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Existence and global stability results for Volterra type fractional Hadamard–Stieltjes partial integral equations

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Abstract. This paper deals with the existence and global stability of solutions of a new class of Volterra partial integral equations of Hadamard–Stieltjes fractional order.

1. Introduction

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. [17]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [4–6], Kilbas *et al.* [16], Miller and Ross [18], Samko *et al.* [22], the papers of Abbas *et al.* [1, 2], Banaś *et al.* [8–10], Darwish *et al.* [14], and the references therein.

In [11], Butzer *et al.* investigate properties of the Hadamard fractional integral and derivative. In [12], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators. In [20], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [22] and the references therein.

Recently, Abbas *et al.* [3] studied some existence and stability results for some classes of nonlinear quadratic Volterra integral equations of Riemann–Liouville fractional order. This paper deals with the global existence and stability of solutions to the following nonlinear quadratic Volterra partial integral equation of Hadamard fractional order,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1} \\ &\quad \times h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d_\xi g_2(x,\xi)d_s g_1(t,s);}{s\xi};\quad (t,x)\in J, \end{aligned} \tag{1}$$

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where $J := [1,\infty) \times [1,b]$, $b > 1$, $r_1, r_2 \in (0,\infty)$, α, β, $\gamma : [1,\infty) \to [1,\infty)$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_1 : \triangle_1 \to \mathbb{R}$, $g_2 : \triangle_2 \to \mathbb{R}$, $h : J_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\triangle_1 = \{(t,s) : 1 \le s \le t\}$, $\triangle_2 = \{(x,\xi) : 1 \le \xi \le x \le b\}$, $J_1 = \{(t,x,s,\xi) : (t,s) \in \triangle_1 \text{ and } (x,\xi) \in \triangle_2\}$, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt; \ \zeta > 0.$$

In the present paper we provide some existence and asymptotic stability of such new class of fractional integral equations. Finally, we present an example illustrating the applicability of the imposed conditions.

This paper initiates the global existence and stability of solutions of such new class of Hadamard integral equations of two independent variables.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([1,+\infty) \times [1,b])$; for $b > 1$, we denote the space of Lebesgue-integrable functions $u : [1,+\infty) \times [1,b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_1^\infty \int_1^b |u(t,x)| dx dt.$$

By $BC := BC(J)$ we denote the Banach space of all bounded and continuous functions from J into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x) \in J} |u(t,x)|.$$

For $u_0 \in BC$ and $\eta \in (0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1. *[16] The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,a],\mathbb{R})$, is defined as*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Example 2.2. *The Hadamard fractional integral of order $q > 0$ for the function $w : [1,e] \to \mathbb{R}$, defined by $w(x) = (\log x)^{\beta-1}$ with $\beta > 0$, is*

$$({}^H I_1^q w)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+q)} (\log x)^{\beta+q-1}.$$

Definition 2.3. *Let $r_1, r_2 \ge 0$, $\sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in L^1(J,\mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression*

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dt ds.$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w : [a,b] \times [c,b] \to \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t \to w(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [7, 19].

If u and φ are two real functions defined on the interval $[a, b]$, then under some conditions (see [7, 19]) we can define the Stieltjes integral (in the Riemann–Stieltjes sense)

$$\int_a^b u(t)d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a, b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [7, 19]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a, b]$.

In what follows we use the following properties of the Stieltjes integral ([21], section 8.13).

(*i*) If u is Stieltjes integrable on the interval $[a, b]$ with respect to a function φ of bounded variation, then

$$\left|\int_a^b u(t)d\varphi(t)\right| \leq \int_a^b |u(t)|d\left(\bigvee_a^t \varphi\right).$$

(*ii*) If u and v are Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $u(t) \leq v(t)$ for $t \in [a, b]$. Then

$$\int_a^b u(t)d\varphi(t) \leq \int_a^b v(t)d\varphi(t).$$

In the sequel we consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t, s),$$

and Hadamard–Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} u(s)d_s g(t, s),$$

where $g : [1, \infty) \times [1, \infty) \to \mathbb{R}$, $q \in (0, \infty)$ and the symbol d_s indicates the integration with respect to s.

Definition 2.4. *Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$ and $r = (r_1, r_2)$. For $w \in L^1(J, \mathbb{R})$, define the Hadamard–Stieltjes partial fractional integral of order r by the expression*

$$({}^{HS}I^r_\sigma w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{w(s, t)}{st}d_t g_2(y, t)d_s g_1(x, s),$$

where $g_1 : [1, \infty)^2 \to \mathbb{R}$, $g_2 : [1, b]^2 \to \mathbb{R}$.

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t, x) = u(t, x). \tag{2}$$

Now we review the concept of attractivity of solutions for equation (1).

Definition 2.5. *[5] Solutions of equation (2) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t, x)$ and $w = w(t, x)$ of equations (2) belonging to $B(u_0, \eta) \bigcap \Omega$, we have that, for each $x \in [1, b]$,*

$$\lim_{t\to\infty}\Big(v(t, x) - w(t, x)\Big) = 0. \tag{3}$$

When the limit (3) is uniform with respect to $B(u_0, \eta)$, solutions of equation (2) are said to be uniformly locally attractive (or equivalently that solutions of (2) are locally asymptotically stable).

Definition 2.6. *[5] The solution $v = v(t, x)$ of equation (2) is said to be globally attractive if (3) holds for each solution $w = w(t, x)$ of (2). If condition (3) is satisfied uniformly with respect to the set Ω, solutions of equation (2) are said to be globally asymptotically stable (or uniformly globally attractive).*

Lemma 2.7. *[13] Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $[1, \infty) \times [1, b]$,
i.e. equicontinuous on every compact of J,
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [1, b]$ there corresponds $T(\epsilon, x) > 0$ such that $|u(t, x) - \lim_{t\to\infty} u(t, x)| < \epsilon$ for any $t \geq T(\epsilon, x)$ and $u \in D$.

Theorem 2.8. *(Schauder's fixed point theorem)[15] Let B be a closed, convex and nonempty subset of a Banach space X. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of X. Then N has at least one fixed point in B.*

3. Existence and Asymptotic Stability Results

In this section, we are concerned with the existence and the asymptotic stability of solutions for the Hadamard partial integral equation (1).

The following hypotheses will be used in the sequel.

(H_1) The function $\alpha : [1, \infty) \to [1, \infty)$ satisfies $\lim_{t\to\infty} \alpha(t) = \infty$,

(H_2) There exist constants $M, L > 0$, and a nondecreasing function $\psi_1 : [0, \infty) \to (0, \infty)$ such that $M < \frac{L}{2}$,

$$|f(t, x, u_1, v_1) - f(t, x, u_2, v_2)| \leq \frac{M(|u_1 - u_2| + |v_1 - v_2|)}{(1 + \alpha(t))(L + |u_1 - u_2| + |v_1 - v_2|)},$$

and

$$|f(t_1, x_1, u, v) - f(t_2, x_2, u, v)| \leq (|t_1 - t_2| + |x_1 - x_2|)\psi_1(|u| + |v|),$$

for each $(t, x), (t_1, x_1), (t_2, x_2) \in J$ and $u, v, u_1, v_1, u_2, v_2 \in \mathbb{R}$,

(H_3) The function $(t, x) \to f(t, x, 0, 0)$ is bounded on J with

$$f^* = \sup_{(t,x)\in[1,\infty)\times[1,b]} f(t, x, 0, 0)$$

and

$$\lim_{t\to\infty} |f(t, x, 0, 0)| = 0; \ x \in [1, b],$$

(H_4) There exist continuous functions $\varphi : J \to \mathbb{R}_+$, $p : J_1 \to \mathbb{R}_+$ and a nondecreasing function $\psi_2 : [0, \infty) \to (0, \infty)$ such that

$$|h(t_1, x_1, s, \xi, u, v) - h(t_2, x_2, s, \xi, u, v)| \leq \varphi(s, \xi)(|t_1 - t_2| + |x_1 - x_2|)\psi_2(|u| + |v|),$$

and

$$|h(t, x, s, \xi, u, v)| \leq \frac{p(t, x, s, \xi)}{1 + \alpha(t) + |u| + |v|},$$

for each (t, x, s, ξ), (t_1, x_1, s, ξ), $(t_2, x_2, s, \xi) \in J_1$ and $u, v \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} \int_1^{\beta(t)} \int_1^x \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} p(t, x, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t, k_1) = 0,$$

for each $x \in [1, b]$,

(H_5) The functions $s \mapsto g_1(t,s)$ and $\xi \mapsto g_2(x,\xi)$ have bounded variations for each fixed $t \in [1,\infty)$ or $x \in [1,b]$, respectively. Moreover, the functions $s \mapsto g_1(1,s)$ and $\xi \mapsto g_2(1,\xi)$ are nondecreasing on $[1,\infty)$ or $[1,b]$ respectively,

(H_6) For each $(t,s),(t_1,s),(t_2,s) \in \Delta_1$, $(x,\xi),(x_1,\xi),(x_2,\xi) \in \Delta_2$, we have

$$\left|\bigvee_{k_2=1}^{x_2} g_2(x_2,k_2) \bigvee_{k_1=1}^{t_2} g_1(t_2,k_1) - \bigvee_{k_2=1}^{x_1} g_2(x_1,k_2)d_s \bigvee_{k_1=1}^{t_1} g_1(t_1,k_1))\right| \to 0,$$

as $t_1 \to t_2$ and $x_1 \to x_2$,

(H_7) $g_1(t,1) = g_2(x,1) = 0$ for any $t \in [1,\infty)$ and any $x \in [1,b]$.

Theorem 3.1. *Assume that hypotheses* $(H_1) - (H_7)$ *hold. Then the integral equation (1) has at least one solution in the space BC. Moreover, solutions of equation (1) are globally asymptotically stable.*

Proof: Set $d^* := \sup_{(t,x)\in J} d(t,x)$ where

$$d(t,x) = \int_1^{\beta(t)} \int_1^x \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \frac{p(t,x,s,\xi)}{\Gamma(r_1)\Gamma(r_2)} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1).$$

From hypothesis (H_4), we infer that d^* is finite. Let us define the operator N such that, for any $u \in BC$,

$$\begin{aligned}(Nu)(t,x) \;=\; & f(t,x,u(t,x),u(\alpha(t),x)) + \int_1^{\beta(t)} \int_1^x \left(\log \tfrac{\beta(t)}{s}\right)^{r_1-1} \left(\log \tfrac{x}{\xi}\right)^{r_2-1} \\ & \times h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \frac{d_\xi g_2(x,\xi) d_s g_1(t,s)}{s\xi\Gamma(r_1)\Gamma(r_2)}; \;\; (t,x) \in J.\end{aligned} \tag{4}$$

By considering the assumptions of this theorem, we infer that $N(u)$ is continuous on J. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t,x) \in J$ we have

$$\begin{aligned}|(Nu)(t,x)| \le\; & |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0)| + |f(t,x,0,0)| \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\ & \times |h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \frac{|d_\xi g_2(x,\xi) d_s g_1(t,s)|}{s\xi} \\ \le\; & \frac{M(|u(t,x)| + |u(\alpha(t),x)|)}{(1+\alpha(t))(L + |u(t,x)| + |u(\alpha(t),x)|)} + |f(t,x,0,0)| \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\ & \times \frac{p(t,x,s,\xi)}{1+\alpha(t) + |u(s,\xi))| + |u(\gamma(s),\xi))|} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) \\ \le\; & \frac{M(|u(t,x)| + |u(\alpha(t),x)|)}{|u(t,x)| + |u(\alpha(t),x)|} + f^* + d^*.\end{aligned}$$

Thus

$$\|N(u)\|_{BC} \le M + f^* + d^*. \tag{5}$$

Hence $N(u) \in BC$. The equation (5) yields that N transforms the ball $B_\eta := B(0,\eta)$ into itself where $\eta = M + f^* + d^*$. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Theorem 2.8. The proof will be given in several steps and cases.

Step 1: *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x)\in J$, we have

$$\begin{aligned}
&|(Nu_n)(t,x)-(Nu)(t,x)| \leq |f(t,x,u_n(t,x),u_n(\alpha(t),x))-f(t,x,u(t,x),u(\alpha(t),x))| \\
&+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\tfrac{\beta(t)}{s}\right|^{r_1-1}\left|\log\tfrac{x}{\xi}\right|^{r_2-1} \\
&\times\sup_{(s,\xi)\in J}|h(t,x,s,\xi,u_n(s,\xi),u_n(\gamma(s),\xi))-h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \\
&\times\tfrac{|d_\xi g_2(x,\xi)d_s g_1(t,s)|}{s\xi} \\
\leq\ &\tfrac{2M}{L}\|u_n-u\|_{BC} \\
&+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\tfrac{\beta(t)}{s}\right|^{r_1-1}\left|\log\tfrac{x}{\xi}\right|^{r_2-1} \\
&\times\|h(t,x,\cdot,\cdot,u_n(\cdot,\cdot),u_n(\gamma(\cdot),\cdot))-h(t,x,\cdot,\cdot,u(\cdot,\cdot),u(\gamma(\cdot),\cdot))\|_{BC} \\
&\times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2)d_s\bigvee_{k_1=1}^{s} g_1(t,k_1).
\end{aligned} \tag{6}$$

Case 1. If $(t,x)\in[1,T]\times[1,b],\ T>1$, then, since $u_n\to u$ as $n\to\infty$ and h,γ are continuous, then (6) gives

$$\|N(u_n)-N(u)\|_{BC}\to 0 \quad \text{as } n\to\infty.$$

Case 2. If $(t,x)\in(T,\infty)\times[1,b],\ T>1$, then from (H_4) and (6), for each $(t,x)\in J$, we have

$$\begin{aligned}
&|(Nu_n)(t,x)-(Nu)(t,x)| \leq \tfrac{2M}{L}\|u_n-u\|_{BC} \\
&+\tfrac{2}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x \left|\log\tfrac{\beta(t)}{s}\right|^{r_1-1}\left|\log\tfrac{x}{\xi}\right|^{r_2-1} \\
&\times p(t,x,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi} g_2(x,k_2)d_s\bigvee_{k_1=1}^{s} g_1(t,k_1). \\
\leq\ &\tfrac{2M}{L}\|u_n-u\|_{BC}+2d(t,x).
\end{aligned}$$

Thus, we get

$$|(Nu_n)(t,x)-(Nu)(t,x)| \leq \frac{2M}{L}\|u_n-u\|_{BC}+2d(t,x). \tag{7}$$

Since $u_n\to u$ as $n\to\infty$ and $t\to\infty$, then (7) gives

$$\|N(u_n)-N(u)\|_{BC}\to 0 \quad \text{as } n\to\infty.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.*
This is clear since $N(B_\eta)\subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on every compact subset $[1,a]\times[1,b]$ of J, $a>0$.*
Let $(t_1,x_1),(t_2,x_2)\in[1,a]\times[1,b]$, $t_1<t_2,\ x_1<x_2$ and let $u\in B_\eta$. Also without loss of generality suppose that $\beta(t_1)\leq\beta(t_2)$. Then, we have

$$\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \\
&\leq |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2))-f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\tfrac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\tfrac{x_2}{\xi}\right|^{r_2-1} \\
&\times|h(t_2,x_2,s,\xi,u(s,\xi),u(\gamma(s),\xi))-h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \\
&\times|d_\xi g_2(x_2,\xi)d_s g_1(t_2,s)|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\\
&\times h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d_\xi g_2(x_2,\xi)d_s g_1(t_2,s)\\
&-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\\
&\times h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d_\xi g_2(x_2,\xi)d_s g_1(t_2,s)\Big|\\
&+\Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\\
&\times h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))(d_\xi g_2(x_2,\xi)d_s g_1(t_2,s)-d_\xi g_2(x_1,\xi)d_s g_1(t_1,s))\Big|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\Big|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\\
&-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\Big||h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|\\
&\times|d_\xi g_2(x_1,\xi)d_s g_1(t_1,s)|.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)|\\
&\leq\frac{M}{L}(|u(t_2,x_2)-u(t_1,x_1)|+|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|)\\
&+(|t_2-t_1|+|x_2-x_1|)\psi_1(2\|u\|_{BC})\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times\varphi(s,\xi)(|t_2-t_1|+|x_2-x_1|)\psi_2(2\|u\|_{BC})d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times|h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times|h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times|h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times|h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))||d_\xi g_2(x_2,\xi)d_s g_1(t_2,s)-d_\xi g_2(x_1,\xi)d_s g_1(t_1,s))|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\Big|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\\
&-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\Big||h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|\\
&\times d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_1,k_1).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)|\\
&\leq\frac{M}{L}(|u(t_2,x_2)-u(t_1,x_1)|+|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|)\\
&+(|t_2-t_1|+|x_2-x_1|)\psi_1(2\eta)\\
&+\frac{(|t_2-t_1|+|x_2-x_1|)\psi_2(2\eta)}{\Gamma(r_1)\Gamma(r_2)}\\
&\times\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\varphi(s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times p(t_1,x_1,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\times p(t_1,x_1,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)
\end{aligned}
$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}$$
$$\times p(t_1,x_1,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{\beta(t_1)}\int_{1}^{x_1}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}p(t_1,x_1,s,\xi)$$
$$\times|d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)-d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_1,k_1))|$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{\beta(t_1)}\int_{1}^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right.$$
$$\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right|p(t_1,x_1,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_1,k_1).$$

From continuity of α,β,φ,p and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the above inequality tends to zero.

Step 4: *$N(B_\eta)$ is equiconvergent.*
Let $(t,x)\in J$ and $u\in B_\eta$, then we have

$$\begin{aligned}
|u(t,x)| &\le \left|f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)+f(t,x,0,0)\right|\\
&\quad+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\right.\\
&\quad\left.\times\, h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d_\xi g_2(x,\xi)d_s g_1(t,s)}{s\xi}\right|\\
&\le \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)}+|f(t,x,0,0)|\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\quad\times\frac{p(t,x,s,\xi)}{1+\alpha(t)+|u(s,\xi)|+|u(\gamma(s),\xi)|}d_\xi g_2(x,\xi)d_s g_1(t,s)\\
&\le \frac{M}{1+\alpha(t)}+|f(t,x,0,0)|\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)(1+\alpha(t))}\int_1^{\beta(t)}\int_1^x\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\quad\times p(t,x,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1)\\
&\le \frac{M}{1+\alpha(t)}+|f(t,x,0,0)|+\frac{d^*}{1+\alpha(t)}.
\end{aligned}$$

Thus, for each $x\in[1,b]$, we get
$$|u(t,x)|\to 0,\ \text{as } t\to+\infty.$$

Hence,
$$|u(t,x)-u(+\infty,x)|\to 0,\ \text{as } t\to+\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.7, we can conclude that $N:B_\eta\to B_\eta$ is continuous and compact. From an application of Theorem 2.8, we deduce that N has a fixed point u which is a solution of the Hadamard integral equation (1).

Step 5: *The uniform global attractivity.*
Let us assume that u_0 is a solution of integral equation (1) with the conditions of this theorem. Consider

the ball $B(u_0,\eta)$ with $\eta^*=\frac{LM^*}{L-2M}$, where

$$\begin{aligned}M^* \;:=\; & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sup_{(t,x)\in J}\Big\{\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\\ & \times|h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))-h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))|\\ & \times d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1);\ u\in BC\Big\}.\end{aligned}$$

Taking $u\in B(u_0,\eta^*)$. Then, we have

$$\begin{aligned}|(Nu)(t,x)-u_0(t,x)| &= |(Nu)(t,x)-(Nu_0)(t,x)|\\ &\le |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))|\\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\\ &\quad\times|h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))-h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))|\\ &\quad\times d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1)\\ &\le \frac{2M}{L}\|u-u_0\|_{BC}+M^*\\ &\le \frac{2M}{L}\eta^*+M^*=\eta^*.\end{aligned}$$

Thus we observe that N is a continuous function such that $N(B(u_0,\eta^*))\subset B(u_0,\eta^*)$. Moreover, if u is a solution of equation (1), then

$$\begin{aligned}|u(t,x)-u_0(t,x)| &= |(Nu)(t,x)-(Nu_0)(t,x)|\\ &\le |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))|\\ &\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^x\left|\log\tfrac{\beta(t)}{s}\right|^{r_1-1}\left|\log\tfrac{x}{\xi}\right|^{r_2-1}\\ &\quad\times|h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))-h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))|\\ &\quad\times d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1).\end{aligned}$$

Hence

$$\begin{aligned}|u(t,x)-u_0(t,x)| \le\; & \tfrac{M}{L}(|u(t,x)-u_0(t,x)|+|u(\alpha(t),x)-u_0(\alpha(t),x)|)\\ & +\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_2^{\beta(t)}\int_1^x\left|\log\tfrac{\beta(t)}{s}\right|^{r_1-1}\left|\log\tfrac{x}{\xi}\right|^{r_2-1}\\ & \times p(t,x,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1).\end{aligned}\tag{8}$$

By using (8) and the fact that $\alpha(t)\to\infty$ as $t\to\infty$, we get

$$\begin{aligned}\lim_{t\to\infty}|u(t,x)-u_0(t,x)| &\le \lim_{t\to\infty}\frac{2L}{\Gamma(r_1)\Gamma(r_2)(L-2M)}\int_1^{\beta(t)}\int_1^x\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\\ &\times\; p(t,x,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1)=0.\end{aligned}$$

Consequently, all solutions of the Hadamard–Volterra–Stieltjes integral equation (1) are globally asymptotically stable.

4. An Example

As an application of our results we consider the following partial Hadamard–Volterra–Stieltjes integral equation of fractional order

$$
\begin{aligned}
u(t,x) &= \tfrac{tx}{10(1+t+t^2+t^3)}(1+2\sin(u(t,x))) + \tfrac{1}{\Gamma^2(\frac{1}{3})}\int_1^t\int_1^x \left(\log\tfrac{t}{s}\right)^{\frac{-2}{3}}\left(\log\tfrac{x}{\xi}\right)^{\frac{-2}{3}} \\
&\quad \tfrac{\ln(1+2x(s\xi)^{-1}|u(s,\xi)|)}{(1+t+2|u(s,\xi)|)^2(1+x^2+t^4)} d_\xi g_2(x,\xi) d_s g_1(t,s);\ (t,x)\in[1,\infty)\times[1,e],
\end{aligned}
\tag{9}
$$

where $r_1 = r_2 = \frac{1}{3},\ \alpha(t)=\beta(t)=\gamma(t)=t,\ g_1(t,s)=s,\ g_2(x,\xi)=\xi;\ s,\xi\in[1,e]$,

$$f(t,x,u,v) = \frac{tx(1+\sin(u)+\sin(v))}{10(1+t)(1+t^2)},$$

and

$$h(t,x,s,\xi,u,v) = \frac{\ln(1+x(s\xi)^{-1}(|u|+|v|))}{(1+t+|u|+|v|)^2(1+x^2+t^4)};$$

for $(t,x),(s,\xi)\in[1,\infty)\times[1,e]$, and $u,v\in\mathbb{R}$.

We can easily check that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function f is continuous and satisfies assumption (H_2), where $M=\frac{1}{10},\ L=1$. Also f satisfies assumption (H_3), with $f^* = \frac{e}{10}$. Next, let us notice that the function h satisfies assumption (H_4), where $p(t,x,s,\xi) = \frac{1}{s\xi(1+x^2+t^4)}$. Also,

$$
\begin{aligned}
&\lim_{t\to\infty}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} p(t,x,s,\xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1)\\
=\ &\lim_{t\to\infty}\frac{x}{1+x^2+t^4}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} \frac{1}{s\xi} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1)\\
=\ &\lim_{t\to\infty}\frac{x}{1+x^2+t^4}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} \frac{d\xi ds}{s\xi}\\
=\ &\lim_{t\to\infty}\frac{9x(\log t)^{\frac{1}{3}}}{1+x^2+t^4} = 0.
\end{aligned}
$$

Hence by Theorem 3.1, the Volterra–Stieltjes equation (9) has at least a solution defined on $[1,\infty)\times[1,e]$ and solutions of this equation are globally asymptotically stable.

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