Then $r + az = a(a^{-1}r + z)$ and since $a^{-1}r \in \mathrm{Rad}(\mathcal{A})$ and $z \in \mathrm{Exp}\mathcal{A}$, it follows from Theorem 5.2 that $a^{-1}r + z \in \mathrm{Exp}\mathcal{A}$. Hence, $r + az \in a \cdot \mathrm{Exp}\mathcal{A} = \mathcal{A}_\gamma$ and so $r \in \mathcal{P}(\mathcal{A}_\gamma)$. Hence, $\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}_\gamma)$. To prove that $\mathcal{P}(\mathcal{A}_\gamma) \subseteq \mathrm{Rad}(\mathcal{A})$, let $x \in \mathcal{P}(\mathcal{A}_\gamma)$. If $az \in \mathcal{A}_\gamma = a \cdot \mathrm{Exp}\mathcal{A}$ with $z \in \mathrm{Exp}\mathcal{A}$, then $a^{-1}x + z = a^{-1}(x + az)$. Since $x \in \mathcal{P}(\mathcal{A}_\gamma)$ and $az \in \mathcal{A}_\gamma$, $x + az = ab$ for some $b \in \mathrm{Exp}\mathcal{A}$. Hence, $a^{-1}x + z = a^{-1} \cdot ab = b \in \mathrm{Exp}\mathcal{A}$. By Theorem 5.2, $a^{-1}x \in \mathrm{Rad}(\mathcal{A})$ and so $x \in \mathrm{Rad}(\mathcal{A})$ because $\mathrm{Rad}(\mathcal{A})$ is an ideal. If we combine our arguments we get $\mathcal{P}(\mathcal{A}_\gamma) = \mathrm{Rad}(\mathcal{A})$. $\square$

In the first part of the proof of Theorem 6.5 we proved that $\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}_\gamma)$. One can also prove this by employing Lemma 2.4.

Let $\mathcal{A}$ be a Banach algebra and let $R = QN(\mathcal{A})$. Note that R is neither an upper nor a lower semiregularity since $\mathcal{A}^{-1} \cap QN(\mathcal{A}) = \emptyset$. However, we have

Corollary 6.6. Let $\mathcal{A}$ be a Banach algebra and let $R = QN(\mathcal{A})$. Then

$$\mathcal{P}(R) = \mathrm{Rad}(\mathcal{A}).$$

Proof. If $0 \neq \alpha \in \mathbb{C}$, then $\alpha R \subseteq R$ because if $a \in R$, then $r(\alpha a) = |\alpha| r(a) = |\alpha| \cdot 0 = 0$. In view of Proposition 2.9 and Theorem 4.4, our result is proved. □

We now investigate the perturbation ideal of $\exp\mathcal{A}$. The corresponding spectrum of $\exp\mathcal{A}$ is

$$e(x) = e(x, \mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda - x \notin \exp\mathcal{A}\}, \quad x \in \mathcal{A}.$$

This spectrum is compact (see [10], Theorem 3.2). Since $\exp\mathcal{A} \subseteq \mathrm{Exp}\mathcal{A} \subseteq \mathcal{A}^{-1}$, we get the following inclusions,

$$\sigma(x) \subseteq \varepsilon(x) \subseteq e(x),$$

thus also showing that $e(x)$ is nonempty. If $\mathcal{A}$ is commutative, then $\varepsilon(x) = e(x)$.

Remark 6.7. If $\mathcal{A}$ is a Banach algebra, then $\mathcal{A} = \exp\mathcal{A} + \exp\mathcal{A}$. For a proof of this statement, see the proof of the Remark 5.1.

Although $\exp\mathcal{A}$ contains a neighbourhood of the identity it is not an upper semiregularity because it is not closed under multiplication. It is also not a lower semiregularity because $\mathcal{A}^{-1} \not\subseteq \exp\mathcal{A}$.

Proposition 6.8. Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathrm{Rad}(\mathcal{A}) + \exp\mathcal{A} \subseteq \mathrm{Exp}\mathcal{A}.$$

Proof. Let $a \in \mathrm{Rad}(\mathcal{A})$ and $e^x \in \exp\mathcal{A}$. Then $e^x + a = e^x(1_\mathcal{A} + e^{-x}a)$. Since $e^{-x}a \in \mathrm{Rad}(\mathcal{A})$, it follows from the Spectral Mapping Theorem that

$$\sigma(1_\mathcal{A} + e^{-x}a) = 1 + \sigma(e^{-x}a) = 1 + \{0\} = \{1\}.$$

Since the spectrum of $1_\mathcal{A} + e^{-x}a$ does not separate 0 from infinity, $1_\mathcal{A} + e^{-x}a = e^y \in \exp\mathcal{A}$ for some $y \in \mathcal{A}$, see ([3], Theorem 3.3.6). It then follows that $e^x(1_\mathcal{A} + a^{-1}x) = e^x e^y \in \mathrm{Exp}\mathcal{A}$. This completes the proof. □

From this last proof, it is worth mentioning that $e^x e^y \in \exp\mathcal{A}$ only if x and y commute, which in general is not the case.

Proposition 6.9. Let $\mathcal{A}$ be a Banach algebra. If $a \in \mathcal{P}(\exp\mathcal{A})$, then

$$1_\mathcal{A} - \exp\mathcal{A} \cdot a \subseteq \mathrm{Exp}\mathcal{A}.$$

Proof. Let $a \in \mathcal{P}(\exp\mathcal{A})$ and $e^x \in \exp\mathcal{A}$. Then

$$\begin{aligned} 1_\mathcal{A} - e^x a &= e^x(e^{-x} - a) \\ &= e^x e^y, \text{ for some } y \in \mathcal{A} \text{ and since } a \in \mathcal{P}(\exp\mathcal{A}), \\ &\in \mathrm{Exp}\mathcal{A}. \end{aligned}$$

□

From this last proof, if we replace e^x by $b \in \mathcal{A}$ and using the fact that $\mathcal{A} = \exp\mathcal{A} + \exp\mathcal{A}$, we get

$$\begin{aligned} 1_{\mathcal{A}} - ba &= 1_{\mathcal{A}} - (e^c + e^d)a, \text{ for some } c, d \in \mathcal{A}; \\ &= 1_{\mathcal{A}} - e^c a - e^d a. \end{aligned}$$

It follows from Proposition 6.8 and Proposition 6.9 that the relationship between $\mathrm{Rad}(\mathcal{A})$ and $\mathcal{P}(\exp\mathcal{A})$ is not clear.

It is worth noting that since $\alpha\exp\mathcal{A} \subseteq \exp\mathcal{A}$, by Lemma 2.3, $\mathcal{P}(\exp\mathcal{A})$ is a subspace but $\mathcal{P}(\exp\mathcal{A})$ is not an ideal: If $\mathcal{P}(\exp\mathcal{A})$ is an ideal, then for $a \in \mathcal{P}(\exp\mathcal{A})$, $e^x e^y a \in \mathcal{P}(\exp\mathcal{A})$ for $x, y \in \mathcal{A}$. Now, since $1_{\mathcal{A}} \in \exp\mathcal{A}$, it then follows that $e^x e^y a + 1_{\mathcal{A}} \in \exp\mathcal{A}$. But $e^x e^y a + 1_{\mathcal{A}} = e^x(e^y a + e^{-x})$. Now, since $e^y a \in \mathcal{P}(\exp\mathcal{A})$, it then follows that $e^y a + e^{-x} \in \exp\mathcal{A}$. Hence $e^x(e^y a + e^{-x}) \in \mathrm{Exp}\mathcal{A}$. In general $e^x(e^y a + e^{-x}) \notin \exp\mathcal{A}$, which leads to a contradiction.

Lastly, we look at the following complements, $C_1 = \mathcal{A} \setminus \mathcal{A}^{-1}$, $C_2 = \mathcal{A} \setminus \mathcal{A}_l^{-1}$, $C_3 = \mathcal{A} \setminus \mathcal{A}_r^{-1}$, $C_4 = \mathcal{A} \setminus H_l$, $C_5 = \mathcal{A} \setminus H_r$, $C_6 = \mathcal{A} \setminus \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$, $C_7 = \mathcal{A} \setminus \mathrm{Exp}\mathcal{A}$ and $C_8 = \mathcal{A} \setminus \mathcal{A}_\gamma$. One is able to deduce that C_i is closed under nonzero scalar multiplication for each $i = 1, 2, \dots, 8$. It then follows by Proposition 2.9 that

Corollary 6.10. Let $\mathcal{A}$ be a Banach algebra. Then

$$\mathcal{P}(C_i) = \mathrm{Rad}(\mathcal{A}) \text{ for each } i = 1, 2, 3, 6, 7, 8;$$

and

$$\mathcal{P}(C_i) \subseteq \mathrm{Rad}(\mathcal{A}) \text{ for each } i = 4, 5.$$

References

[1] G.R. Allan, An analytic approach to the spectral characterization of the radical, Studia Mathematica 186 (2008), 179–188.
[2] B. Aupetit, Spectral characterizations of the radical in Banach and Jordan-Banach algebras, Mathematical Proceedings of the Cambridge Philosophic Society 114 (1993), 31–35.
[3] B. Aupetit, A Primer on Spectral Theory, (second edition), Springer-Verlag, New York, 1991.
[4] R.M. Brits, On the multiplicative spectral characterization of the Jacobson radical, Quaestiones Mathematicae 31 (2008), 179–188.
[5] R. Harte, The Exponential Spectrum in Banach Algebras, Proceedings of the American Mathematical Society 58 (1976), 114–118.
[6] V. Kordula and V. M*ü*ller, On the Axiomatic Theory of Spectrum, Studia Mathematica 119 (1996), 109–128.
[7] A. Lebow and M. Schechter, Semigroups of Operators and Measures of Noncompactness, Journal of Functional Analysis 7 (1971), 1–26.
[8] S. Mouton, On the Boundary Spectrum in Banach Algebras, Bulletin of the Australian Mathematical Society 74 (2006), 239–246.
[9] S. Mouton, Mapping and Continuity Properties of the Boundary Spectrum in Banach Algebras, Illinois Journal of Mathematics 53 (2009), 757–767.
[10] H. du T. Mouton, S. Mouton and H. Raubenheimer, Ruston Elements and Fredholm Theory Relative to Arbitrary Homomorphisms, Quaestiones Mathematicae 34 (2011), 341–359.
[11] V. M*ü*ller, Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras, Operator Theory Advances and Applications, Vol. 139, Birkh*ä*user Verlag, Basel-Boston-Berlin, 2007.
[12] H. Raubenheimer and A. Swartz, Radius Preserving (Semi)Regularities in Banach Algebras, Quaestiones Mathematicae 42 (2019), 811–822.
[13] H. Raubenheimer and A. Swartz, Regularity-type Properties of the Boundary Spectrum in Banach Algebras, The Rocky Mountain Journal of Mathematics 49 (2019), 2747–2754.
[14] J. Zemánek, Properties of the spectral radius in Banach algebras, PhD thesis, Inst. Math., Polish Acad. Sci., Warszawa, 1977.
[15] J. Zemánek, A note on the radical of a Banach algebra, Manuscripta Mathematica 20 (1977), 191–196.
[16] J. Zemánek, Spectral radius characterizations of commutativity in Banach algebras, Studia Mathematica 61 (1977), 257–268.