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Positive fuzzy quasi-orders on semigroups

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Abstract. In this paper we study certain types of positive fuzzy quasi-orders on semigroups and their links with completely semiprime fuzzy ideals, completely prime fuzzy ideals, and fuzzy filters. We characterize various properties of a positive fuzzy quasi-order Q in terms of the properties of its left and right eigen spaces $\mathscr{I}_Q$ and $\mathscr{C}_Q$, the solution sets of eigen fuzzy set equations corresponding to Q. We also demonstrate certain applications of the obtained results to semilattice decompositions of semigroups. The results of this paper shed new light on the known links between positive (crisp) quasi-orders, completely semiprime ideals, completely prime ideals, and filters of semigroups, and make these links much clearer.

1. Introduction

Positive quasi-orders on semigroups have been introduced by B. M. Schein in [42], in the study of semigroups of binary relations, and later they have been studied in different contexts, mostly in relation to semilattice decompositions of semigroups. Semilattice decompositions are one of the most powerful tools used in the study of structural properties of semigroups. They were first defined and studied by A. H. Clifford in [18], where completely regular semigroups were characterized as semilattices of simple semigroups. Particular impetus to the study of semilattice decompositions was given by the general results of T. Tamura and N. Kimura [52] and T. Tamura [46], according to which each semigroup possess the greatest semilattice decomposition, whose components are semilattice indecomposable semigroups. This result initiated intensive study of the greatest semilattice decompositions and the corresponding smallest semilattice congruences. From the point of view of this paper, the most interesting characterizations of the smallest semilattice congruence are those given by means of completely prime ideals and filters (M. Petrich [35]), completely semiprime ideals (M. Ćirić and S. Bogdanović [13, 14]), and the relation that we call here Tamura's quasi-order (T. Tamura [48, 49]).

The study of Tamura's quasi-order has prompted more general research on positive quasi-orders and their role in semilattice decompositions, which was conducted in [8–10, 12, 14, 37–40, 47–51]. It is worth mentioning that T. Tamura established in [51] an isomorphism between the lattice of semilattice congruences

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on a semigroup and the lattice of positive lower-potent half-congruences on this semigrop. In the same paper he also showed that positive lower-potent half-congruences are the same as positive quasi-orders having the cm-property. Both concepts, the lower-potency and the cm-property, were introduced by Tamura. On the other hand, Ćirić and Bogdanović in [14] linked positivity, lower-potency and cm-property with filters, completely prime and completely semiprime ideals.

In this paper we discuss the mentioned problems from the point of view of fuzzy set theory. The main tool we use are fuzzy quasi-orders. They originated as a generalization of ordinary (crisp) quasi-orders, and like their crisp counterparts, they have wide applications in numerous fields of mathematics and computer science. It is worth mentioning that fuzzy quasi-orders have recently been used as a major tool in reducing the number of states of fuzzy automata [17, 44] and in the positional analysis of fuzzy social networks [15, 28, 45]. Also, they naturally emerge as the greatest solutions to various systems of fuzzy relation inequations and equations [25, 26, 28, 45] and the major tool used in computing the greatest solutions to systems of eigen fuzzy set inequations and equations [29, 30]. Here we study positive fuzzy quasi-orders on semigroups and their certain varieties, such as those which are lower-potent and those that have the cm-property or cp-property. It should be noted that the lower-potency and cm-property are generalizations of the concepts already used in the study of ordinary quasi-orders, whereas the cp-property is a brand new concept. We also investigate the links of all these fuzzy quasi-orders with fuzzy ideals, consistent fuzzy subsets and related concepts, such as completely semiprime and completely prime fuzzy ideals and fuzzy filters, and demonstrate their applications in semilattice decompositions of semigroups.

The paper is organized as follows. After this section and Section 2, in which we introduce preliminary notions and results, in the consequent sections we present the main results of the paper. First, in Theorem 3.1, we prove the equivalence of the twelve conditions that define positive fuzzy quasi-orders, and then we provide theorems that characterize their special types (with cm-property and cp-property, lower-potent, etc.). Some of the results obtained are generalizations of the well-known results concerning ordinary quasi-orders, but many of them are brand new and have not had their crisp counterparts so far. It should be emphasized that, by Theorem 3.7, we establish the duality between fuzzy ideals and consistent fuzzy subsets, and assuming that the underlying structure of membership values is linearly ordered, we establish the duality between completely prime fuzzy ideals and fuzzy filters. In Section 4 we first prove that the lattice of positive fuzzy quasi-orders can be dually (anti-isomorphically) embedded into the latices of fuzzy ideals and consistent fuzzy subsets (Theorems 4.1 and 4.2), and then we provide the construction of positive lower-potent fuzzy quasi-orders, starting from collections of completely semiprime fuzzy ideals, and the construction of positive fuzzy quasi-orders with the cm-property, starting from collections of completely prime fuzzy ideals and fuzzy filters (Theorems 4.4, 4.6 and 4.7). We also give new characterizations of positive fuzzy quasi-orders with the cm-property and cp-property (Theorems 4.5 and 4.8).

In Section 5 we provide certain applications to semilattice decompositions of semigroups. By Theorem 5.1 we establish a link between positive lower-potent fuzzy half-congruences and semilattice fuzzy congruences, which is bidirectional if the membership values are taken from a complete Heyting algebra, and by Theorem 5.2 we show that under the same conditions the concepts of a positive fuzzy quasi-order with the cm-property and a positive lower-potent fuzzy half-congruence are identical. By Theorem 5.4 we provide a characterization of the smallest semilattice fuzzy congruence through completely semiprime fuzzy ideals, completely prime fuzzy ideals, and fuzzy filters, whereas Theorem 5.5 provides a version of the well-known Prime ideal theorem for completely semiprime fuzzy ideals. It should be noted that in lattice theory, ring theory and semigroup theory Prime ideal theorem is usually proved using the Zorn's lemma arguments, but here we give a proof in which Zorn's lemma is not used.

Finally, we point out that the results of this paper shed new light on the known links between positive quasi-orders, completely semiprime ideals, completely prime ideals, and filters of semigroups, and make these links much clearer.

2. Preliminaries

A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,

(L3) $\otimes$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z. \tag{1}$$

Moreover, $\mathscr{L}$ is called a *complete residuated lattice* if it satisfies (L2), (L3), and

(L1') $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1.

The operations $\otimes$ (called *multiplication*) and $\to$ (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values.

If $\mathscr{L}$ is a complete residuated lattice, then for all $x, y, z \in L$ and any $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subseteq L$ the following holds:

$$x \otimes (x \to y) \leqslant y, \tag{3}$$

$$x \leqslant y \;\text{ if and only if }\; x \to y = 1, \tag{4}$$

$$x \leqslant y \;\text{ implies }\; x \otimes z \leqslant y \otimes z, \tag{5}$$

$$x \leqslant y \;\text{ implies }\; z \to x \leqslant z \to y \;\text{ and }\; y \to z \leqslant x \to z, \tag{6}$$

$$(x \to y) \otimes (y \to z) \leqslant (x \to z), \tag{7}$$

$$x \otimes \bigvee_{i\in I} y_i = \bigvee_{i\in I} (x \otimes y_i), \tag{8}$$

$$x \otimes \bigwedge_{i\in I} y_i \leqslant \bigwedge_{i\in I} (x \otimes y_i), \tag{9}$$

$$\bigwedge_{i\in I} (x_i \to y) = \Big(\bigvee_{i\in I} x_i\Big) \to y, \tag{10}$$

$$\bigvee_{i\in I} (x_i \to y) \leqslant \Big(\bigwedge_{i\in I} x_i\Big) \to y, \tag{11}$$

$$\bigwedge_{i\in I} (x \to y_i) = x \to \Big(\bigwedge_{i\in I} y_i\Big), \tag{12}$$

$$\bigvee_{i\in I} (x \to y_i) \leqslant x \to \Big(\bigvee_{i\in I} y_i\Big), \tag{13}$$

$$\bigwedge_{i\in I} (x_i \to y_i) \leqslant \Big(\bigwedge_{i\in I} x_i\Big) \to \Big(\bigwedge_{i\in I} y_i\Big), \tag{14}$$

For other properties of complete residuated lattices we refer to [2, 3].

The most studied and applied structures of truth values, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* (where $x \otimes y = \max(x + y - 1, 0)$, $x \to y = \min(1 - x + y, 1)$), the *Goguen* (*product*) *structure* ($x \otimes y = x \cdot y$, $x \to y = 1$ if $x \leqslant y$, and $= y/x$ otherwise), and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \to y = 1$ if $x \leqslant y$, and $= y$ otherwise). More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \to y = \bigvee\{u \in [0, 1] \,|\, u \otimes x \leqslant y\}$ (cf. [3]). Another important set of truth values is the set $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \to a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*. Let us note that all of the above mentioned structures are linearly ordered.

A residuated lattice $\mathscr{L}$ satisfying $x \otimes y = x \wedge y$, for all $x, y \in L$, is called a *Heyting algebra*, and a complete residuated lattice satisfying this condition is called a *complete Heyting algebra*.

Let $\mathscr{L}$ be a complete residuated lattice. A *fuzzy subset* of a set A is any function from A into L. For a fuzzy subset $f : A \to L$ and $a \in A$, we say that $f(a)$ is the *membership degree* of a in f (names such as *membership value* or *truth degree* are also used). The set of all fuzzy subsets of A taking membership values in L, i.e., the set of all functions from A to L is denoted by L^A. Let $f, g \in L^A$. The *equality* of f and g is defined as the usual equality of functions, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. If so, we say that *f is contained in g*. Endowed with this partial order L^A forms a complete lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of a family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are functions from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x), \tag{15}$$

for every $x \in A$, the least element is the empty set $\emptyset$ and the greatest one is the whole set A. Let us denote this lattice by $\mathscr{F}(A) = (L^A, \vee, \wedge, \emptyset, A)$.

A *crisp subset* of a set A is a fuzzy subset which takes values only in the two-element set $\{0, 1\}$. If f is a crisp subset of A, then expressions "$f(x) = 1$" and "$x \in f$" have the same meaning, i.e., f is considered as an ordinary subset of A. The *crisp part* of a fuzzy subset f of A is a crisp subset $f^c : A \to L$ defined by $f^c(a) = 1$, if $f(a) = 1$, and $f^c(a) = 0$, if $f(a) < 1$, i.e., $f^c = \{x \in A \,|\, f(x) = 1\}$. A fuzzy subset f of A is *normalized* (or *modal*, in some sources) if $f(x) = 1$ for at least one $x \in A$, i.e., if its crisp part is non-empty.

A *fuzzy relation* on A is any function from $A \times A$ into L, that is to say, any fuzzy subset of $A \times A$, and equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets. Let us note that when $U \leqslant V$, for two fuzzy relations U and V on A, we say that *U is contained in V*. The set of all fuzzy relations on a set A is denoted by $\mathscr{R}(A)$.

For fuzzy relations $U, V \in \mathscr{R}(A)$, their *composition* is a fuzzy relation $U \circ V$ on A defined by

$$(U \circ V)(a, b) = \bigvee_{c\in A} U(a, c) \otimes V(c, b), \tag{16}$$

for all $a, b \in A$, and for $f \in \mathscr{F}(A)$ and $U \in \mathscr{R}(A)$, the *compositions* $f \circ U$ and $U \circ f$ are fuzzy subsets of A defined by

$$(f \circ U)(a) = \bigvee_{b\in A} f(b) \otimes U(b, a), \quad (U \circ f)(a) = \bigvee_{b\in A} U(a, b) \otimes f(b), \tag{17}$$

for any $a \in A$. Finally, for $f, g \in \mathscr{F}(A)$ we write

$$f \circ g = \bigvee_{a\in A} f(a) \otimes g(a). \tag{18}$$

The value $f \circ g$ can be interpreted as the "degree of overlapping" of f and g.

For arbitrary $U, V, W \in \mathscr{R}(A)$ we have that

$$(U \circ V) \circ W = U \circ (V \circ W), \tag{19}$$

$$U \leqslant V \text{ implies } U \circ W \leqslant V \circ W \text{ and } W \circ U \leqslant W \circ V, \tag{20}$$

and for arbitrary $f, g \in \mathscr{F}(A)$ and $U, V \in \mathscr{R}(A)$ it can be easily verified that

$$(f \circ U) \circ V = f \circ (U \circ V), \quad (f \circ U) \circ g = f \circ (U \circ g), \tag{21}$$

and hence, the parentheses in (21) can be omitted. For $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers (without zero included), an n-th power of a fuzzy relation $U \in \mathscr{R}(A)$ is a fuzzy relation U^n on A defined inductively by $U^1 = U$ and $U^{n+1} = U^n \circ U$. We also define U^0 to be the ordinary equality relation on A.

For all $U, V \in \mathscr{R}(A)$ and all $\{U_i\}_{i\in I}, \{V_i\}_{i\in I} \subseteq \mathscr{R}(A)$ we have that

$$U \circ \Big(\bigvee_{i\in I} V_i\Big) = \bigvee_{i\in I}(U \circ V_i), \ \ \Big(\bigvee_{i\in I} U_i\Big) \circ V = \bigvee_{i\in I}(U_i \circ V), \tag{22}$$

$$U \circ \Big(\bigwedge_{i\in I} V_i\Big) \leqslant \bigwedge_{i\in I}(U \circ V_i), \ \ \Big(\bigwedge_{i\in I} U_i\Big) \circ V \leqslant \bigwedge_{i\in I}(U_i \circ V), \tag{23}$$

and then the system $(\mathscr{R}(A), \wedge, \vee, \circ, \emptyset, \nabla_A, \Delta_A)$ forms a quantale, where ∇_A is the universal relation on A and Δ_A is the equality relation on A, i.e., for all $a, b \in A$ we have that $\nabla_A(a, b) = 1$, and $\Delta_A(a, b) = 1$, if $a = b$, and $\Delta_A(a, b) = 0$, if $a \neq b$ (for the definition of a quantale we refer to [26]).

Notice that if A is a finite set with n elements, then U and V can be treated as $n \times n$ fuzzy matrices over $\mathscr{L}$ and $U \circ V$ is the matrix product, whereas $f \circ U$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix U, and $U \circ f$ as the product of an $n \times n$ matrix U and an $n \times 1$ matrix f^t (the transpose of f), i.e., as vector-matrix products. Also, for $f, g \in \mathscr{F}(A)$ we can interpret $f \circ g$ as the scalar product of vectors f and g.

Let R be a fuzzy relation on a set A and let f be a fuzzy subset of A. The *left residual* of f by R is a fuzzy subset f/R of A defined by

$$(f/R)(a) = \bigwedge_{b\in A} R(a, b) \to f(b), \tag{24}$$

for each $a \in A$, and the *right residual* of f by R is a fuzzy subset $R\backslash f$ of A defined by

$$(R\backslash f)(a) = \bigwedge_{b\in A} R(b, a) \to f(b), \tag{25}$$

for each $a \in A$. We think of the left residual f/R as what remains of f on the left after "dividing" f on the right by R, and of the right residual $R\backslash f$ as what remains of f on the right after "dividing" f on the left by R. It is easy to check that

$$f \circ R \leqslant g \ \Leftrightarrow \ f \leqslant g/R, \quad R \circ f \leqslant g \ \Leftrightarrow \ f \leqslant R\backslash g, \tag{26}$$

for all fuzzy subsets f and g of A and every fuzzy relation R on A. We call (26) the *residuation property*.

For a fuzzy relation R on A, the fuzzy relation R^{-1} on A defined by $R^{-1}(a, b) = R(b, a)$, for every $a, b \in A$, is called the *inverse* of R. A fuzzy relation R on A is said to be

(R) *reflexive* (or *fuzzy reflexive*) if $\Delta_A \leqslant R$, i.e., if $R(a, a) = 1$, for every $a \in A$;

(S) *symmetric* (or *fuzzy symmetric*) if $R^{-1} \leqslant R$, i.e., if $R(a, b) = R(b, a)$, for all $a, b \in A$;

(T) *transitive* (or *fuzzy transitive*) if $R \circ R \leqslant R$, i.e., if for all $a, b, c \in A$ we have

$$R(a, b) \otimes R(b, c) \leqslant R(a, c).$$

For a fuzzy relation R on a set A, a fuzzy relation R^∞ on A defined by $R^\infty = \bigvee_{n\in\mathbb{N}} R^n$ is the least transitive fuzzy relation on A containing R, and it is called the *transitive closure* of R.

A reflexive and transitive fuzzy relation on A is called a *fuzzy quasi-order*, and a reflexive and transitive crisp relation on A is called a *quasi-order*. In some sources quasi-orders and fuzzy quasi-orders are called *preorders* and *fuzzy preorders*. Let us note that a reflexive fuzzy relation R is a fuzzy quasi-order if and only if $R^2 = R$. A reflexive, symmetric and transitive fuzzy relation on A is called a *fuzzy equivalence*, and a reflexive, symmetric and transitive crisp relation on A is called an *equivalence*. A fuzzy equivalence E on A is called a *fuzzy equality* if for any $a, b \in A$, $E(a, b) = 1$ implies $a = b$. If Q is a fuzzy quasi-order on A, then $E_Q = Q \wedge Q^{-1}$ is the greatest fuzzy equivalence contained in Q, and it is called the *natural fuzzy equivalence* of Q.

With respect to the ordering of fuzzy relations, the set $\mathscr{Q}(A)$ of all fuzzy quasi-orders on a set A, and the set $\mathscr{E}(A)$ of all fuzzy equivalences on A, form complete lattices. The meet both in $\mathscr{Q}(A)$ and $\mathscr{E}(A)$ is the ordinary intersection of fuzzy relations, but in the general case, the joins in $\mathscr{Q}(A)$ and $\mathscr{E}(A)$ do not coincide with the ordinary union of fuzzy relations. Namely, if $\{R_i\}_{i\in I}$ is a family of fuzzy quasi-orders (resp. fuzzy equivalences) on A, then its join in $\mathscr{Q}(A)$ (resp. in $\mathscr{E}(A)$) is $(\bigvee_{i\in I} R_i)^\infty$, the transitive closure of the union of this family.

Let Q be a fuzzy quasi-order on a set A. For each $a \in A$, the *Q-afterset* of a is the fuzzy subset aQ of A defined by $(aQ)(x) = Q(a,x)$, for any $x \in A$, and the *Q-foreset* of a is the fuzzy subset Qa of A defined by $(Qa)(x) = Q(x,a)$, for any $x \in A$ (cf. [1, 6, 20, 29, 33, 44]). The set of all Q-aftersets will be denoted by A/Q, and the set of all Q-foresets by $A\backslash Q$. If E is a fuzzy equivalence, then for every $a \in A$ we have that $aE = Ea$, and we usually denote this fuzzy set by E_a and we call it the *equivalence class* of a with respect to E (cf. [16]). The set of all equivalence classes of E is denoted by A/E and called the *factor set* of A with respect to E. We have the following:

Theorem 2.1 ([29, 44]). *Let Q be a fuzzy quasi-order on a set A and E the natural fuzzy equivalence of Q. Then*
(a) *For arbitrary $a, b \in A$ the following statements are equivalent:*

(i) $E(a,b) = 1$;

(ii) $E_a = E_b$;

(iii) $aQ = bQ$;

(iv) $Qa = Qb$.

(b) *Functions $E_a \mapsto aQ$ and $E_a \mapsto Qa$ are bijective functions of A/E onto A/Q and of A/E onto $A\backslash Q$, respectively.*

According to the previous theorem, the sets A/Q, $A\backslash Q$ and A/E have the same cardinality. This cardinality will be called the *index* of Q, and it will be denoted by $\mathrm{ind}(Q)$.

If A is a finite set with n elements and a fuzzy quasi-order Q on A is treated as an $n \times n$ fuzzy matrix over $\mathscr{L}$, then Q-aftersets are row vectors, whereas Q-foresets are column vectors of this matrix.

It is important to note that a fuzzy quasi-order on a set A is uniquely determined both by the family of all its aftersets and the family of all its foresets, since it can be reconstructed from these families as follows

$$Q(a,b) = \bigwedge_{c\in A} Q(c,a) \to Q(c,b) = \bigwedge_{c\in A} cQ(a) \to cQ(b) \tag{27}$$

$$= \bigwedge_{c\in A} Q(b,c) \to Q(a,c) = \bigwedge_{c\in A} Qc(b) \to Qc(a), \tag{28}$$

for all $a, b \in A$ (for the proof we refer to [21, 53]). Besides, we have that

$$E_Q(a,b) = \bigwedge_{c\in A} Q(c,a) \leftrightarrow Q(c,b) = \bigwedge_{c\in A} cQ(a) \leftrightarrow cQ(b) \tag{29}$$

$$= \bigwedge_{c\in A} Q(b,c) \leftrightarrow Q(a,c) = \bigwedge_{c\in A} Qc(b) \leftrightarrow Qc(a), \tag{30}$$

for all $a, b \in A$.

For any fuzzy subset f of A, let fuzzy relations Q_f, Q^f, and E_f on A be defined by

$$Q_f(a,b) = f(a) \to f(b), \qquad Q^f(a,b) = f(b) \to f(a), \qquad E_f(a,b) = f(a) \leftrightarrow f(b), \tag{31}$$

for all $a, b \in A$. We have that Q_f and Q^f are fuzzy quasi-orders, and E_f is a fuzzy equivalence on A. In particular, if f is a normalized fuzzy subset of A, then it is an afterset of Q_f, a foreset of Q^f, and an equivalence class of E_f.

For undefined notions and notation from fuzzy set theory we refer to [2, 3], for those from semigroup theory we refer to [11, 24], and for those concerning ordered sets and lattices we refer to [4, 19, 41].

3. Fuzzy ideals, consistent fuzzy subsets and fuzzy quasi-orders

Let S be a semigroup and f a fuzzy subset of S. We say that f is a *fuzzy ideal* of S if $f(a) \vee f(b) \leqslant f(ab)$, for all $a, b \in S$, or equivalently, if $f(a) \leqslant f(ab)$ and $f(b) \leqslant f(ab)$, for all $a, b \in S$, and it is a *consistent fuzzy subset* of S if $f(ab) \leqslant f(a) \wedge f(b)$, for all $a, b \in S$, or equivalently, if $f(ab) \leqslant f(a)$ and $f(ab) \leqslant f(b)$, for all $a, b \in S$.

On the other hand, f is said to be a *fuzzy subsemigroup* of S if $f(a) \wedge f(b) \leqslant f(ab)$, for all $a, b \in S$, and a *completely prime fuzzy subset* of S if $f(ab) \leqslant f(a) \vee f(b)$, for all $a, b \in S$.

Lastly, f is a *completely prime fuzzy ideal* of S if it is both a fuzzy ideal and a completely prime fuzzy subset, i.e., if $f(ab) = f(a) \vee f(b)$, for all $a, b \in S$, and it is a *fuzzy filter* of S if $f(ab) = f(a) \wedge f(b)$, for all $a, b \in S$.

It is easy to verify that f is both a fuzzy ideal and a consistent fuzzy subset if and only if it is a *fuzzy point*, i.e., if there exists $\lambda \in L$ such that $f(a) = \lambda$, for every $a \in S$. Similarly, f is both a fuzzy subsemigroup and a completely prime fuzzy subset if and only if it is a fuzzy point. It should be noted that fuzzy filters can be viewed as homomorphisms of S into the meet-subsemilattice of $\mathscr{L}$, and completely prime fuzzy ideals as homomorphisms of S into the join-subsemilattice of $\mathscr{L}$.

A slight modification of the definition of a completely prime fuzzy subset gives the notion of a completely semiprime fuzzy subset. Namely, a fuzzy subset f of a semigroup S is said to be a *completely semiprime fuzzy subset* of S if $f(a^2) \leqslant f(a)$, for every $a \in S$. If f is both a fuzzy ideal and a completely semiprime fuzzy subset, then it is said to be a *completely semiprime fuzzy ideal* of S. Clearly, every completely prime fuzzy subset is completely semiprime.

Let S^1 denote the semigroup $S \cup \{1\}$ arising from S by the adjunction of an identity element 1, unless S already has an identity, in which case $S^1 = S$. The *division relation* on a semigroup S is a crisp relation $D : S \times S \to \{0, 1\} \subseteq L$ defined by

$$D(a, b) = \begin{cases} 1 & \text{if } b = paq, \text{ for some } p, q \in S^1, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

The first theorem in this section associates fuzzy ideals, consistent fuzzy subsets and fuzzy quasi-orders on a semigroup.

Theorem 3.1. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *$f \circ Q$ is a fuzzy ideal of S, for every fuzzy subset f of S;*

(ii) *f/Q is a fuzzy ideal of S, for every fuzzy subset f of S;*

(iii) *every Q-afterset is a fuzzy ideal of S;*

(iv) *Q satisfies the inequality*

$$Q(c, a) \vee Q(c, b) \leqslant Q(c, ab), \qquad \text{for all } a, b, c \in S; \tag{33}$$

(v) *$Qa \vee Qb \leqslant Qab$, for all $a, b \in S$;*

(vi) *$Q \circ f$ is a consistent fuzzy subset of S, for every fuzzy subset f of S;*

(vii) *$Q\backslash f$ is a consistent fuzzy subset of S, for every fuzzy subset f of S;*

(viii) *every Q-foreset is a consistent fuzzy subset of S;*

(ix) *Q satisfies the inequality*

$$Q(ab, c) \leqslant Q(a, c) \wedge Q(b, c), \qquad \text{for all } a, b, c \in S; \tag{34}$$

(x) *$abQ \leqslant aQ \wedge bQ$, for all $a, b \in S$;*

(xi) *Q satisfies the equalities*

$$Q(a, ab) = Q(b, ab) = 1, \qquad \text{for all } a, b \in S; \tag{35}$$

(xii) *Q contains the division relation on S.*

Proof. It is enough to prove sequences of implications (i)⇒(iii)⇒(iv)⇒(i), (ix)⇒(ii)⇒(iii) and (v)⇒(xi)⇒(iv), and the equivalences (iv)⇔(v), since (vi)⇒(viii)⇒(ix)⇒(vi), (iv)⇒(vii)⇒(viii) and (ix)⇔(x) can be proved in a similar manner. The verification of (xi)⇔(xii) is straightforward.

(i)⇒(iii). Consider an arbitrary $a \in S$. For each $b \in S$ we have that

$$((aQ) \circ Q)(b) = \bigvee_{c \in S} (aQ)(c) \otimes Q(c,b) = \bigvee_{c \in S} Q(a,c) \otimes Q(c,b) = Q^2(a,b) = Q(a,b) = (aQ)(b),$$

so $(aQ) \circ Q = aQ$, and according to (i), aQ is a fuzzy ideal of S.

(iii)⇒(iv). This follows directly from definitions of a Q-afterset and a fuzzy ideal.

(iv)⇒(i). Notice first that (33) is equivalent to $Q(a,b) \leqslant Q(a,bc)$ and $Q(a,c) \leqslant Q(a,bc)$, for all $a,b,c \in S$. Consider now an arbitrary fuzzy subset f of S and arbitrary $a,b \in S$. Then

$$(f \circ Q)(a) = \bigvee_{c \in S} f(c) \otimes Q(c,a) \leqslant \bigvee_{c \in S} f(c) \otimes Q(c,ab) = (f \circ Q)(ab),$$

and similarly, $(f \circ Q)(b) \leqslant (f \circ Q)(ab)$. Therefore, we conclude that $f \circ Q$ is a fuzzy ideal of S.

(iv)⇔(v). This equivalence can be easily verified.

(ix)⇒(ii). Since the residuum operation is antitone in the first argument, for any fuzzy subset f of S and all $a,b \in S$ we obtain that

$$(f/Q)(a) = \bigwedge_{c \in S} Q(a,c) \to f(c) \leqslant \bigwedge_{c \in S} Q(ab,c) \to f(c) = (f/Q)(ab),$$

and

$$(f/Q)(b) = \bigwedge_{c \in S} Q(b,c) \to f(c) \leqslant \bigwedge_{c \in S} Q(ab,c) \to f(c) = (f/Q)(ab).$$

Hence, f/Q is a fuzzy ideal of S.

(ii)⇒(iii). Consider arbitrary $a,b \in S$. For each $c \in S$ we have that $Q(a,b) \otimes Q(b,c) \leqslant Q(a,c)$, which is equivalent to $Q(a,b) \leqslant Q(b,c) \to Q(a,c)$, whence it follows

$$Q(a,b) \leqslant \bigwedge_{c \in S} Q(b,c) \to Q(a,c) \leqslant Q(b,b) \to Q(a,b) = 1 \to Q(a,b) = Q(a,b),$$

and therefore,

$$Q(a,b) = \bigwedge_{c \in S} Q(b,c) \to Q(a,c).$$

Now we obtain that

$$((aQ)/Q)(b) = \bigwedge_{c \in S} Q(b,c) \to (aQ)(c) = \bigwedge_{c \in S} Q(b,c) \to Q(a,c) = Q(a,b) = (aQ)(b).$$

Hence, for each $a \in S$ we have that $(aQ)/Q = aQ$, and according to (ii), aQ is a fuzzy ideal of S.

(v)⇒(xi). Consider arbitrary $a,b \in S$. According to (v) we have that $Qa \leqslant Qab$, which implies that

$$1 = Qa(a) \leqslant Qab(a) = Q(a,ab).$$

Thus, $Q(a,ab) = 1$. In the exactly same way we show that $Q(b,ab) = 1$.

(xi)⇒(iv). Let $a,b,c \in S$. Because of the assumption $Q(b,bc) = 1$ and the transitivity of Q, we get

$$Q(c,a) = Q(c,a) \otimes Q(a,ab) \leqslant Q(c,ab),$$

and analogously $Q(c,b) \leqslant Q(c,ab)$. Thus, $Q(c,a) \vee Q(c,b) \leqslant Q(c,ab)$.

This completes the proof of the theorem. □

A fuzzy quasi-order on a semigroup S satisfying any of the twelve equivalent conditions of Theorem 3.1 (particularly the condition (35)) will be called a *positive fuzzy quasi-order*.

Let us note that condition (xi) of Theorem 3.1 means that the crisp part of Q is a positive crisp quasi-order (cf. [14]). In other words, a fuzzy quasi-order Q on a semigroup S is positive if and only if its crisp part is a positive crisp quasi-order on S.

Theorem 3.2. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *$f \circ Q$ is a completely prime fuzzy subset of S, for every fuzzy subset f of S;*

(ii) *every Q-afterset is a completely prime fuzzy subset of S;*

(iii) *Q satisfies the inequality*

$$Q(c, ab) \leqslant Q(c, a) \vee Q(c, b), \qquad \text{for all } a, b, c \in S; \tag{36}$$

(iv) *$Qab \leqslant Qa \vee Qb$, for all $a, b \in S$;*

(v) *Q satisfies the equality*

$$Q(ab, a) \vee Q(ab, b) = 1, \qquad \text{for all } a, b \in S. \tag{37}$$

Proof. The implication (i)⇒(ii) can be proved in the same way as (i)⇒(iii) in Theorem 3.1, and the implication (ii)⇒(iii) and the equivalence (iii)⇔(iv) are straightforward.

(iii)⇒(i). Consider an arbitrary fuzzy subset f of S and arbitrary $a, b \in S$. According to (36) and (8) we have that

$$\begin{aligned}(f \circ Q)(ab) &= \bigvee_{c\in S} f(c) \otimes Q(c, ab) \leqslant \bigvee_{c\in S} f(c) \otimes \big(Q(c, a) \vee Q(c, b)\big) = \bigvee_{c\in S} \big(f(c) \otimes Q(c, a)\big) \vee \big(f(c) \otimes Q(c, b)\big) \\ &= \Big(\bigvee_{c\in S} (f(c) \otimes Q(c, a)\Big) \vee \Big(\bigvee_{c\in S} f(c) \otimes Q(c, b)\Big) = (f \circ Q)(a) \vee (f \circ Q)(b),\end{aligned}$$

and thus, $f \circ Q$ is a completely prime fuzzy subset of S.

(iii)⇒(v). For arbitrary $a, b \in S$ we have that

$$1 = Q(ab, ab) \leqslant Q(ab, a) \vee Q(ab, b),$$

so we conclude that (37) holds.

(v)⇒(iii). Consider arbitrary $a, b, c \in S$. Then

$$\begin{aligned}Q(c, ab) &= Q(c, ab) \otimes 1 = Q(c, ab) \otimes \big(Q(ab, a) \vee Q(ab, b)\big) = \big(Q(c, ab) \otimes Q(ab, a)\big) \vee \big(Q(c, ab) \otimes Q(ab, b)\big) \\ &\leqslant Q(c, a) \vee Q(c, b),\end{aligned}$$

and hence, we have that (36) holds. □

A fuzzy quasi-order on a semigroup S satisfying any of the equivalent conditions of Theorem 3.2 (particularly the condition (36)) is said to have the *cp-property* (the abbreviation for *complete primeness property*).

Theorem 3.3. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *every Q-foreset is a fuzzy subsemigroup of S;*

(ii) *Q satisfies the inequality*

$$Q(a, c) \wedge Q(b, c) \leqslant Q(ab, c), \qquad \text{for all } a, b, c \in S; \tag{38}$$

(iii) *$aQ \wedge bQ \leqslant abQ$, for all $a, b \in S$.*

Proof. The equivalence (i)⇔(ii) follows directly from definitions of a Q-foreset and a fuzzy subsemigroup, and the equivalence (ii)⇔(iii) is straightforward. □

A fuzzy quasi-order on a semigroup satisfying the inequality (38) is said to have the *cm-property* (the abbreviation for *common multiple property*).

It is worth emphasizing once again that the inequalities (33) and (34) are equivalent to each other. However, we will show that the reverse inequalities (36) and (38) are not necessarily equivalent.

Theorem 3.4. *Let Q be a fuzzy quasi-order on a semigroup S.*
If Q has the cp-property, then it also has the cm-property, but the reverse implication does not necessarily hold.

Proof. Let Q have the cp-property. According to this property and equalities (27) and (10) we have that

$$\begin{aligned} Q(a,c) \wedge Q(b,c) &= \Big(\bigwedge_{d\in S} Q(d,a) \to Q(d,c)\Big) \wedge \Big(\bigwedge_{d\in S} Q(d,b) \to Q(d,c)\Big) \\ &= \bigwedge_{d\in S}\Big(Q(d,a) \to Q(d,c)\Big) \wedge \Big(Q(d,b) \to Q(d,c)\Big) = \bigwedge_{d\in S}\Big(Q(d,a) \vee Q(d,b)\Big) \to Q(d,c) \\ &\leqslant \bigwedge_{d\in S} Q(d,ab) \to Q(d,c) = Q(ab,c), \end{aligned}$$

which means that Q has the cm-property.

On the other hand, in any commutative semigroup the division relation has the cm-property. However, if S is a null semigroup, i.e., S has a zero 0 and $xy = 0$, for all $x, y \in S$, and S has at least two non-zero elements a and b, then the division relation on S does not have the cp-property since

$$D(ab,a) \vee D(ab,b) = D(0,a) \vee D(0,b) = 0.$$

Therefore, the cm-property does not necessarily imply the cp-property. □

The next theorem characterizes positive fuzzy quasi-orders having the cm-property.

Theorem 3.5. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *every Q-foreset is a fuzzy filter of S;*

(ii) *Q is positive and has the cm-property, that is, it satisfies the equality*

$$Q(ab,c) = Q(a,c) \wedge Q(b,c), \qquad \textit{for all } a,b,c \in S; \tag{39}$$

(iii) *$aQ \wedge bQ = abQ$, for all $a, b \in S$.*

Proof. This theorem follows directly from Theorems 3.1 and 3.3. □

A fuzzy quasi-order Q on a set A is said to be *prelinear* if for arbitrary $a, b \in A$ we have $Q(a,b) \vee Q(b,a) = 1$, and it is said to be *linear* if for all $a, b \in A$ we have $Q(a,b) = 1$ or $Q(b,a) = 1$. It is clear that any linear fuzzy quasi-order is prelinear, and if the underlying complete residuated lattice $\mathscr{L}$ is linearly ordered, then these two notions coincide.

Now we characterize positive fuzzy quasi-orders having the cp-property.

Theorem 3.6. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *$f \circ Q$ is a completely prime fuzzy ideal of S, for every fuzzy subset f of S;*

(ii) *every Q-afterset is a completely prime fuzzy ideal of S;*

(iii) *Q is positive and has the cp-property;*

(iv) *Q is positive, prelinear and has the cm-property;*

(v) *Q satisfies the equalities*

$$Q(ab, c) = Q(a, c) \wedge Q(b, c), \quad Q(c, ab) = Q(c, a) \vee Q(c, b), \qquad \text{for all } a, b, c \in S; \tag{40}$$

(vi) $Qab = Qa \vee Qb$, *for all* $a, b \in S$.

Proof. The equivalence of statements (i), (ii), (iii) and (vi) follows directly from Theorems 3.1 and 3.2, while the equivalence (iii)⇔(v) follows from Theorem 3.4. It remains to prove the equivalence of (iii) and (iv).

(iii)⇔(iv). If Q is positive and has the cm-property, then by (39) we obtain that

$$Q(ab, b) \vee Q(ab, a) = \big(Q(a, b) \wedge Q(b, b)\big) \vee \big(Q(a, a) \wedge Q(b, a)\big) = Q(a, b) \vee Q(b, a). \tag{41}$$

Now, if Q is positive and has the cp-property, then it also has the cm-property, so by (37) and (41) it follows that Q is prelinear. Conversely, if Q is positive, prelinear and has the cm-property, then (41) yields (37), so we conclude that Q has the cp-property. □

An important connection between fuzzy ideals and consistent fuzzy subsets, completely prime fuzzy subsets and fuzzy subsemigroups, as well as between completely prime fuzzy ideals and fuzzy filters is given by the following theorem.

Theorem 3.7. *Let S be a semigroup, let f be a fuzzy subset of S and $\lambda \in L$ a fixed scalar, and let a fuzzy subset g of S be defined by $g(a) = f(a) \to \lambda$, for each $a \in S$. Then the following is true:*

(a) *If f is a fuzzy ideal then g is a consistent fuzzy subset;*

(b) *If f is a consistent fuzzy subset then g is a fuzzy ideal;*

(c) *If f is a completely prime fuzzy subset then g is a fuzzy subsemigroup;*

(d) *If f is a completely prime fuzzy ideal then g is a fuzzy filter.*

If the membership values are taken in a linearly ordered complete residuated lattice, then the following is also true:

(e) *If f is a fuzzy subsemigroup then g is a consistent fuzzy subset;*

(f) *If f is a fuzzy filter then g is a completely prime fuzzy ideal.*

Proof. a) Let f be a fuzzy ideal of S. According to (6) and (10), for arbitrary $a, b \in S$ we have that

$$g(ab) = f(ab) \to \lambda \leqslant (f(a) \vee f(b)) \to \lambda = (f(a) \to \lambda) \wedge (f(b) \to \lambda) = g(a) \wedge g(b),$$

and thus, g is a consistent fuzzy subset of S.

In a similar way we prove c) and d).

b) Let f be a consistent fuzzy subset of S. According to (6) and (11), for arbitrary $a, b \in S$ we get

$$g(ab) = f(ab) \to \lambda \geqslant (f(a) \wedge f(b)) \to \lambda \geqslant (f(a) \to \lambda) \vee (f(b) \to \lambda) = g(a) \vee g(b),$$

and we conclude that g is a fuzzy ideal of S.

Further, suppose that the underlying complete residuated lattice $\mathscr{L}$ is linearly ordered.

e) Let f be a fuzzy subsemigroup of S. Consider arbitrary $a, b \in S$. Since the underlying complete residuated lattice $\mathscr{L}$ is linearly ordered, we can distinguish two cases: $f(a) \leqslant f(b)$ and $f(b) \leqslant f(a)$.

If $f(a) \leqslant f(b)$, then $f(a) = f(a) \wedge f(b) \leqslant f(ab)$, and according to (6) we get

$$g(b) = f(b) \to \lambda \leqslant f(a) \to \lambda = g(a),$$

and

$$g(ab) = f(ab) \to \lambda \leqslant f(a) \to \lambda = g(a) = g(a) \vee g(b).$$

In the same way we show that $f(b) \leqslant f(a)$ implies $g(ab) \leqslant g(a) \vee g(b)$. Thus, we conclude that g is a completely prime fuzzy subset of S.

The claim f) follows directly from b) and e). □

It should be noted that the *negation* in a residuated lattice is given by $\neg x = x \to 0$, which means that the fuzzy set g, defined in the previous theorem by $g(a) = f(a) \to \lambda$, is some kind of a *generalized complement* of f (the usual complement is obtained for $\lambda = 0$).

We prove the following theorem in an analogous way as the previous one.

Theorem 3.8. *Let S be a semigroup, let f be a fuzzy subset of S and $\lambda \in L$ a fixed scalar, and let a fuzzy subset g of S be defined by $g(a) = \lambda \to f(a)$, for each $a \in S$. Then the following is true:*

(a) *If f is a fuzzy ideal (resp. consistent fuzzy subset, fuzzy subsemigroup, fuzzy filter) of S, then g has the same property.*

If the membership values are taken in a linearly ordered complete residuated lattice, then the following is also true:

(b) *If f is a completely prime fuzzy subset (resp. completely prime fuzzy ideal) of S, then g has the same property.*

This section will be completed by a theorem which characterizes fuzzy quasi-orders whose aftersets are completely semiprime fuzzy subsets and ideals. The theorem can be proved in a similar way as Theorem 3.2, so its proof will be omitted.

Theorem 3.9. *Let Q be a fuzzy quasi-order on a semigroup S. Then the following statements are equivalent:*

(i) *$f \circ Q$ is a completely semiprime fuzzy subset of S, for every fuzzy subset f of S;*

(ii) *every Q-afterset is a completely semiprime fuzzy subset of S;*

(iii) *Q satisfies the inequality*

$$Q(b, a^2) \leqslant Q(b, a), \qquad \text{for all } a, b \in S; \tag{42}$$

(iv) *$Qa^2 \leqslant Qa$, for each $a \in S$;*

(v) *Q satisfies the inequality*

$$Q(a, b) \leqslant Q(a^2, b), \qquad \text{for all } a, b \in S; \tag{43}$$

(vi) *Q satisfies the equality*

$$Q(a^2, a) = 1, \qquad \text{for each } a \in S. \tag{44}$$

In addition, if Q is positive, then the term "fuzzy set" in (i) *and* (ii) *can be replaced by the term "fuzzy ideal", and* (44) *is equivalent to*

$$Q(a^n, a) = 1, \qquad \text{for all } a \in S \text{ and } n \in \mathbb{N}. \tag{45}$$

A fuzzy quasi-order satisfying any of the equivalent conditions of Theorem 3.9 is called *lower-potent*. Each fuzzy quasi-order having the cm-property is lower-potent, because (38) implies (43).

4. Eigen spaces of a positive fuzzy quasi-order

Let R be a fuzzy relation on a set A. A fuzzy subset f of A is called a *left eigen fuzzy set* of R if $f \circ R = f$, and the set of all left eigen fuzzy sets of R is called the *left eigen space* of R. Similarly, f is called a *right eigen fuzzy set* of R if $R \circ f = f$, and the set of all right eigen fuzzy sets of R is called the *right eigen space* of R. Here we consider (left and right) eigen spaces of positive fuzzy quasi-orders.

Let us consider the complete lattice $\mathscr{F}(A) = (L^A, \vee, \wedge, \emptyset, A)$ of all fuzzy subsets of a set A with membership values in the complete residuated lattice $\mathscr{L}$. For any $\lambda \in L$ and $f \in L^A$ let us define the *scalar multiplication*

λf as follows: $\lambda f(a) = \lambda \otimes f(a)$, for any $a \in A$. With respect to this scalar multiplication, the monoid $(L^A, \vee, \emptyset)$ forms a left L-semimodule (cf. [22, 23]), i.e., for all $\lambda, \lambda_1, \lambda_2 \in L$ and $f, f_1, f_2 \in L^A$ the following is true:

$$(\lambda_1 \otimes \lambda_2) f = \lambda_1(\lambda_2 f), \tag{46}$$

$$\lambda(f_1 \vee f_2) = \lambda f_1 \vee \lambda f_2, \tag{47}$$

$$(\lambda_1 \vee \lambda_2) f = \lambda_1 f \vee \lambda_2 f, \tag{48}$$

$$1f = f, \tag{49}$$

$$\lambda\emptyset = 0f = \emptyset. \tag{50}$$

It is easy to see that (47) and (48) hold for arbitrary joins, including the infinite ones. The lattice $\mathscr{F}(A)$ equipped with this scalar multiplication will be denoted by $\mathscr{F}_\otimes(A)$ and called the *$\mathscr{L}$-lattice of fuzzy subsets* of the set A. A fuzzy subset $f \in L^A$ is said to be a *linear combination* of fuzzy subsets $f_i \in L^A$ $(i \in I)$ if there exist scalars $\lambda_i \in L$ $(i \in I)$ such that f is expressed in the form

$$f = \bigvee_{i \in I} \lambda_i f_i. \tag{51}$$

Any subset of L^A which is closed under scalar multiplication and arbitrary meets and joins, and it contains the least and the greatest element of $\mathscr{F}(A)$ will be called a *complete $\mathscr{L}$-sublattice* of $\mathscr{F}_\otimes(A)$.

A subset $\mathscr{X}$ of L^A is called a *closure system* in $\mathscr{F}(A)$ if $A \in \mathscr{X}$ and $\mathscr{X}$ is closed under arbitrary meets. If $\mathscr{X}$ is a closure system, then for any $a \in A$ the family $\{f \in \mathscr{X} \mid f(a) = 1\}$ is non-empty, since it contains A, so

$$\mathscr{X}_a = \bigwedge \Big\{ f \in \mathscr{X} \;\Big|\; f(a) = 1 \Big\} \in \mathscr{X}. \tag{52}$$

The fuzzy set $\mathscr{X}_a$ will be called the *principal element* of $\mathscr{C}$ generated by a, and the set $\mathscr{P}(\mathscr{X}) = \{\mathscr{X}_a \mid a \in A\}$ will be called the *principal part* of $\mathscr{X}$. If for arbitrary $a, b \in A$ we have that

$$\mathscr{X}_a(b) = I(\mathscr{X}_a, \mathscr{X}_b) = \bigwedge_{c \in A} \mathscr{X}_a(c) \to \mathscr{X}_b(c), \tag{53}$$

then we say that the principal elements of $\mathscr{X}$ satisfy the *inclusion property*, and if

$$\mathscr{X}_a(b) = I(\mathscr{X}_b, \mathscr{X}_a) = \bigwedge_{c \in A} \mathscr{X}_b(c) \to \mathscr{X}_a(c), \tag{54}$$

then we say that the principal elements of $\mathscr{C}$ satisfy the *reverse inclusion property*.

Let $I(S)$ denote the set of all fuzzy ideals of a semigroup S. It is easy to verify that $I(S)$ is a complete $\mathscr{L}$-sublattice of $\mathscr{F}_\otimes(S)$, and this $\mathscr{L}$-lattice is denoted by $\mathscr{I}_\otimes(S)$ and called the *$\mathscr{L}$-lattice of fuzzy ideals* of S.

For any positive fuzzy quasi-order Q on a semigroup S we define a collection $\mathscr{I}_Q$ of fuzzy ideals of S by

$$\mathscr{I}_Q = \Big\{ f \in I(S) \;\Big|\; f \circ Q = f \Big\}.$$

In other words, $\mathscr{I}_Q$ is the left eigen space of Q.

Left eigen spaces of positive fuzzy quasi-orders are characterized by the next theorem. The proof of the first part of the theorem is similar to the proof of Theorem 6.4 from [29], but for the sake of completeness we give a a full proof.

Theorem 4.1. *Let $\mathscr{I}$ be a collection of fuzzy ideals of a semigroup S. Then there exists a positive fuzzy quasi-order Q on S such that $\mathscr{I} = \mathscr{I}_Q$ if and only if the following hold:*

(1) *$\mathscr{I}$ is a complete $\mathscr{L}$-sublattice of $\mathscr{I}_\otimes(S)$;*

(2) *every element of $\mathscr{I}$ can be expressed as a linear combination of principal elements of $\mathscr{I}$;*

(3) *principal elements of $\mathscr{I}$ satisfy the reverse inclusion property.*

Furthermore, the function $Q \mapsto \mathscr{I}_Q$ is a dual order isomorphism of the lattice of positive fuzzy quasi-orders on S onto the partially ordered set of all complete $\mathscr{L}$-sublattices of $\mathscr{I}_\otimes(S)$ satisfying conditions (2) *and* (3)*, i.e.,*

$$Q_1 \leqslant Q_2 \quad\Leftrightarrow\quad \mathscr{I}_{Q_2} \subseteq \mathscr{I}_{Q_1}, \tag{55}$$

for arbitrary positive fuzzy quasi-orders Q_1 and Q_2 on S.

Proof. Let $\mathscr{I} = \mathscr{I}_Q$, for some positive fuzzy quasi-order Q on S. For any family $\{f_i\}_{i\in I} \subseteq \mathscr{I}_Q$ we have that

$$\Big(\bigvee_{i\in I} f_i\Big) \circ Q = \bigvee_{i\in I} f_i \circ Q = \bigvee_{i\in I} f_i, \qquad \Big(\bigwedge_{i\in I} f_i\Big) \circ Q \leqslant \bigwedge_{i\in I} f_i \circ Q = \bigwedge_{i\in I} f_i \leqslant \Big(\bigwedge_{i\in I} f_i\Big) \circ Q,$$

which means that $\mathscr{I}_Q$ is closed under arbitrary joins and meets, and it is clear that it is closed under scalar multiplication. Therefore, $\mathscr{I}_Q$ is a complete $\mathscr{L}$-sublattice of $\mathscr{I}_\otimes(S)$.

Next, for arbitrary $a, b \in S$ we have that

$$(aQ \circ Q)(b) = \bigvee_{c\in S} aQ(c) \otimes Q(c,b) = \bigvee_{c\in S} Q(a,c) \otimes Q(c,b) = Q(a,b) = aQ(b),$$

so $aQ \circ Q = aQ$. This means that $aQ \in \mathscr{I}_Q$, for each $a \in S$. It is clear that $aQ(a) = 1$, for every $a \in S$. Let $a \in S$ and let $f \in \mathscr{I}_Q$ such that $f(a) = 1$. Then for any $b \in S$ we have that

$$f(b) = (f \circ Q)(b) = \bigvee_{c\in S} f(c) \otimes Q(c,b) \geqslant f(a) \otimes Q(a,b) = Q(a,b) = aQ(b),$$

which implies that $aQ \leqslant f$. Therefore, $aQ = \mathscr{I}_a$, for every $a \in S$. Now, according to (28), for arbitrary $a, b \in S$ we obtain that

$$\bigwedge_{c\in S} \mathscr{I}_b(c) \to \mathscr{I}_a(c) = \bigwedge_{c\in S} bQ(c) \to aQ(c) = \bigwedge_{c\in S} Qc(b) \to Qc(a) = Q(a,b) = \mathscr{I}_a(b),$$

which means that principal elements of $\mathscr{I}_Q$ satisfy the reverse inclusion property.

Finally, for arbitrary $f \in \mathscr{I}_Q$ and $a \in A$ we have that

$$f(a) = (f \circ Q)(a) = \bigvee_{b\in S} f(b) \otimes Q(b,a) = \bigvee_{b\in S} f(b) \otimes \mathscr{I}_b(a) = \Big(\bigvee_{b\in S} f(b)\mathscr{I}_b\Big)(a),$$

and this implies

$$f = \bigvee_{b\in S} f(b)\mathscr{I}_b.$$

Thus, every element of $\mathscr{I}$ can be expressed as a linear combination of principal elements of $\mathscr{I}$. Accordingly, we have proved that (1), (2) and (3) hold.

Conversely, let $\mathscr{I}$ satisfy conditions (1), (2) and (3). Let us define a fuzzy relation Q on S by

$$Q(a,b) = \mathscr{I}_a(b) = \bigwedge_{c\in S} \mathscr{I}_b(c) \to \mathscr{I}_a(c), \tag{56}$$

for all $a, b \in S$. It is clear that Q is reflexive. Consider arbitrary $a, b, c, d \in S$. According to (7), we have that

$$Q(a,b) \otimes Q(b,c) \leqslant (\mathscr{I}_b(d) \to \mathscr{I}_a(d)) \otimes (\mathscr{I}_c(d) \to \mathscr{I}_b(d)) \leqslant \mathscr{I}_c(d) \to \mathscr{I}_a(d),$$

and since this holds for every $d \in S$, we conclude that

$$Q(a,b) \otimes Q(b,c) \leqslant \bigwedge_{d\in S} \mathscr{I}_c(d) \to \mathscr{I}_a(d) = Q(a,c).$$

Therefore, Q is transitive, i.e., Q is a fuzzy quasi-order. According to (56), for an arbitrary $a \in S$ we have that $\mathscr{I}_a = aQ$, and by Theorem 3.1 we obtain that Q is a positive fuzzy quasi-order.

Consider an arbitrary $f \in \mathscr{I}_Q$, i.e., a fuzzy ideal of S satisfying $f \circ Q = f$. Then for each $a \in S$ we have

$$f(a) = \bigvee_{b\in S} f(b) \otimes Q(b,a) = \bigvee_{b\in S} f(b) \otimes \mathscr{I}_b(a) = \Big(\bigvee_{b\in S} f(b)\mathscr{I}_b\Big)(a),$$

whence

$$f = \bigvee_{b\in S} f(b)\mathscr{I}_b.$$

Since $\mathscr{I}_b \in \mathscr{I}$, for every $b \in S$, and $\mathscr{I}$ is closed under scalar products and arbitrary joins, we conclude that $f \in \mathscr{I}$. Therefore, $\mathscr{I}_Q \subseteq \mathscr{I}$.

On the other hand, let $f \in \mathscr{I}$. According to (2), there are scalars $\lambda_c \in L$, $c \in S$, such that

$$f = \bigvee_{c\in S} \lambda_c \mathscr{I}_c,$$

and for any $a \in S$ we have that

$$\begin{aligned}(f \circ Q)(a) &= \bigvee_{b\in S} f(b) \otimes Q(b,a) = \bigvee_{b\in S}\Big(\bigvee_{c\in S} \lambda_c \otimes \mathscr{I}_c(b)\Big) \otimes Q(b,a) = \bigvee_{c\in S} \lambda_c \otimes \Big(\bigvee_{b\in S} Q(c,b) \otimes Q(b,a)\Big)\\ &= \bigvee_{c\in S} \lambda_c \otimes Q(c,a) = \bigvee_{c\in S} \lambda_c \otimes \mathscr{I}_c(a) = f(a).\end{aligned}$$

Thus, $f \in \mathscr{I}_Q$, so $\mathscr{I} \subseteq \mathscr{I}_Q$. Consequently, we have proved that $\mathscr{I} = \mathscr{I}_Q$.

Consider arbitrary positive fuzzy quasi-orders Q_1 and Q_2 on S. If $Q_1 \leqslant Q_2$ and $f \in \mathscr{I}_{Q_2}$, i.e., $f \circ Q_2 = f$, then $f \leqslant f \circ Q_1 \leqslant f \circ Q_2 = f$, so $f \circ Q_1 = f$, and hence, $f \in \mathscr{I}_{Q_1}$. Conversely, let $\mathscr{I}_{Q_2} \subseteq \mathscr{I}_{Q_1}$. Then for each $a \in S$ we have that $aQ_2 \in \mathscr{I}_{Q_2} \subseteq \mathscr{I}_{Q_1}$, which means that $aQ_2 \circ Q_1 = aQ_2$. Thus, for arbitrary $a, b \in S$ we have that

$$Q_2(a,b) = aQ_2(b) = (aQ_2 \circ Q_1)(b) = \bigvee_{c\in S} aQ_2(c) \otimes Q_1(c,b) \geqslant aQ_2(a) \otimes Q_1(a,b) = Q_1(a,b),$$

whence $Q_1 \leqslant Q_2$. Therefore, we have proved that (57) holds. □

It is worth noting that in the crisp case conditions (2) and (3) in Theorem 4.1 are trivially satisfied. Namely, in this case condition (3) means that $b \in \mathscr{I}_a$ if and only if $\mathscr{I}_b \subseteq \mathscr{I}_a$, and condition (2) means that every $f \in \mathscr{I}$ is the union of all principal elements of $\mathscr{I}$ contained in f. Moreover, in the crisp case the only scalars are 0 and 1, and according to (49) and (50), closeness under scalar multiplication is redundant.

Next we consider the lattice of consistent fuzzy subsets. For any non-empty set A, let $\mathscr{F}'_\otimes(A)$ denote the lattice $\mathscr{F}(A)$ equipped with the scalar multiplication written on the right (a right L-semimodule), i.e., $f\lambda(a) = f(a) \otimes \lambda$, for all $f \in L^A$, $\lambda \in L$ and $a \in A$. It is easy to see that in this case conditions dual to (46)–(50) are satisfied.

Let $C(S)$ denote the set of all consistent fuzzy subsets of a semigroup S. It is easy to verify that $C(S)$ is a complete $\mathscr{L}$-sublattice of $\mathscr{F}'_\otimes(S)$, and this $\mathscr{L}$-lattice will be denoted by $\mathscr{C}_\otimes(S)$ and called the *$\mathscr{L}$-lattice of consistent fuzzy subsets* of S.

For any positive fuzzy quasi-order Q on S we define a collection $\mathscr{C}_Q$ of consistent fuzzy subsets of S by

$$\mathscr{C}_Q = \Big\{f \in C(S) \;\Big|\; Q \circ f = f\Big\}.$$

In other words, $\mathscr{C}_Q$ is the right eigen space of Q.

The proof of the following theorem is similar to the proof of the Theorem 4.1 and it will be omitted.

Theorem 4.2. *Let $\mathscr{C}$ be a collection of consistent fuzzy subsets of a semigroup S. Then there exists a positive fuzzy quasi-order Q on S such that $\mathscr{C} = \mathscr{C}_Q$ if and only if the following hold:*

(1) *$\mathscr{C}$ is a complete $\mathscr{L}$-sublattice of $\mathscr{C}_\otimes(S)$;*

(2) *every element of $\mathscr{C}$ can be expressed as a linear combination of principal elements of $\mathscr{C}$;*

(3) *principal elements of $\mathscr{C}$ satisfy the inclusion property.*

Furthermore, the function $Q \mapsto \mathscr{C}_Q$ is a dual order isomorphism of the lattice of positive fuzzy quasi-orders on S onto the partially ordered set of all complete $\mathscr{L}$-sublattices of $\mathscr{C}_\otimes(S)$ satisfying conditions (2) and (3), i.e.,

$$Q_1 \leqslant Q_2 \quad\Leftrightarrow\quad \mathscr{C}_{Q_2} \subseteq \mathscr{C}_{Q_1}, \tag{57}$$

for arbitrary positive fuzzy quasi-orders Q_1 and Q_2 on S.

For any fuzzy subset f of a semigroup S and any fuzzy quasi-order Q on S, by the reflexivity of Q and the residuation property (26) we get the following chain of equivalences:

$$f \circ Q = f \quad\Leftrightarrow\quad f \circ Q \leqslant f \quad\Leftrightarrow\quad f \leqslant f/Q \quad\Leftrightarrow\quad f = f/Q.$$

Therefore, $f \circ Q = f$ if and only if $f = f/Q$, and analogously, $Q \circ f = f$ if and only if $f = Q\backslash f$. This means that for any positive fuzzy quasi-order Q on S we have that

$$\mathscr{I}_Q = \Big\{ f \in I(S) \;\Big|\; f \circ Q = f \Big\} = \Big\{ f \in I(S) \;\Big|\; f/Q = f \Big\}, \tag{58}$$

$$\mathscr{C}_Q = \Big\{ f \in C(S) \;\Big|\; Q \circ f = f \Big\} = \Big\{ f \in C(S) \;\Big|\; Q\backslash f = f \Big\}. \tag{59}$$

It is important to note that functions $f \mapsto f \circ Q$ and $f \mapsto Q \circ f$ are closure operators on the lattice $\mathscr{F}(S)$, and $f \mapsto f/Q$ and $f \mapsto Q\backslash f$ are opening operators on $\mathscr{F}(S)$ (under assumption that Q is a fuzzy quasi-order, cf. [5, 7, 29]), so $\mathscr{I}_Q$ is the set of all closed elements w.r.t. $f \mapsto f \circ Q$ and open elements w.r.t. $f \mapsto f/Q$, whereas $\mathscr{C}_Q$ is the set of all closed elements w.r.t. $f \mapsto Q \circ f$ and open elements w.r.t. $f \mapsto Q\backslash f$.

The following theorem characterizes positive fuzzy quasi-orders which determine complete $\mathscr{L}$-sublattices of $\mathscr{F}_\otimes(S)$ consisting of completely semiprime fuzzy ideals.

Theorem 4.3. *Let Q be a positive quasi-fuzzy order on a semigroup S. Then $\mathscr{I}_Q$ consists of completely semiprime fuzzy ideals if and only if Q is lower-potent.*

Proof. If $\mathscr{I}_Q$ consists of completely semiprime fuzzy ideals, then all Q-aftersets are completely semiprime fuzzy ideals, and according to Theorem 3.9 we have that Q is lower-potent.

Conversely, if Q is lower-potent, by Theorem 3.9 it follows that principal elements of $\mathscr{I}_Q$ are completely semiprime fuzzy ideals, and since every element of $\mathscr{I}_Q$ can be represented as a linear combination of principal elements and the set of all completely semiprime fuzzy ideals is closed under linear combinations, we conclude that every element of $\mathscr{I}_Q$ is a completely semiprime fuzzy ideal. □

The following theorem provides the construction of a positive lower-potent fuzzy quasi-order starting from a collection of completely semiprime fuzzy ideals.

Theorem 4.4. *Let $\mathscr{X}$ be a collection of completely semiprime fuzzy ideals of a semigroup S and let Q be a fuzzy relation on S defined by*

$$Q(a,b) = \bigwedge_{f \in \mathscr{X}} f(a) \to f(b), \qquad \text{for all } a, b \in S. \tag{60}$$

Then Q is a positive lower-potent fuzzy quasi-order on S such that $\mathscr{X} \subseteq \mathscr{I}_Q$.

If $\mathscr{X}$ is the collection of all completely semiprime fuzzy ideals of S, then $\mathscr{X} = \mathscr{I}_Q$ and Q is the smallest positive lower-potent fuzzy quasi-order on S.

Proof. In accordance with the remarks made at the end of Section 2 (see (31)) we have that Q is the meet of the family of fuzzy quasi-orders $\{Q_f\}_{f\in\mathscr{X}}$, so Q is also a fuzzy quasi-order.

For a fixed $a \in S$ we have that

$$aQ(b) = \bigwedge_{f\in\mathscr{X}} f(a) \to f(b),$$

for each $b \in S$, and by Theorem 3.8 we obtain that aQ is a fuzzy ideal, for each $a \in S$. Hence, by Theorem 3.1 we conclude that Q is a positive fuzzy quasi-order.

From (4) we conclude that $f(a^2) \to f(a) = 1$, for all $a \in S$ and $f \in \mathscr{X}$, which means that $Q(a^2, a) = 1$, for each $a \in S$. Consequently, Q is a lower-potent fuzzy quasi-order.

According to (9) and (3), for arbitrary $g \in \mathscr{X}$ and $a \in S$ we have that

$$\begin{aligned}(g \circ Q)(a) &= \bigvee_{b\in S} g(b) \otimes Q(b,a) = \bigvee_{b\in S} g(b) \otimes \left(\bigwedge_{f\in\mathscr{X}} f(b) \to f(a) \right) \leqslant \bigvee_{b\in S} \left(\bigwedge_{f\in\mathscr{X}} g(b) \otimes \big(f(b) \to f(a) \big) \right) \\ &\leqslant \bigvee_{b\in S} g(b) \otimes \big(g(b) \to g(a) \big) \leqslant g(a),\end{aligned}$$

which yields $g \circ Q \leqslant g$. Since the opposite inequality is an immediate consequence of the reflexivity of Q, we conclude that $g \circ Q = g$, and thus $g \in \mathscr{I}_Q$, which finally yields $\mathscr{X} \subseteq \mathscr{I}_Q$.

Furthermore, let $\mathscr{X}$ be the collection of all completely semiprime fuzzy ideals of S. As we have just shown, $\mathscr{X} \subseteq \mathscr{I}_Q$, and since Theorem 4.3 implies the opposite inclusion, we conclude that $\mathscr{X} = \mathscr{I}_Q$. Let R be an arbitrary positive lower-potent fuzzy quasi-order on S. Then from Theorem 3.9 it follows that $cR \in \mathscr{X}$, for each $c \in S$, and according to (27) we get

$$Q(a,b) = \bigwedge_{f\in\mathscr{X}} f(a) \to f(b) \leqslant \bigwedge_{c\in S} cR(a) \to cR(b) = R(a,b),$$

for all $a, b \in S$. Therefore, $Q \leqslant R$. This completes the proof of the theorem. □

The next theorem gives new characterizations of positive fuzzy quasi-orders having the cm-property in terms of properties of the corresponding lattices of fuzzy ideals and consistent fuzzy subsets. Note that the theorem is proved under assumption that the underlying complete residuated lattice is linearly ordered, and it is an open problem whether it can be proved without this assumption. However, if we take into account that in the practical application of the theory of fuzzy sets, the most commonly used structures of truth-values are those defined by left-continuous t-norms on the real unit interval, which are linearly ordered, this assumption does not seem too restrictive.

Theorem 4.5. *Let Q be a positive quasi-fuzzy order on a semigroup S, and let the underlying complete residuated lattice $\mathscr{L}$ be linearly ordered. Then the following statements are equivalent:*

(i) *Q has the cm-property;*

(ii) *every principal element of $\mathscr{C}_Q$ is a fuzzy filter;*

(iii) *every element of $\mathscr{I}_Q$ can be expressed as the meet of some family of completely prime fuzzy ideals from $\mathscr{I}_Q$.*

Proof. (i)⇔(ii). This follows directly from Theorem 3.5.

(ii)⇒(iii). Consider an arbitrary $f \in \mathscr{I}_Q$. By (58) it follows that $f = f/Q$, and for each $a \in S$ we have that

$$f(a) = (f/Q)(a) = \bigwedge_{b\in S} Q(a,b) \to f(b) = \bigwedge_{b\in S} Qb(a) \to f(b) = \bigwedge_{b\in S} f_b(a),$$

where for each $b \in S$ we put $f_b(a) = Qb(a) \to f(b)$. By assumption, Qb is a fuzzy filter, and by Theorem 3.7 we obtain that f_b is a completely prime fuzzy ideal of S. Therefore, we have expressed f as the meet of the family $\{f_b\}_{b\in S}$ of completely prime fuzzy ideals of S.

(iii)⇒(i). In order to prove that Q has the cm-property, consider arbitrary $a, b \in S$. By $abQ \in \mathscr{I}_Q$ and the assumption (4) we obtain that

$$abQ = \bigwedge_{i\in I} f_i,$$

where for each $i \in I$, f_i is a completely prime fuzzy ideal of S, and we have that

$$\bigwedge_{i\in I} f_i(ab) = abQ(ab) = Q(ab, ab) = 1,$$

which implies that $f_i(ab) = 1$, for each $i \in I$. Since f_i is a completely prime fuzzy ideal of S we get

$$f_i(a) \vee f_i(b) = f_i(ab) = 1,$$

for each $i \in I$, and as the assumption of the theorem states that $\mathscr{L}$ is linearly ordered, we have that $f_i(a) = 1$ or $f_i(b) = 1$, for any $i \in I$. Now, let us set $I_a = \{i \in I \,|\, f_i(a) = 1\}$ and $I_b = \{i \in I \,|\, f_i(b) = 1\}$. Then $I = I_a \cup I_b$, and without loss of generality we can suppose that $I_a \neq \emptyset$ and $I_b \neq \emptyset$ (we can take S to be one of f_i's). Since aQ and bQ are principal elements of $\mathscr{I}_Q$, we have that $aQ \leqslant f_i$ and $bQ \leqslant f_j$, for all $i \in I_a$ and $j \in I_b$, whence

$$aQ \wedge bQ \leqslant \Big(\bigwedge_{i\in I_a} f_i\Big) \wedge \Big(\bigwedge_{j\in I_b} f_j\Big) = \bigwedge_{i\in I} f_i = abQ.$$

This means that $aQ \wedge bQ \leqslant abQ$, for all $a, b \in S$, and in accordance with Theorem 3.3 we conclude that Q has the cm-property. □

We further prove the following:

Theorem 4.6. *Let $\mathscr{X}$ be a collection of completely prime fuzzy ideals of a semigroup S and let Q be a fuzzy relation on S defined as in (60). Then Q is a positive fuzzy quasi-order on S having the cm-property such that $\mathscr{X} \subseteq \mathscr{I}_Q$.*

If $\mathscr{X}$ is the collection of all completely prime fuzzy ideals of S, then Q is the smallest positive fuzzy quasi-order on S having the cm-property.

Proof. According to Theorem 4.4, Q is a positive lower-potent fuzzy quasi-order and $\mathscr{X} \subseteq \mathscr{I}_Q$.

Let us consider arbitrary $a, b, c \in S$. Using (10) we get

$$\begin{aligned} Q(a,c) \wedge Q(b,c) &= \Big(\bigwedge_{f\in\mathscr{X}} f(a) \to f(c)\Big) \wedge \Big(\bigwedge_{f\in\mathscr{X}} f(b) \to f(c)\Big) = \bigwedge_{f\in\mathscr{X}} \big(f(a) \to f(c)\big) \wedge \big(f(b) \to f(c)\big) \\ &= \bigwedge_{f\in\mathscr{X}} \big(f(a) \vee f(b)\big) \to f(c) = \bigwedge_{f\in\mathscr{X}} f(ab) \to f(c) = Q(ab, c). \end{aligned}$$

Hence, Q has the cm-property.

Furthermore, let $\mathscr{X}$ be the collection of all completely prime fuzzy ideals of S, and let R be an arbitrary positive fuzzy quasi-order on S with the cm-property. According to Theorem 4.5, for any $c \in S$ we have that

$$cR = \bigwedge_{i\in I_c} f_i,$$

where $\{f_i\}_{i\in I_c}$ is some family of completely prime fuzzy ideals. Put $I = \bigcup_{c\in S} I_c$. Using (27), (14) and the fact that $\{f_i\}_{i\in I} \subseteq \mathscr{X}$ we get

$$\begin{aligned} R(a,b) &= \bigwedge_{c\in S} cR(a) \to cR(b) = \bigwedge_{c\in S}\Big(\Big(\bigwedge_{i\in I_c} f_i(a)\Big) \to \Big(\bigwedge_{i\in I_c} f_i(b)\Big)\Big) \geqslant \bigwedge_{c\in S}\bigwedge_{i\in I_c}\big(f_i(a) \to f_i(b)\big) = \bigwedge_{i\in I} f_i(a) \to f_i(b) \\ &\geqslant \bigwedge_{f\in\mathscr{X}} f(a) \to f(b) = Q(a,b). \end{aligned}$$

Thus, we have proved that $Q \leqslant R$, and we conclude that Q is the smallest positive fuzzy quasi-order on S having the cm-property. □

The following theorem gives the construction of a positive fuzzy quasi-order having the cm-property starting from a collection of fuzzy filters.

Theorem 4.7. *Let $\mathscr{X}$ be a collection of fuzzy filters of a semigroup S and let Q be a fuzzy relation on S defined by*

$$Q(a,b) = \bigwedge_{f\in\mathscr{X}} f(b) \to f(a), \qquad \text{for all } a,b \in S. \tag{61}$$

Then Q is a positive fuzzy quasi-order on S having the cm-property such that $\mathscr{X} \subseteq \mathscr{C}_Q$.

If $\mathscr{X}$ is the collection of all fuzzy filters of S, then Q is the smallest positive fuzzy quasi-order on S having the cm-property.

Proof. In an analogous way as in Theorem 4.4 we conclude that Q is a fuzzy quasi-order.

Consider arbitrary $a,b,c \in S$. According to (12) we get

$$\begin{aligned} Q(a,c) \wedge Q(b,c) &= \Big(\bigwedge_{f\in\mathscr{X}} f(c) \to f(a)\Big) \wedge \Big(\bigwedge_{f\in\mathscr{X}} f(c) \to f(b)\Big) = \bigwedge_{f\in\mathscr{X}} \big(f(c) \to f(a)\big) \wedge \big(f(c) \to f(b)\big) \\ &= \bigwedge_{f\in\mathscr{X}} f(c) \to \big(f(a) \wedge f(b)\big) = \bigwedge_{f\in\mathscr{X}} f(c) \to f(ab) = Q(ab,c), \end{aligned}$$

and on the basis of Theorem 3.5 we have that Q is a positive fuzzy quasi-order with the cm-property.

Moreover, in an analogous way as in Theorem 4.4 we show that $\mathscr{X} \subseteq \mathscr{C}_Q$, and if $\mathscr{X}$ is the collection of all fuzzy filters of S, we show that Q is the smallest positive fuzzy quasi-order on S having the cm-property. □

The last theorem of this section gives new characterizations of positive fuzzy quasi-orders having the cp-property in terms of properties of the corresponding lattices of fuzzy ideals and consistent fuzzy subsets.

Theorem 4.8. *Let Q be a positive quasi-fuzzy order on a semigroup S. Then the following statements are equivalent:*

(i) *Q has the cp-property;*

(ii) *$\mathscr{I}_Q$ consists of completely prime fuzzy ideals;*

(iii) *$\mathscr{C}_Q$ consists of fuzzy filters.*

Proof. (i)⇔(ii). This follows directly from Theorems 3.6 and 4.1, and the fact that the set of all completely prime fuzzy ideals is closed under linear combinations.

(ii)⇒(iii). Consider an arbitrary $f \in \mathscr{C}_Q$. By (59) it follows that $f = Q\backslash f$, and for any $a \in S$ we get

$$f(a) = (Q\backslash f)(a) = \bigwedge_{b\in S} Q(b,a) \to f(b) = \bigwedge_{b\in S} bQ(a) \to f(b).$$

For each $b \in S$ we have that bQ is a completely prime fuzzy ideal, and according to Theorem 3.7, a fuzzy subset f_b of S defined by $f_b(a) = bQ(a) \to f(b)$ is a fuzzy filter. Therefore, f is the meet of a family of fuzzy filters $\{f_b\}_{b\in S}$, and since the set of all fuzzy filters is closed under meets, we conclude that f is a fuzzy filter.

(iii)⇒(i). Let $\mathscr{C}_Q$ consist of fuzzy filters. To prove that Q has the cp-property, we will prove that (40) holds. By assumption, every Q-foreset is a fuzzy filter, and by Theorem 3.5 we obtain that (39) holds. On the other hand, by Theorem 3.1 it follows that (33) also holds, and hence, to prove (40) it is sufficient to prove inequality opposite to the inequality (33).

Consider arbitrary $a,b \in S$. Set $f = Qa \vee Qb$. Since $\mathscr{C}_Q$ is closed under joins we conclude that $f \in \mathscr{C}_Q$, and thus, f is a fuzzy filter of S. Further,

$$\begin{aligned} \bigwedge_{c\in S} Q(c,ab) \to f(c) &= (Q\backslash f)(ab) = f(ab) = f(a) \wedge f(b) = \big(Qa(a) \vee Qb(a)\big) \wedge \big(Qa(b) \vee Qb(b)\big) \\ &= \big(1 \vee Qb(a)\big) \wedge \big(Qa(b) \vee 1\big) = 1 \wedge 1 = 1, \end{aligned}$$

which means that $Q(c,ab) \to f(c) = 1$, for every $c \in S$, that is, $Q(c,ab) \leqslant f(c)$, for every $c \in S$. Therefore,

$$Q(c,ab) \leqslant f(c) = Qa(c) \vee Qb(c) = Q(c,a) \vee Q(c,b),$$

for all $a,b,c \in S$, which was to be proved. □

5. Application to semilattice decompositions of semigroups

A fuzzy relation R on a semigroup S is *compatible* if for all $a, b, c \in S$ we have that $R(a, b) \leqslant R(ac, bc)$ and $R(a, b) \leqslant R(ca, cb)$. A compatible fuzzy equivalence is called a *fuzzy congruence*, and a compatible fuzzy quasi-order is called a *fuzzy half-congruence*. Clearly, in the crisp case these notions come down to notions of a *congruence* and a *half-congruence*. If Q is a fuzzy half-congruence on a semigroup S, it is easy to check that its natural fuzzy equivalence E_Q is a fuzzy congruence.

If E is a fuzzy congruence on a semigroup S, the binary operation on S can be transferred to the factor set S/E by letting $E_aE_b = E_{ab}$, for all $a, b \in S$. Because of the compatibility of E, that operation is well-defined and S/E becomes a semigroup called the *factor semigroup* of S with respect to E. If Q is a fuzzy half-congruence, in the same way we can define an operation on the set S/Q of all Q-aftersets, by letting $(aQ)(bQ) = (ab)Q$, for all $a, b \in S$, and and operation on the set $S\backslash Q$ of all Q-foresets, by letting $(Qa)(Qb) = Q(ab)$, for all $a, b \in S$. However, we are essentially getting nothing new because, in light of Theorem 2.1, the *afterset semigroup* S/Q, the *foreset semigroup* $S\backslash Q$, and the factor semigroup S/E_Q, are isomorphic to each other.

A fuzzy congruence E on a semigroup S is called a *semilattice fuzzy congruence* on S if the corresponding factor semigroup S/E is a semilattice, i.e., an idempotent commutative semigroup. It is easy to verify that this holds if and only if $E(a^2, a) = 1$, for each $a \in S$, and $E(ab, ba) = 1$, for all $a, b \in S$.

Analogously, a fuzzy half-congruence Q on a semigroup S is called a *semilattice fuzzy half-congruence* on S if the corresponding afterset semigroup S/Q (resp. foreset semigoup $S\backslash Q$, factor semigroup S/E_Q) is a semilattice. In accordance with Theorem 2.1, this is true if and only if $Q(a^2, a) = Q(a, a^2) = 1$, for each $a \in S$, and $Q(ab, ba) = 1$, for all $a, b \in S$.

The following theorem shows the relationship between semilattice fuzzy congruences and positive lower-potent fuzzy half-congruences.

Theorem 5.1. *For any positive lower-potent fuzzy half-congruence on a semigroup S, its natural fuzzy equivalence is a semilattice fuzzy congruence.*

If the membership values are taken from a complete Heyting algebra, then the converse is also true, i.e., any semilattice fuzzy congruence on S can be represented as the natural fuzzy equivalence of some positive lower-potent fuzzy half-congruence.

Proof. Let Q be a positive lower-potent half-congruence on S and let $E = Q \wedge Q^{-1}$. For arbitrary $a, b, c \in S$ we have that

$$E(a, b) \leqslant Q(a, b) \leqslant Q(ac, bc) \quad \text{and} \quad E(a, b) = E(b, a) \leqslant Q(b, a) \leqslant Q(bc, ac),$$

and therefore, $E(a, b) \leqslant Q(ac, bc) \wedge Q(bc, ac) = E(ac, bc)$. In the same way we show that $E(a, b) \leqslant E(ca, cb)$. Hence, E is a fuzzy congruence.

Next, for an arbitrary $a \in S$ we have that $Q(a^2, a) = 1$, because of the lower-potency of Q, and $Q(a, a^2) = 1$, because of the positivity of Q. This means that $E(a^2, a) = 1$. Let us now consider arbitrary $a, b \in S$. Due to the positivity of Q we get $Q(ab, bab) = 1$ and $Q(bab, (ba)^2) = 1$, which yields $Q(ab, (ba)^2) = 1$, because of the transitivity. Since $Q((ba)^2, ba) = 1$, we conclude that $Q(ab, ba) = 1$, which is what we wanted to prove. Therefore, E is a semilattice fuzzy congruence.

Further, let the underlying complete residuated lattice $\mathscr{L}$ be a complete Heyting algebra, that is, let the multiplication operation $\otimes$ and the meet operation $\wedge$ coincide, and let E be an arbitrary semilattice fuzzy congruence on S. Define a fuzzy relation Q on S as follows

$$Q(a, b) = \bigvee_{u \in S} E(ua, b), \qquad \text{for all } a, b \in S.$$

We will prove that Q is a positive lower-potent half-congruence on S and $E = Q \wedge Q^{-1}$.

First, for each $a \in S$ we have that

$$Q(a, a) = \bigvee_{u \in S} E(ua, a) \geqslant E(a^2, a) = 1,$$

which means that Q is reflexive. Next, consider arbitrary $a, b \in S$. Due to the compatibility and transitivity of E, for each pair $u, v \in S$ we have that

$$E(ua, b) \wedge E(vb, c) \leqslant E(vua, vb) \wedge E(vb, c) \leqslant E(vua, c),$$

whence

$$\begin{aligned} Q(a,b) \wedge Q(b,c) &= \Big(\bigvee_{u\in S} E(ua,b)\Big) \wedge \Big(\bigvee_{v\in S} E(vb,c)\Big) \leqslant \bigvee_{u\in S}\Big(\bigvee_{v\in S} E(ua,b) \wedge E(vb,c)\Big) \leqslant \bigvee_{u,v\in S} E(vua,c) \\ &\leqslant \bigvee_{w\in S} E(wa,c) = Q(a,c), \end{aligned}$$

and we conclude that Q is transitive. Therefore, Q is a fuzzy quasi-order.

Further, for arbitrary $a, b \in S$ we have that

$$Q(a, ab) = \bigvee_{u\in S} E(ua, ab) \geqslant E(ba, ab) = 1,$$

whence $Q(a, ab) = 1$, and in a similar way we get $Q(b, ab) = 1$. Thus, Q is positive. Besides, for an arbitrary $a \in S$ we have

$$Q(a^2, a) = \bigvee_{u\in S} E(ua^2, a) \geqslant E(a^3, a) = 1,$$

which means that Q is lower-potent.

Due to the compatibility of E, for arbitrary $a, b, c \in S$ we have that

$$Q(a, b) = \bigvee_{u\in S} E(ua, b) \leqslant \bigvee_{u\in S} E(uac, bc) = Q(ac, bc),$$

and similarly, $Q(a, b) \leqslant Q(ca, cb)$. Hence, Q is a positive lower-potent fuzzy half-congruence on S.

To prove that $Q \wedge Q^{-1} = E$, consider arbitrary $a, b \in S$. First, we have that

$$E(a, b) = E(a^2, b) \leqslant \bigvee_{u\in S} E(ua, b) = Q(a, b),$$

and analogously, $E(a, b) = E(b, a) \leqslant Q(b, a)$, so $E(a, b) \leqslant Q(a, b) \wedge Q(b, a)$.

On the other hand, due to the compatibility and transitivity of E, for each pair $u, v \in S$ we have

$$E(ua, b) \leqslant E(vbua, vb^2) = E(uvab, vb) \qquad \text{and} \qquad E(vb, a) \leqslant E(uavb, ua^2) = E(uvab, ua),$$

whence

$$\begin{aligned} &E(ua, b) \wedge E(vb, a) \leqslant E(uvab, vb) \wedge E(vb, a) \leqslant E(uvab, a), \\ &E(ua, b) \wedge E(vb, a) \leqslant E(uvab, ua) \wedge E(ua, b) \leqslant E(uvab, b), \end{aligned}$$

which yields

$$E(ua, b) \wedge E(vb, a) \leqslant E(uvab, a) \wedge E(uvab, b) = E(a, uvab) \wedge E(uvab, b) \leqslant E(a, b).$$

Now we have that

$$Q(a,b) \wedge Q(b,a) = \Big(\bigvee_{u\in S} E(ua,b)\Big) \wedge \Big(\bigvee_{v\in S} E(vb,a)\Big) \leqslant \bigvee_{u\in S}\Big(\bigvee_{v\in S} E(ua,b) \wedge E(vb,a)\Big) \leqslant E(a,b),$$

and therefore, $Q(a, b) \wedge Q(b, a) \leqslant E(a, b)$. Hence, $Q \wedge Q^{-1} = E$. This completes the proof of the theorem. □

As we noted above, every fuzzy quasi-order having the cm-property is lower-potent. The following theorem explains in more detail the connection between cm-property and lower-potency.

Theorem 5.2. *Every positive fuzzy quasi-order on a semigroup S which has the cm-property is a lower-potent fuzzy half-congruence.*

If the membership values are taken from a complete Heyting algebra, then the converse is also true, i.e., every positive lower-potent fuzzy half-congruence on S has the cm-property.

Proof. Let Q be a positive fuzzy quasi-order on S having the cm-property. We can easily derive from (39) that Q is lower-potent.

To prove the compatibility of Q consider arbitrary $a, b, c \in S$. According to (39) and (33) we have that

$$Q(a,b) \leqslant Q(a,b) \vee Q(a,c) \leqslant Q(a,bc) = Q(a,bc) \wedge 1 = Q(a,bc) \wedge Q(b,bc) = Q(ac,bc),$$

and in a similar way we prove that $Q(a,b) \leqslant Q(ca,cb)$. Thus, Q is a fuzzy half-congruence.

Conversely, let the underlying complete residuated lattice $\mathscr{L}$ be a complete Heyting algebra, and let Q be a lower-potent fuzzy half-congruence on S. To prove that Q has the cm-proprety, let us consider arbitrary $a, b, c \in S$. Due to the compatibility, lower-potency and transitivity of Q we have that

$$Q(a,c) \leqslant Q(ab,cb) \quad \text{and} \quad Q(b,c) \leqslant Q(cb,c^2) = Q(cb,c^2) \wedge Q(c^2,c) \leqslant Q(cb,c),$$

whence it follows that

$$Q(a,c) \wedge Q(b,c) \leqslant Q(ab,cb) \wedge Q(cb,c) \leqslant Q(ab,c).$$

Therefore, Q has the cm-property. This completes the proof of the theorem. □

As an immediate consequence of the previous theorem we obtain the well-known result of T. Tamura [51] according to which a positive (crisp) quasi-order on a semigroup has the cm-property if and only if it is a lower-potent half-congruence.

Remark 5.3. The reason why in the converse parts of Theorems 5.1 and 5.2 we required that membership values be taken from a Heyting algebra is that the concepts used in their proofs are defined by two different types of fuzzy conjunction. Specifically, transitivity is defined by the multiplication operation in $\mathscr{L}$ (strong conjunction), while cm-property and natural fuzzy equivalence are defined by the meet operation in $\mathscr{L}$ (weak conjunction). This creates difficulties in proving the converse parts of Theorems 5.1 and 5.2, which disappear when it is assumed that membership values are taken from a Heyting algebra, where these two operations (conjunctions) coincide.

It should be pointed out that the use of the multiplication operation in defining the transitivity, as well as the composition of fuzzy relations, is of enormous importance, given that many significant properties of fuzzy relations are ensured by the distributivity of the multiplication over arbitrary joins.

On the other hand, in many sources dealing with fuzzy algebra, with complete residuated lattices as the underlying structures of membership values, the concept of a fuzzy subalgebra was defined by means of the multiplication operation (for instance, see [27]). However, such an approach would not be appropriate here. The use of the meet operation is crucial in defining a consistent fuzzy subset, fuzzy subsemigroup and fuzzy filter, because it provides a duality between consistent fuzzy subsets and fuzzy ideals, as well as other related concepts considered in Theorems 3.7 and 3.8, and links semilattice decompositions of semigroups with fuzzy filters. In particular, this enables semilattice homomorphic images of a semigroup to be characterized as semilattices of fuzzy filters which are principal elements in sublattices of the lattice of fuzzy consistent subsets corresponding to positive fuzzy quasi-orders having the cm-property.

According to Theorem 3.1, the ordinary (crisp) division relation D, defined by (32), is the smallest positive fuzzy quasi-order on a semigroup S. Moreover, Theorem 3.1 says that the set of all positive fuzzy quasi-orders on S is the principal dual ideal of the lattice of fuzzy quasi-orders on S generated by D (or the principal filter, as it is said in many sources). On the other hand, the crisp relation $P : S \times S \to \{0, 1\} \subseteq L$ defined by

$$P(a, b) = \begin{cases} 1 & \text{if } a = b^n, \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \tag{62}$$

is the smallest fuzzy quasi-order on S having the cm-property. Indeed, for an arbitrary fuzzy quasi-order Q on S we have that $Q(b^n, b) = 1$, for all $b \in S$ and $n \in \mathbb{N}$, and if $P(a, b) = 1$, for some $a, b \in S$, that is, if $a = b^n$, for some $n \in \mathbb{N}$, then $Q(a, b) = Q(b^n, b) = 1$, which means that $P \leqslant Q$.

As each fuzzy quasi-order with the cm-property is lower-potent, we have that P is lower-potent, but it is clear that P is not the smallest lower-potent fuzzy quasi-order on S, since the smallest lower-potent fuzzy quasi-order is obtained when in (62) we replace "$a = b^n$, for some $n \in \mathbb{N}$" with $a = b^2$.

T. Tamura in [48] defined a relation on a semigroup S that is the composition of the division relation D and the relation P, namely

$$(D \circ P)(a, b) = \begin{cases} 1 & \text{if } paq = b^n, \text{ for some } p, q \in S \text{ and } n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \tag{63}$$

and a quasi-order $T = (D \circ P)^\infty$, the transitive closure of $D \circ P$. We will call $D \circ P$ *Tamura's relation*, whereas T will be called *Tamura's quasi-order*. In the mentioned paper Tamura proved that, on an arbitrary semigroup S, the natural equivalence of Tamura's quasi-order T is the smallest semilattice congruence. He also noted that T is the smallest positive lower-potent quasi-order on S, as well as the smallest positive lower-potent half congruence (cf. [49–51]). It is important to note that the crisp part of an arbitrary positive lower-potent fuzzy quasi-order is a positive lower-potent quasi-order. Since this quasi-order is contained in the original fuzzy quasi-order, we can conclude that the smallest positive lower-potent fuzzy quasi-order on S is a crisp quasi-order. Thus, Tamura's quasi-order T is also the smallest positive lower-potent fuzzy quasi-order on S. As we have already noted, Tamura in [51] proved that a positive quasi-order has the cm-property if and only if it is a lower-potent half-congruence, which means that T is the smallest positive quasi-order on S having the cm-property. Furthermore, T is the smallest positive fuzzy quasi-order on S having the cm-property. Similarly, the smallest semilattice fuzzy congruence on S is a crisp congruence, and it coincides with the natural equivalence of Tamura's quasi-order T. These facts will be used in the proof of the following theorem:

Theorem 5.4. *The smallest semilattice congruence E on a semigroup S can be represented as follows:*

$$E(a, b) = \bigwedge_{f \in \mathscr{I}^{cs}} f(a) \leftrightarrow f(b) = \bigwedge_{f \in \mathscr{I}^{cp}} f(a) \leftrightarrow f(b) = \bigwedge_{f \in \mathscr{C}^f} f(a) \leftrightarrow f(b), \qquad \text{for all } a, b \in S, \tag{64}$$

where $\mathscr{I}^{cs}$ denotes the collection of all completely semiprime fuzzy ideals, $\mathscr{I}^{cp}$ the collection of all completely prime fuzzy ideals, and $\mathscr{C}^f$ the collection of all fuzzy filters of S.

Proof. Let E be the smallest semilattice congruence on S, and for an arbitrary $\mathscr{X} \subseteq \mathscr{F}(S)$ let

$$F(a, b) = \bigwedge_{f \in \mathscr{X}} f(a) \leftrightarrow f(b).$$

Then F is the natural fuzzy equivalence of the fuzzy quasi-order Q defined as in (60) or (61).

When $\mathscr{X} = \mathscr{I}^{\mathrm{cs}}$, by Theorem 4.4 we obtain that Q is the smallest positive lower-potent fuzzy quasi-order on S, that is, Q is equal to Tamura's quasi-order T. This means that $F = Q \wedge Q^{-1} = T \wedge T^{-1} = E$.

If $\mathscr{X} = \mathscr{I}^{\mathrm{cp}}$, then by Theorem 4.6 we have that Q is the smallest positive fuzzy quasi-order on S having the cm-property, and according to the above given remark, Q is contained in T, which means that F is contained in E. In addition, by Theorems 5.1 and 5.2 we obtain that F is a semilattice fuzzy congruence, and consequently, its crisp part is a semilattice congruence on S, so both the crisp part of F and F contain E. Hence, we conclude that $F = E$.

In the same way we prove the case when $\mathscr{X} = \mathscr{C}^{\mathrm{f}}$. $\square$

We also have that the following is true.

Theorem 5.5. *Let the considered structure $\mathscr{L}$ of membership values be a linearly ordered complete residuated lattice. Then every completely semiprime fuzzy ideal of a semigroup S is the intersection of some family of completely prime fuzzy ideals of S.*

Proof. As we have already shown above, Tamura's quasi-order T is the smallest positive lower-potent fuzzy quasi-order on S, and also, it is the smallest positive fuzzy quasi-order on S having the cm-property. According to Theorem 4.4, $\mathscr{I}_T$ is the collection of all completely semiprime fuzzy ideals of S, and by Theorem 4.5 we obtain that every completely semiprime fuzzy ideal of S is the intersection of some family of completely prime fuzzy ideals of S. □

Let us note that the above theorem is an immediate generalization of the well-known theorem which asserts that every completely semiprime ideal of a semigroup is an intersection of some family of completely prime ideals of this semigroup. This theorem was proved in some special cases by Š. Schwarz [43] and K. Iséki [31], and in the general case by M. Petrich [35]. The same result, without use of Zorn's lemma arguments, was proved by Ćirić and Bogdanović in [13, 14]. The related result in Theory of lattices is known as *Prime ideal theorem*, and for the related results for rings we refer to W. Krull [32] and N. H. McCoy [34].

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