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# Existence of solutions to nonlinear impulsive fuzzy differential equations

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**Abstract.** In this article, we establish the existence and uniqueness of ($c$1)-differentiable and ($c$2)-differentiable solutions to first-order nonlinear impulsive fuzzy differential equations under generalized Hukuhara differentiability using the contraction mappings principle. In particular, ($c$1)-differentiable solutions are written as hyperbolic cosine and sine functions with impulsive terms, which is the main difficulty. An example is provided to prove our results.

## 1. Introduction

Impulsive differential equations are used in many fields (see [1–9]). Existence theory of fuzzy differential equations was studied in [10–15]. Generalized derivatives further expand the original $H$-differentiable fuzzy numerical function class. Within this framework, fuzzy differential equations allow solutions with decreasing length (diameter) of the support sets. However, some scholars believe that under this differentiability, the uniqueness of the solution of fuzzy differential equations with an initial value is destroyed. In fact, there are two solutions with increasing and decreasing support sets in a neighborhood of a point, but it is this seemingly shortcoming that helps in considering fuzzy problems because in an actual problem the suitable solution can be selected according to the characteristics of the problem itself. We have a basic knowledge of the general state (convergence or divergence) in the study of physical, biological or medical problems, so one would choose the solution that conforms to the actual situation. Liu et al. [16, 17] adopted the variation of constant formula to present the representation of ($c$1)-differentiable and ($c$2)-differentiable solutions for first-order linear impulsive fuzzy differential equations with constant coefficients. Vu and Hoa [18] considered the existence and uniqueness of solutions to nonlinear impulsive fuzzy functional differential equations under generalized Hukuhara differentiability using the contraction mappings principle.

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Motivated by [16–18], we study first-order nonlinear impulsive fuzzy differential equations (described by a linear part with a nonlinear perturbation) and discuss existence and uniqueness of $(c1)$-differentiable and $(c2)$-differentiable solutions. In particular, $(c1)$-differentiable solutions are written as hyperbolic cosine and sine functions with impulsive terms, which is the main difficulty. The remainder of this paper consists of the following three parts. In Section 2, we introduce notations, definitions and theorems for fuzzy sets. In Section 3, we give some conditions about the existence result of the solution for nonlinear impulsive fuzzy differential equations. In the last section an example is given to prove our conclusions.

## 2. Preliminary

Let $\aleph = [0, \daleth]$. Now $C(\aleph, \mathbb{R}_F)$ denotes the space of all continuous functions from $\aleph$ into $\mathbb{R}_F$. Let $PC(\aleph, \mathbb{R}_F) := \{\nu : \aleph \to \mathbb{R}_F : \nu \in C((\iota_k, \iota_{k+1}], \mathbb{R}_F),\ k \in \mathbb{M}_0$ and $\exists\ \nu(\iota_k^-)$ and $\nu(\iota_k^+), k \in \mathbb{M}_0$, with $\nu(\iota_k^-) = \nu(\iota_k)\}$, where $\mathbb{M}_0 := \mathbb{M} \cup \{0\}$, $\mathbb{M} = \{1, 2, \cdots, m\}$, and $\iota_k < \iota_{k+1}$ for any $k \in \mathbb{M}_0$; here $0 = \iota_0 < \iota_1 < \iota_2 < \cdots < \iota_k < \iota_m < \iota_{m+1} = \daleth$.

We have collected a number of symbols and concepts that will be used throughout the text. For more detailed information, see [20, 22].

Denote by $\mathbb{R}_F := \{\chi \mid \chi : \mathbb{R} \to [0, 1]\}$ the class of the fuzzy subsets of the real axis satisfying $(X1)$-$(X4)$:

$(X_1)$ $\chi$ is normal (i.e., $\exists\ \kappa_0 \in \mathbb{R}$ s.t. $\chi(\kappa_0) = 1$).

$(X_2)$ $\chi$ is convex fuzzy set (i.e., $\chi(\xi s + (1 - \xi)s_1) \geq \min\{\chi(s), \chi(s_1)\}$), $\forall s, s_1 \in \mathbb{R}$ and $\xi \in [0, 1]$.

$(X_3)$ $\chi$ is upper semicontinuous on $\mathbb{R}$.

$(X_4)$ $[\chi]^0 = \overline{\{x \in \mathbb{R} : \chi(x) > 0\}}$ is compact.

Let $\alpha \in (0, 1]$. Consider the $\alpha-$level set of $\chi \in \mathbb{R}_F$ by $[\chi]^\alpha = \{s \in \mathbb{R} \mid \chi(s) \geq \alpha\}$, which is a nonempty compact interval $\forall \alpha \in (0, 1]$. We use $[\chi]^\alpha = [\underline{\chi}_\alpha, \overline{\chi}_\alpha]$ to denote explicitly the $\alpha-$level set of $\chi$ and use $\underline{\chi}_\alpha$ and $\overline{\chi}_\alpha$ denote the lower and upper branches of $\chi$, respectively. $diam([\chi]^\alpha) = \overline{\chi}_\alpha - \underline{\chi}_\alpha$ denote the length of $\chi$.

$\forall \alpha \in [0, 1]$, $\tilde{u}, \tilde{v} \in \mathbb{R}_F$ and $\xi \in \mathbb{R}$, we define $\tilde{u} + \tilde{v}$ and $\xi\tilde{u}$ as $[\tilde{u} + \tilde{v}]^\alpha = [\tilde{u}]^\alpha + [\tilde{v}]^\alpha = [\underline{\tilde{u}}_\alpha + \underline{\tilde{v}}_\alpha, \overline{\tilde{u}}_\alpha + \overline{\tilde{v}}_\alpha]$ and $[\xi\tilde{u}]^\alpha = \xi[\tilde{u}]^\alpha$.

Define $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$ and consider the Hausdorff distance $D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\{|\underline{\tilde{u}}_\alpha - \underline{\tilde{v}}_\alpha|, |\overline{\tilde{u}}_\alpha - \overline{\tilde{v}}_\alpha|\}$ (see [14]). Then $(\mathbb{R}_F, D)$ is a complete metric space (see [15]) and (i) $D(\tilde{u} + \tilde{e}, \tilde{v} + \tilde{e}) = D(\tilde{u}, \tilde{v})$, $\forall \tilde{u}, \tilde{v}, \tilde{e} \in \mathbb{R}_F$, (ii) $D(\varsigma\tilde{u}, \varsigma\tilde{v}) = |\varsigma| D(\tilde{u}, \tilde{v})$, $\forall \varsigma \in \mathbb{R}, \tilde{u}, \tilde{v} \in \mathbb{R}_F$, (iii) $D(\tilde{u} + \tilde{e}, \tilde{v} + \tilde{\varrho}) \leq D(\tilde{u}, \tilde{v}) + D(\tilde{e}, \tilde{\varrho})$, $\forall \tilde{u}, \tilde{v}, \tilde{e}, \tilde{\varrho} \in \mathbb{R}_F$ are satisfied.

**Definition 2.1.** *(see [20]) Let $\Omega : [o_1, \pi_1] \to \mathbb{R}_F$ be measurable and integrably bounded. The integral of $\Omega$ over $[o_1, \pi_1]$, express as $\int_{o_1}^{\pi_1} \Omega(\iota)d\iota$, its levelwise is expressed as follows*

$$\Big[\int_{o_1}^{\pi_1} \Omega(\iota)d\iota\Big]^\alpha := \int_{o_1}^{\pi_1} [\Omega(\iota)]^\alpha d\iota = \Big\{\int_{o_1}^{\pi_1} \tilde{\Omega}(\iota)d\iota \mid \tilde{\Omega} : [o_1, \pi_1] \to \mathbb{R}_F \text{ is a measurable selection for } [\Omega(\cdot)]^\alpha\Big\},$$

$\forall \alpha \in [0, 1]$.

In this article, we apply $\ominus$ to denote the $H$-difference. We must take note of $a_1 \ominus a_2 \neq a_1 + (-1)a_2 := a_1 - a_2$.

**Definition 2.2.** *(see [20]) Let $\varpi : \aleph \to \mathbb{R}_F$ and take a fixed $n_0 \in \aleph$. If $\varpi$ is differentiable at $n_0$, then $\exists \varpi'(n_0) \in \mathbb{R}_F$ such that*

*(c1) $\forall p > 0$ sufficiently close to $0$, the H-difference $\varpi(n_0 + p) \ominus \varpi(n_0)$, $\varpi(n_0) \ominus \varpi(n_0 - p)$ exist and the following limits (in the metric D)*

$$\lim_{p \to 0^+} \frac{\varpi(n_0 + p) \ominus \varpi(n_0)}{p} = \lim_{p \to 0^+} \frac{\varpi(n_0) \ominus \varpi(n_0 - p)}{p} = \varpi'(n_0),$$

*holds or*

*(c2) $\forall p > 0$ sufficiently close to 0, the H-difference $\varpi(n_0) \ominus \varpi(n_0 + p)$, $\varpi(n_0 - p) \ominus \varpi(n_0)$ exist and the following limits (in the metric D)*

$$\lim_{p\to 0^+} \frac{\varpi(n_0) \ominus \varpi(n_0 + p)}{-p} = \lim_{p\to 0^+} \frac{\varpi(n_0 - p) \ominus \varpi(n_0)}{-p} = \varpi'(n_0).$$

*holds.*

**Definition 2.3.** *(See [20]) Let $\varpi : \aleph \to \mathbb{R}_F$. $\varpi$ is (c1)-differentiable on $\aleph$ if $\varpi$ is differentiable Case (c1) of in Definition 2.2 and can be denoted $D_1\varpi$. Similarly, (c2)-differentiability denote by $D_2\varpi$.*

**Theorem 2.4.** *(see [20]) Let $\varpi : \aleph \to \mathbb{R}_F$ and put $[\varpi(\iota)]^\alpha = [\tilde{\jmath}_\alpha(\iota), \tilde{\ell}_\alpha(\iota)]$ for each $\alpha \in [0, 1]$.*
*(i) If $\varpi$ is (c1)-differentiable then $\tilde{\jmath}_\alpha$ and $\tilde{\ell}_\alpha$ are differentiable functions and $[D_1\varpi(\iota)]^\alpha = [\tilde{\jmath}'_\alpha(\iota), \tilde{\ell}'_\alpha(\iota)]$.*
*(ii) If $\varpi$ is (c2)-differentiable then $\tilde{\jmath}_\alpha$ and $\tilde{\ell}_\alpha$ are differentiable functions and we have $[D_2\varpi(\iota)]^\alpha = [\tilde{\ell}'_\alpha(\iota), \tilde{\jmath}'_\alpha(\iota)]$.*

**Theorem 2.5.** *(see [21]) Let $\mathfrak{I} : \aleph \to \mathbb{R}_F$ and we suppose that the derivative $\mathfrak{I}'$ is integrable over $\aleph$. Then $\forall \iota \in \aleph$, we obtain*
*(a) if $\mathfrak{I}$ is (c1)-differentiable, then $\mathfrak{I}(\iota) = \mathfrak{I}(b) + \int_b^\iota \mathfrak{I}'(s)ds$;*
*(b) if $\mathfrak{I}$ is (c2)-differentiable, then $\mathfrak{I}(\iota) = \mathfrak{I}(b) \ominus \int_b^\iota -\mathfrak{I}'(s)ds$.*

**Theorem 2.6.** *(see [20]) Let $\varpi$ be (c2)-differentiable on $\aleph$ and assume that the derivative $\varpi'$ is integrable over $\aleph$. Then for each $\iota \in \aleph$ we have*

$$\varpi(\iota) = \varpi(a) \ominus \int_a^\iota -\varpi'(\tau)d\tau.$$

**Theorem 2.7.** *(see [23]) Let $\mathfrak{I} : \aleph \to \mathbb{R}_F$ be continuous. Define $\Xi(\iota) := \sigma \ominus \int_v^\iota -\mathfrak{I}(s)ds$, $\iota \in \aleph$, where $\sigma \in \mathbb{R}_F$ is such that the preceeding H-difference exist on $\aleph$. Then $\Xi(\iota)$ is (c2)-differentiable and $\Xi'(\iota) = \mathfrak{I}(\iota)$.*

Consider the following conditions (here $\varpi : \mathbb{R} \to \mathbb{R}_F$):
(H1) For a given $\iota \in \aleph$, $\varpi(\iota + h) \ominus \varpi(\iota)$ and $\varpi(\iota) \ominus \varpi(\iota - h)$ exist for $h \to 0^+$;
(H2) For a given $\iota \in \aleph$, $\varpi(\iota) \ominus \varpi(\iota + h)$ and $\varpi(\iota - h) \ominus \varpi(\iota)$ exist for $h \to 0^+$.

## 3. Main results

### 3.1. Existence results

We consider the following systems:

$$\begin{cases} \nu'(\iota) = a\nu(\iota) + \daleth(\iota, \nu(\iota)),\ \iota \in [0, \urcorner], \iota \neq \iota_k, \\ \Delta\nu(\iota_k) = c_k\nu(\iota_k^-) + \daleth_k,\ k \in \aleph, \\ \nu(0) = \nu_0 \in \mathbb{R}_F, \end{cases} \tag{1}$$

where $a < 0$, $0 = \iota_0 < \iota_1 < \iota_2 < \cdots < \iota_k < \iota_m < \iota_{m+1} = \urcorner$, and $\daleth : [0, \urcorner] \times \mathbb{R}_F \to \mathbb{R}_F$, $1 + c_k < 0$, $\daleth_k \in \mathbb{R}_F$.

In this paper, we consider Case: ($c1$)-differentiable and Case: ($c2$)-differentiable.

We give the following lemma, which transfers the result for linear fuzzy differential equations in [16, 18] to the nonlinear case.

**Lemma 3.1.** *Assume that the function $\daleth : \aleph \times \mathbb{R}_F \to \mathbb{R}_F$ is continuous. $\nu : \aleph \to \mathbb{R}_F$ is a solution of problem (1) if and only if $\nu$ is piecewise continuous on $\iota \in \aleph$ and it satisfies (L1) or (L2):*

*(L1) $\nu$ is (c1)-differentiable case:*

$$
\nu(\iota) = \begin{cases}
\cosh(a\iota)\nu_0 + \sinh(a\iota)\nu_0 \\
+ \int_0^{\iota} [\cosh(a(\iota - s))\gimel(s, \nu(s)) + \sinh(a(\iota - s))\gimel(s, \nu(s))]ds, \ \iota \in [0, \iota_1], \\
(1 + c_1)\sinh(a\iota)\nu_0 + (1 + c_1)\cosh(a\iota)\nu_0 \\
+ \int_0^{\iota_1} [(1 + c_1)\sinh(a(\iota - s))\gimel(s, \nu(s)) + (1 + c_1)\cosh(a(\iota - s))\gimel(s, \nu(s))]ds \\
+ \int_{\iota_1}^{\iota} [\cosh(a(\iota - s))\gimel(s, \nu(s)) + \sinh(a(\iota - s))\gimel(s, \nu(s))]ds \\
+ \cosh(a(\iota - \iota_1^+))\gimel_1 + \sinh(a(\iota - \iota_1^+))\gimel_1, \iota \in (\iota_1, \iota_2], \\
\cdots \\
\Big[\cosh(a(\iota - \iota_k^+))\varphi_1 + \sinh(a(\iota - \iota_k^+))\varphi_2\Big]\nu_0 \\
+\Big[\cosh(a(\iota - \iota_k^+))\varphi_2 + \sinh(a(\iota - \iota_k^+))\varphi_1\Big]\nu_0 \\
+ \sum_{i=1}^{k-1} \int_{\iota_{i-1}}^{\iota_i} \Big\{\Big[(1 + c_i)\sinh(a(\iota - \iota_k^+) + a(\iota_i - s))m_1 \\
+(1 + c_i)\cosh(a(\iota - \iota_k^+) + a(\iota_i - s))m_2\Big]\gimel(s, \nu(s)) \\
+\Big[(1 + c_i)\cosh(a(\iota - \iota_k^+) + a(\iota_i - s))m_1 \\
+(1 + c_i)\sinh(a(\iota - \iota_k^+) + a(\iota_i - s))m_2\Big]\gimel(s, \nu(s))\Big\}ds \\
+ \int_{\iota_{k-1}}^{\iota_k} \Big\{\Big[(1 + c_k)\sinh(a(\iota - \iota_k^+))\cosh(a(\iota_k - s)) \\
+(1 + c_k)\cosh(a(\iota - \iota_k^+))\sinh(a(\iota_k - s))\Big]\gimel(s, \nu(s)) \\
+\Big[(1 + c_k)\sinh(a(\iota - \iota_k^+))\sinh(a(\iota_k - s)) \\
+(1 + c_k)\cosh(a(\iota - \iota_k^+))\cosh(a(\iota_k - s))\Big]\gimel(s, \nu(s))\Big\}ds \\
+ \int_{\iota_k}^{\iota} \Big[\cosh(a(\iota - s))\gimel(s, \nu) + \sinh(a(\iota - s))\gimel(s, \nu(s))\Big]ds \\
+ \sum_{i=1}^{k-1} \Big\{\Big[\cosh(a(\iota - \iota_k^+))m_1 + \sinh(a(\iota - \iota_k^+))m_2\Big]\gimel_i \\
+\Big[\cosh(a(\iota - \iota_k^+))m_2 + \sinh(a(\iota - \iota_k^+))m_1\Big]\gimel_i\Big\} \\
+ \cosh(a(\iota - \iota_k^+))\gimel_k + \sinh(a(\iota - \iota_k^+))\gimel_k, \\
\iota \in (\iota_k, \iota_{k+1}],
\end{cases} \tag{2}
$$

*where*

$$
\varphi_1 = \frac{\varsigma_{j1} + \varsigma_{j2}}{2}, \ \varphi_2 = \frac{\varsigma_{j1} - \varsigma_{j2}}{2}, \ m_1 = \frac{r_{j1} + r_{j2}}{2}, \ m_2 = \frac{r_{j1} - r_{j2}}{2},
$$

$$
\varsigma_{j1} = \prod_{j=k}^{1} [(1 + c_j)\sinh(a(\iota_j - \iota_{j-1}^+)) + (1 + c_j)\cosh(a(\iota_j - \iota_{j-1}^+))],
$$

$$
\varsigma_{j2} = \prod_{j=k}^{1} [(1 + c_j)\sinh(a(\iota_j - \iota_{j-1}^+)) - (1 + c_j)\cosh(a(\iota_j - \iota_{j-1}^+))],
$$

$$
r_{j1} = \prod_{j=k}^{i+1} [(1 + c_j)\sinh(a(\iota_j - \iota_{j-1}^+)) + (1 + c_j)\cosh(a(\iota_j - \iota_{j-1}^+))],
$$

$$
r_{j2} = \prod_{j=k}^{i+1} [(1 + c_j)\sinh(a(\iota_j - \iota_{j-1}^+)) - (1 + c_j)\cosh(a(\iota_j - \iota_{j-1}^+))].
$$

*(L2) $\nu$ is (c2)-differentiable case:*

$$\nu(\iota) = \begin{cases} e^{a\iota}\nu_0 \ominus \int_0^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds, \ \iota\in[0,\iota_1], \\ e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-\iota_0^+)}\nu_0 \ominus \Big\{ \int_0^{\iota_1}(-1)e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-s)}\gimel(s,\nu(s))ds \\ +\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\} + e^{a(\iota-\iota_1^+)}\gimel_1, \iota\in(\iota_1,\iota_2], \\ \cdots \\ e^{a(\iota-\iota_k^+)}\prod\limits_{j=k}^{1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\nu_0 \ominus \Big\{ \sum\limits_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}(-1)e^{a(\iota-\iota_k^+)}\prod\limits_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)} \\ \times(1+c_i)e^{a(\iota_i-s)}\gimel(s,\nu(s))ds + \int_{\iota_{k-1}}^{\iota_k}(-1)e^{a(\iota-\iota_k^+)}(1+c_k)e^{a(\iota_k-s)}\gimel(s,\nu(s))ds \\ +\int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\} + \sum\limits_{i=1}^{k-1}e^{a(\iota-\iota_k^+)}\prod\limits_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\gimel_i + e^{a(\iota-\iota_k^+)}\gimel_k, \\ \iota\in(\iota_k,\iota_{k+1}], \end{cases} \tag{3}$$

*provided that the Hukuhara differences exist in* (3).

*Proof.* ***Case 1.*** *$\nu$ is (c1)-differentiable.*
*If $\nu$ satisfies* (1)*, then it satisfies* (2)*. Indeed, if $\iota\in[0,\iota_1]$ we can get*

$$\nu'(\iota) = a\nu(\iota) + \gimel(\iota,\nu(\iota)).$$

*From [19, Lemma 3.2] we can get*

$$\nu(\iota) = \cosh(a\iota)\nu_0 + \sinh(a\iota)\nu_0 + \int_0^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds.$$

*Then*

$$\nu(\iota_1) = \cosh(a\iota_1)\nu_0 + \sinh(a\iota_1)\nu_0 + \int_0^{\iota_1}[\cosh(a(\iota_1-s))\gimel(s,\nu(s)) + \sinh(a(\iota_1-s))\gimel(s,\nu(s))]ds.$$

*If $\iota\in(\iota_1,\iota_2]$ then [19, Lemma 3.2] implies that*

$$\begin{aligned} \nu(\iota) &= \cosh(a(\iota-\iota_1^+))\nu(\iota_1^+) + \sinh(a(\iota-\iota_1^+))\nu(\iota_1^+) \\ &\quad + \int_{\iota_1}^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds \\ &= (1+c_1)\sinh(a\iota)\nu_0 + (1+c_1)\cosh(a\iota)\nu_0 \\ &\quad + \int_0^{\iota_1}[(1+c_1)\sinh(a(\iota-s))\gimel(s,\nu(s)) + (1+c_1)\cosh(a(\iota-s))\gimel(s,\nu(s))]ds \\ &\quad + \int_{\iota_1}^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds \\ &\quad + \cosh(a(\iota-\iota_1^+))\gimel_1 + \sinh(a(\iota-\iota_1^+))\gimel_1. \end{aligned}$$

*Repeat the above procedures, we get $\iota\in(\iota_k,\iota_{k+1}]$, $k\in\aleph$, by using [19, Lemma 3.2], we can get*

$$\begin{aligned} \nu(\iota) &= \cosh(a(\iota-\iota_k^+))\nu(\iota_k^+) + \sinh(a(\iota-\iota_k^+))\nu(\iota_k^+) \\ &\quad + \int_{\iota_k}^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds \\ &= \Big[\cosh(a(\iota-\iota_k^+))\varphi_1 + \sinh(a(\iota-\iota_k^+))\varphi_2\Big]\nu_0 \\ &\quad + \Big[\cosh(a(\iota-\iota_k^+))\varphi_2 + \sinh(a(\iota-\iota_k^+))\varphi_1\Big]\nu_0 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}\bigg\{\bigg[(1+c_i)\sinh(a(\iota-\iota_k^+)+a(\iota_i-s))m_1\\
&+(1+c_i)\cosh(a(\iota-\iota_k^+)+a(\iota_i-s))m_2\bigg]\gimel(s,\nu(s))\\
&+\bigg[(1+c_i)\cosh(a(\iota-\iota_k^+)+a(\iota_i-s))m_1\\
&+(1+c_i)\sinh(a(\iota-\iota_k^+)+a(\iota_i-s))m_2\bigg]\gimel(s,\nu(s))\bigg\}ds\\
&+\int_{\iota_{k-1}}^{\iota_k}\bigg\{\bigg[(1+c_k)\sinh(a(\iota-\iota_k^+))\cosh(a(\iota_k-s))\\
&+(1+c_k)\cosh(a(\iota-\iota_k^+))\sinh(a(\iota_k-s))\bigg]\gimel(s,\nu(s))\\
&+\bigg[(1+c_k)\sinh(a(\iota-\iota_k^+))\sinh(a(\iota_k-s))\\
&+(1+c_k)\cosh(a(\iota-\iota_k^+))\cosh(a(\iota_k-s))\bigg]\gimel(s,\nu(s))\bigg\}ds\\
&+\int_{\iota_k}^{\iota}\bigg[\cosh(a(\iota-s))\gimel(s,\nu(s))+\sinh(a(\iota-s))\gimel(s,\nu(s))\bigg]ds\\
&+\sum_{i=1}^{k-1}\bigg\{\bigg[\cosh(a(\iota-\iota_k^+))m_1+\sinh(a(\iota-\iota_k^+))m_2\bigg]\gimel_i\\
&+\bigg[\cosh(a(\iota-\iota_k^+))m_2+\sinh(a(\iota-\iota_k^+))m_1\bigg]\gimel_i\bigg\}\\
&+\cosh(a(\iota-\iota_k^+))\gimel_k+\sinh(a(\iota-\iota_k^+))\gimel_k.
\end{aligned}
$$

*If $\nu$ satisfies* (2)*, then it satisfies* (1)*. Indeed, if $\iota\in[0,\iota_1]$, it is easy to see that*

$$\nu(0)=\cosh(0)\nu_0+\sinh(0)\nu_0=\nu_0.$$

*According to Lemma 3.2 in [19] we can get*

$$\nu'(\iota)=a\nu(\iota)+\gimel(\iota,\nu(\iota)),\ \ \textit{for}\ \ \iota\in[0,\iota_1].$$

*If $\iota\in(\iota_1,\iota_2]$ and $\iota-h\in(\iota_1,\iota_2]$ with $h>0$ sufficiently small, we can get*

$$
\begin{aligned}
&\nu(\iota)\ominus\nu(\iota-h)\\
=\;&(1+c_1)\sinh(a\iota)\nu_0\ominus(1+c_1)\sinh(a(\iota-h))\nu_0\\
&+(1+c_1)\cosh(a\iota)\nu_0\ominus(1+c_1)\cosh(a(\iota-h))\nu_0\\
&+\int_0^{\iota_1}[(1+c_1)\sinh(a(\iota-s))\gimel(s,\nu(s))\ominus(1+c_1)\sinh(a(\iota-h-s))\gimel(s,\nu(s))]ds\\
&+\int_0^{\iota_1}[(1+c_1)\cosh(a(\iota-s))\gimel(s,\nu(s))\ominus(1+c_1)\cosh(a(\iota-h-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota_1}^{\iota-h}[\cosh(a(\iota-s))\gimel(s,\nu(s))\ominus\cosh(a(\iota-h-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota_1}^{\iota-h}[\sinh(a(\iota-s))\gimel(s,\nu(s))\ominus\sinh(a(\iota-h-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota-h}^{\iota}\cosh(a(\iota-s))\gimel(s,\nu(s))ds+\int_{\iota-h}^{\iota}\sinh(a(\iota-s))\gimel(s,\nu(s))ds
\end{aligned}
\tag{4}
$$

$$+[\cosh(a(\iota-\iota_1^+))\gimel_1\ominus\cosh(a(\iota-h-\iota_1^+))\gimel_1]+[\sinh(a(\iota-\iota_1^+))\gimel_1\ominus\sinh(a(\iota-h-\iota_1^+))\gimel_1],$$

*and for* $\iota+h\in(\iota_1,\iota_2]$,

$$\begin{aligned}
&\nu(\iota+h)\ominus\nu(\iota)\\
=\;&(1+c_1)\sinh(a(\iota+h))\nu_0\ominus(1+c_1)\sinh(a\iota)\nu_0 \qquad (5)\\
&+(1+c_1)\cosh(a(\iota+h)))\nu_0\ominus(1+c_1)\cosh(a\iota)\nu_0\\
&+\int_0^{\iota_1}[(1+c_1)\sinh(a(\iota+h-s))\gimel(s,\nu(s))\ominus(1+c_1)\sinh(a(\iota-s))\gimel(s,\nu(s))]ds\\
&+\int_0^{\iota_1}[(1+c_1)\cosh(a(\iota+h-s))\gimel(s,\nu(s))\ominus(1+c_1)\cosh(a(\iota-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota_1}^{\iota}[\cosh(a(\iota+h-s))\gimel(s,\nu(s))\ominus\cosh(a(\iota-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota_1}^{\iota}[\sinh(a(\iota+h-s))\gimel(s,\nu(s))\ominus\sinh(a(\iota-s))\gimel(s,\nu(s))]ds\\
&+\int_{\iota}^{\iota+h}\cosh(a(\iota+h-s))\gimel(s,\nu(s))ds+\int_{\iota}^{\iota+h}\sinh(a(\iota+h-s))\gimel(s,\nu(s))ds\\
&+[\cosh(a(\iota+h-\iota_1^+))\gimel_1\ominus\cosh(a(\iota-\iota_1^+))\gimel_1]+[\sinh(a(\iota+h-\iota_1^+))\gimel_1\ominus\sinh(a(\iota-\iota_1^+))\gimel_1].
\end{aligned}$$

*Multiplying both sides of the equations* (4) *and* (5) *by* $\frac{1}{h}$ *and taking the limit as* $h\to0^+$, *we can get*

$$\lim_{h\to0^+}\frac{\nu(\iota)\ominus\nu(\iota-h)}{h}=a\nu(\iota)+\gimel(\iota,\nu(\iota))$$

*and*

$$\lim_{h\to0^+}\frac{\nu(\iota+h)\ominus\nu(\iota)}{h}=a\nu(\iota)+\gimel(\iota,\nu(\iota)).$$

*Thus* $\nu(\iota)$ *is (c1)-differentiable on* $(\iota_1,\iota_2]$ *and consequently* $\nu'(\iota)=a\nu(\iota)+\gimel(\iota,\nu(\iota))$, *for each* $\iota\in(\iota_1,\iota_2]$. *Repeat the above procedures, then we obtain* $\iota\in(\iota_k,\iota_{k+1}]$, $k\in\aleph$, *we obtain* $\nu'(\iota)=a\nu(\iota)+\gimel(\iota,\nu(\iota))$, *for each* $\iota\in(\iota_k,\iota_{k+1}]$. *Also, we can easily see that* $\Delta\nu(\iota_k)=c_k\nu(\iota_k^-)+\gimel_k$, $k\in\aleph$.

***Case 2.*** $\nu$ *is (c2)-differentiable.*

*If* $\nu$ *satisfied* (1), *then it satisfies* (3). *Indeed, if* $\iota\in[0,\iota_1]$ *we have* $\nu'(\iota)=a\nu(\iota)+\gimel(\iota,\nu(\iota))$. *According to [19, Lemma 3.2] we have* $\nu(\iota)=e^{a\iota}\nu_0\ominus\int_0^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds$. *Then* $\nu(\iota_1)=e^{a\iota_1}\nu_0\ominus\int_0^{\iota_1}(-1)e^{a(\iota_1-s)}\gimel(s,\nu(s))ds$. *If* $\iota\in(\iota_1,\iota_2]$ *then [19, Lemma 3.2] means that*

$$\begin{aligned}
\nu(\iota)&=e^{a(\iota-\iota_1^+)}\nu(\iota_1^+)\ominus\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\\
&=e^{a(\iota-\iota_1^+)}\Big[(1+c_1)[e^{a\iota_1}\nu_0\ominus\int_0^{\iota_1}(-1)e^{a(\iota_1-s)}\gimel(s,\nu(s))ds]+\gimel_1\Big]\ominus\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\\
&=e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-\iota_0^+)}\nu_0\ominus\Big\{\int_0^{\iota_1}(-1)e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-s)}\gimel(s,\nu(s))ds\\
&\quad+\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\}+e^{a(\iota-\iota_1^+)}\gimel_1.
\end{aligned}$$

*Repeat the above procedures, if* $\iota\in(\iota_k,\iota_{k+1}]$, $k\in\aleph$, *then using [19, Lemma 3.2], we can get*

$$\begin{aligned}
\nu(\iota)&=e^{a(\iota-\iota_k^+)}\nu(\iota_k^+)\ominus\int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\\
&=e^{a(\iota-\iota_k^+)}[(1+c_k)\nu(\iota_k)+\gimel_k]\ominus\int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds
\end{aligned}$$

$$= e^{a(\iota-\iota_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\nu_0 \ominus \Big\{\sum_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}(-1)e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}$$
$$\times(1+c_i)e^{a(\iota_i-s)}\gimel(s,\nu(s))ds + \int_{\iota_{k-1}}^{\iota_k}(-1)e^{a(\iota-\iota_k^+)}(1+c_k)e^{a(\iota_k-s)}\gimel(s,\nu(s))ds$$
$$+\int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\} + \sum_{i=1}^{k-1}e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\gimel_i + e^{a(\iota-\iota_k^+)}\gimel_k.$$

*If $\nu$ satisfies* (3)*, then it satisfies* (1)*. Indeed, if $\iota \in [0,\iota_1]$ it is easy to see that $\nu(0) = e^0\nu_0 = \nu_0$ and the Hukuhara difference $e^{a\iota}\nu_0 \ominus \int_0^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu)ds$ exists. From Lemma 3.2 in [19] we obtain*

$$\nu'(\iota) = a\nu(\iota) + \gimel(\iota,\nu(\iota)), \ \text{for} \ \ \iota \in [0,\iota_1].$$

*If $\iota \in (\iota_1,\iota_2]$ and $\iota - h \in (\iota_1,\iota_2]$ with $h > 0$ enough small, we have*

$$\begin{aligned}\nu(\iota-h)\ominus\nu(\iota) &= \Big\{e^{a(\iota-h-\iota_1^+)}\nu(\iota_1^+)\ominus\int_{\iota_1}^{\iota-h}(-1)e^{a(\iota-h-s)}\gimel(s,\nu(s))ds\Big\}\\ &\quad\ominus\Big\{e^{a(\iota-\iota_1^+)}\nu(\iota_1^+)\ominus\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\}\\ &= \Big(e^{a(\iota-h-\iota_1^+)}\nu(\iota_1^+)\ominus e^{a(\iota-\iota_1^+)}\nu(\iota_1^+)\Big)\ominus\Big(\int_{\iota_1}^{\iota-h}(-1)e^{a(\iota-h-s)}\gimel(s,\nu(s))ds \qquad (6)\\ &\quad\ominus\int_{\iota_1}^{\iota-h}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\ominus\int_{\iota-h}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big)\end{aligned}$$

*and for $\iota + h \in (\iota_1,\iota_2]$,*

$$\begin{aligned}\nu(\iota)\ominus\nu(\iota+h) &= \Big\{e^{a(\iota-\iota_1^+)}\nu(\iota_1^+)\ominus\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds\Big\}\\ &\quad\ominus\Big\{e^{a(\iota+h-\iota_1^+)}\nu(\iota_1^+)\ominus\int_{\iota_1}^{\iota+h}(-1)e^{a(\iota+h-s)}\gimel(s,\nu(s))ds\Big\}\\ &= \Big(e^{a(\iota-\iota_1^+)}\nu(\iota_1^+)\ominus e^{a(\iota+h-\iota_1^+)}\nu(\iota_1^+)\Big)\ominus\Big(\int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds \qquad (7)\\ &\quad\ominus\int_{\iota_1}^{\iota}(-1)e^{a(\iota+h-s)}\gimel(s,\nu(s))ds\ominus\int_{\iota}^{\iota+h}(-1)e^{a(\iota+h-s)}\gimel(s,\nu(s))ds\Big).\end{aligned}$$

*The equations* (6) *and* (7) *both sides multiplied by $\frac{1}{-h}$ and taking the limit as $h \to 0^+$, we obtain*

$$\lim_{h\to0^+}\frac{\nu(\iota)\ominus\nu(\iota+h)}{-h} = a\nu(\iota) + \gimel(\iota,\nu(\iota))$$

*and $\lim_{h\to0^+}\frac{\nu(\iota-h)\ominus\nu(\iota)}{-h} = a\nu(\iota) + \gimel(\iota,\nu(\iota))$. Thus $\nu(\iota)$ is (c2)-differentiable on $(\iota_1,\iota_2]$ and consequently*

$$\nu'(\iota) = a\nu(\iota) + \gimel(\iota,\nu(\iota)), \ \text{for each} \ \ \iota \in (\iota_1,\iota_2].$$

*Continue this process so if $\iota \in (\iota_k,\iota_{k+1}]$, $k = 1,2,\cdots,m$, then*

$$\nu'(\iota) = a\nu(\iota) + \gimel(\iota,\nu(\iota)), \ \forall\iota \in (\iota_k,\iota_{k+1}].$$

*Also, we can easily see that $\Delta\nu(\iota_k) = c_k\nu(\iota_k^-) + \gimel_k$, $k \in \aleph$. The proof is complete.* □

***Theorem 3.2.*** *((c1)-differentiable case). Suppose* $\gimel : [0,\daleth]\times\mathbb{R}_F \to \mathbb{R}_F$ *is continuous and satisfies (F1)-(F3):*

(F1) $\exists L_1 > 0$ *such that* $D[\gimel(\iota,\psi),\gimel(\iota,\phi)] \le L_1 D[\psi,\phi]$, *for all* $\psi,\phi\in\mathbb{R}_F$ *and* $\forall\iota\in[0,\daleth]$;

(F2) $\exists M_1 > 0$ *and* $r > 0$ $(r + a > 0)$ *such that* $D[\gimel(\iota,\hat{0}),\hat{0}] \le M_1 e^{r\iota}$, $\forall\iota\in[0,\daleth]$;

(F3) $\exists K_1 > 0$ *such that* $D[\gimel_k,\hat{0}] \le K_1$ *for* $k\in\{1,\dots,m\}$.

*Then* (1) *has a unique solution provided that*

$$L_1 < \frac{r}{(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}} e^{(r-a)\daleth}},$$

*where* $a < 0$, $\hat{c} = \max\{|\,1 + c_k\,|; k = 1,2,\cdots,m\}$, $\hat{\iota} = \max\{\iota_k - \iota_{k-1}; k = 1,2,\cdots,m+1\}$.

*Proof.* *Define a metric on* $PC[\aleph,\mathbb{R}_F]$ *by*

$$D_{ra}(u,v) = \sup_{\iota\in(\iota_k,\iota_{k+1}],k\in\mathbb{M}_0} D[u(\iota),v(\iota)]e^{-(r-a)\iota},$$

*for* $u,v\in PC[\aleph,\mathbb{R}_F]$, *and* $(PC[\aleph,\mathbb{R}_F],D_{ar})$ *is a complete metric space (see [4]). Define an operator* $\mathbb{P}$ *on* $PC[\aleph,\mathbb{R}_F]$ *by*

$$\mathbb{P}\nu(\iota) = \begin{cases}
\cosh(a\iota)\nu_0 + \sinh(a\iota)\nu_0 \\
+\int_0^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds,\ \iota\in[0,\iota_1], \\
(1+c_1)\sinh(a\iota)\nu_0 + (1+c_1)\cosh(a\iota)\nu_0 \\
+\int_0^{\iota_1}[(1+c_1)\sinh(a(\iota-s))\gimel(s,\nu) + (1+c_1)\cosh(a(\iota-s))\gimel(s,\nu(s))]ds \\
+\int_{\iota_1}^{\iota}[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))]ds \\
+\cosh(a(\iota-\iota_1^+))\gimel_1 + \sinh(a(\iota-\iota_1^+))\gimel_1,\ \iota\in(\iota_1,\iota_2], \\
\cdots \\
\Big[\cosh(a(\iota-\iota_k^+))\varphi_1 + \sinh(a(\iota-\iota_k^+))\varphi_2\Big]\nu_0 \\
+\Big[\cosh(a(\iota-\iota_k^+))\varphi_2 + \sinh(a(\iota-\iota_k^+))\varphi_1\Big]\nu_0 \\
+\sum_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}\Big\{\Big[(1+c_i)\sinh(a(\iota-\iota_k^+) + a(\iota_i - s))m_1 \\
+(1+c_i)\cosh(a(\iota-\iota_k^+) + a(\iota_i - s))m_2\Big]\gimel(s,\nu(s)) \\
+\Big[(1+c_i)\cosh(a(\iota-\iota_k^+) + a(\iota_i - s))m_1 \\
+(1+c_i)\sinh(a(\iota-\iota_k^+) + a(\iota_i - s))m_2\Big]\gimel(s,\nu(s))\Big\}ds \\
+\int_{\iota_{k-1}}^{\iota_k}\Big\{\Big[(1+c_k)\sinh(a(\iota-\iota_k^+))\cosh(a(\iota_k - s)) \\
+(1+c_k)\cosh(a(\iota-\iota_k^+))\sinh(a(\iota_k - s))\Big]\gimel(s,\nu(s)) \\
+\Big[(1+c_k)\sinh(a(\iota-\iota_k^+))\sinh(a(\iota_k - s)) \\
+(1+c_k)\cosh(a(\iota-\iota_k^+))\cosh(a(\iota_k - s))\Big]\gimel(s,\nu(s))\Big\}ds \\
+\int_{\iota_k}^{\iota}\Big[\cosh(a(\iota-s))\gimel(s,\nu(s)) + \sinh(a(\iota-s))\gimel(s,\nu(s))\Big]ds \\
+\sum_{i=1}^{k-1}\Big\{\Big[\cosh(a(\iota-\iota_k^+))m_1 + \sinh(a(\iota-\iota_k^+))m_2\Big]\gimel_i \\
+\Big[\cosh(a(\iota-\iota_k^+))m_2 + \sinh(a(\iota-\iota_k^+))m_1\Big]\gimel_i\Big\} \\
+\cosh(a(\iota-\iota_k^+))\gimel_k + \sinh(a(\iota-\iota_k^+))\gimel_k, \\
\iota\in(\iota_k,\iota_{k+1}].
\end{cases}$$

*Let* $\nu \in PC[\aleph, \mathbb{R}_F]$*. Then with* $\rho = D_{ra}[\nu, \hat{0}]$ *we have* $D[\nu(\iota), \hat{0}] \leq \rho e^{(r-a)\iota}$ *for any* $\iota \in (\iota_k, \iota_{k+1}]$, $k \in \mathbb{M}_0$*. Note* $\mathbb{P}\nu \in PC[\aleph, \mathbb{R}_F]$ *and note for* $\iota \in [0, \iota_1]$ *we can get*

$$\begin{aligned}
D[(\mathbb{P}\nu)(\iota), \hat{0}] &\leq (\cosh(a\iota) - \sinh(a\iota))D[\nu_0, \hat{0}] \\
&\quad + \int_0^\iota (\cosh(a(\iota - s)) - \sinh(a(\iota - s)))D[\daleth(s, \nu(s)), \hat{0}]ds \\
&\leq D[e^{-a\iota}\nu_0, \hat{0}] + \int_0^\iota e^{-a(\iota-s)}\Big[L_1 D[\nu, \hat{0}] + M_1 e^{rs}\Big]ds \\
&\leq D[e^{-a\iota}\nu_0, \hat{0}] + \frac{L_1\rho e^{-a\iota}}{r}(e^{r\iota} - 1) + \frac{M_1 e^{-a\iota}}{r+a}(e^{(r+a)\iota} - 1)(r + a > 0),
\end{aligned}$$

*and for* $\iota \in (\iota_k, \iota_{k+1}]$, $k \in \aleph$*, we have*

$$\begin{aligned}
& D[(\mathbb{P}\nu)(\iota), \hat{0}] \\
\leq\ & D\Big[e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})]\nu_0, \hat{0}\Big] \\
& + L_1\sum_{i=1}^{k-1} e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))\frac{\rho}{r}e^{-a\iota_i}\Big(e^{r\iota_i} - e^{r\iota_{i-1}}\Big) \\
& + \frac{M_1}{r+a}\sum_{i=1}^{k-1} e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))e^{-a\iota_i}\Big(e^{(r+a)\iota_i} - e^{(r+a)\iota_{i-1}}\Big) \\
& + \frac{\rho L_1}{r}(-(1+c_k))e^{-a\iota}\Big(e^{r\iota_k} - e^{r\iota_{k-1}}\Big) + \frac{M_1(-(1+c_k))e^{-a\iota}}{r+a}\Big(e^{(a+r)\iota_k} - e^{(a+r)\iota_{k-1}}\Big) \\
& + \frac{L_1\rho e^{-a\iota}}{r}\Big(e^{r\iota} - e^{r\iota_k}\Big) + \frac{M_1 e^{-a\iota}}{r+a}\Big(e^{(r+a)\iota} - e^{(r+a)\iota_k}\Big) \\
& + D\Big[\sum_{i=1}^{k-1} e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})]\daleth_i, \hat{0}\Big] + D[e^{-a(\iota-\iota_k^+)}\daleth_k, \hat{0}],
\end{aligned}$$

*so*

$$\begin{aligned}
& \sup_{\iota\in[0,\iota_1]} D[(\mathbb{P}\nu)(\iota), \hat{0}]e^{-(r-a)\iota} \\
=\ & \sup_{\iota\in[0,\iota_1]}\Big\{D[e^{-a\iota}\nu_0, \hat{0}] + \frac{L_1\rho e^{-a\iota}}{r}(e^{r\iota} - 1) + \frac{M_1 e^{-a\iota}}{r+a}(e^{(r+a)\iota} - 1)\Big\}e^{-(r-a)\iota} \\
\leq\ & D[\nu_0, \hat{0}] + \frac{L_1\rho}{r} + \frac{M_1}{r+a},
\end{aligned}$$

*and for* $k \in \aleph$*, we have*

$$\begin{aligned}
& \sup_{\iota\in(\iota_k,\iota_{k+1}]} D[(\mathbb{P}\nu)(\iota), \hat{0}]e^{-(r-a)\iota} \\
=\ & \sup_{\iota\in(\iota_k,\iota_{k+1}]}\Big\{D\Big[e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})]\nu_0, \hat{0}\Big] \\
& + L_1\sum_{i=1}^{k-1} e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))\frac{\rho}{r}e^{-a\iota_i}\Big(e^{r\iota_i} - e^{r\iota_{i-1}}\Big)
\end{aligned}$$

$$+\frac{M_1}{r+a}\sum_{i=1}^{k-1}e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))e^{-a\iota_i}\left(e^{(r+a)\iota_i}-e^{(r+a)\iota_{i-1}}\right)$$
$$+\frac{\rho L_1}{r}(-(1+c_k))e^{-a\iota}\left(e^{r\iota_k}-e^{r\iota_{k-1}}\right)+\frac{M_1(-(1+c_k))e^{-a\iota}}{r+a}\left(e^{(a+r)\iota_k}-e^{(a+r)\iota_{k-1}}\right)$$
$$+\frac{L_1\rho e^{-a\iota}}{r}\left(e^{r\iota}-e^{r\iota_k}\right)+\frac{M_1e^{-a\iota}}{r+a}\left(e^{(r+a)\iota}-e^{(r+a)\iota_k}\right)$$
$$+D\Big[\sum_{i=1}^{k-1}e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})]\daleth_i,\hat{0}\Big]+D[e^{-a(\iota-\iota_k^+)}\daleth_k,\hat{0}]\Big\}e^{-(r-a)\iota}$$
$$\leq D\Big[\hat{c}^m e^{-am\hat{\iota}}\nu_0,\hat{0}\Big]+\frac{M_1(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}}e^{-a\urcorner}}{r+a}$$
$$+\frac{L_1(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}}\rho e^{(r-a)\urcorner}}{r}+K_1m\hat{c}^{m-1}e^{-a(m-1)\hat{\iota}}.$$

*Now, let* $\mathbb{P}x,\ \mathbb{P}\nu\in PC[\aleph,\mathbb{R}_F]$. *For* $\iota\in[0,\iota_1]$, *we have*

$$D[(\mathbb{P}\nu)(\iota),(\mathbb{P}x)(\iota)]$$
$$= D\Big[\cosh(a\iota)\nu_0+\sinh(a\iota)\nu_0+\int_0^\iota\Big[\cosh(a(\iota-s))\daleth(s,\nu(s))+\sinh(a(\iota-s))\daleth(s,\nu(s))\Big]ds,$$
$$\cosh(a\iota)\nu_0+\sinh(a\iota)\nu_0+\int_0^\iota\Big[\cosh(a(\iota-s))\daleth(s,x(s))+\sinh(a(\iota-s))\daleth(s,x(s))\Big]ds\Big]$$
$$\leq \int_0^\iota\Big[\cosh(a(\iota-s))D[\daleth(s,\nu(s)),\daleth(s,x(s))]+|\sinh(a(\iota-s)|\,D[\daleth(s,\nu(s)),\daleth(s,x(s))]\Big]ds$$
$$= \int_0^\iota e^{-a(\iota-s)}LD[\nu(s),x(s)]e^{-(r-a)s}e^{(r-a)s}ds\leq\frac{L_1D_{ra}[\nu,x]}{r}e^{-a\iota}(e^{r\iota}-1)$$

*and thus (recall* $a<0$*)*

$$\sup_{\iota\in[0,\iota_1]}D[(\mathbb{P}\nu)(\iota),(\mathbb{P}x)(\iota)]e^{-(r-a)\iota}$$
$$= \sup_{\iota\in[0,\iota_1]}\Big\{\frac{L_1D_{ra}[\nu,x]}{r}e^{-a\iota}(e^{r\iota}-1)\Big\}e^{-(r-a)\iota}\leq\frac{L_1}{r}D_{ra}[\nu,x]. \tag{8}$$

*Similarly, for* $\iota\in(\iota_k,\iota_{k+1}]$, $k\in\aleph$, *we have*

$$D[(\mathbb{P}\nu)(\iota),(\mathbb{P}x)(\iota)]$$
$$\leq L_1\sum_{i=1}^{k-1}e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))\frac{1}{r}e^{-a\iota_i}\left(e^{r\iota_i}-e^{r\iota_{i-1}}\right)D_{ra}[\nu,x]$$
$$+\frac{L_1}{r}(-(1+c_k))e^{-a\iota}\left(e^{r\iota_k}-e^{r\iota_{k-1}}\right)D_{ra}[\nu,x]+\frac{L_1e^{-a\iota}}{r}\left(e^{r\iota}-e^{r\iota_k}\right)D_{ra}[\nu,x],$$

*and so*

$$\sup_{\iota\in(\iota_k,\iota_{k+1}]}D[(\mathbb{P}\nu)(\iota),(\mathbb{P}x)(\iota)]e^{-(r-a)\iota}$$
$$\leq \sup_{\iota\in(\iota_k,\iota_{k+1}]}\Big\{L_1\sum_{i=1}^{k-1}e^{-a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[(1+c_j)(-e^{-a(\iota_j-\iota_{j-1}^+)})](-(1+c_i))\frac{1}{r}e^{-a\iota_i}\left(e^{r\iota_i}-e^{r\iota_{i-1}}\right)D_{ra}[\nu,x]$$
$$+\frac{L_1}{r}(-(1+c_k))e^{-a\iota}\left(e^{r\iota_k}-e^{r\iota_{k-1}}\right)D_{ra}[\nu,x]+\frac{L_1e^{-a\iota}}{r}\left(e^{r\iota}-e^{r\iota_k}\right)D_{ra}[\nu,x]\Big\}e^{-(r-a)\iota} \tag{9}$$

$$= \frac{L_1(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}} e^{(r-a)\daleth}}{r} D_{ra}[\nu, x].$$

*From* (8) *and* (9) *one has*

$$D_{ra}[(\mathbb{P}\nu),(\mathbb{P}x)] \leq \frac{L_1}{r} D_{ra}[\nu,x], \text{ for } \iota \in [0,\iota_1]$$

*and*

$$D_{ra}[(\mathbb{P}\nu),(\mathbb{P}x)] \leq \frac{L_1(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}} e^{(r-a)\daleth}}{r} D_{ra}[\nu,x],$$

*for* $\iota \in (\iota_k, \iota_{k+1}]$, $k \in \aleph$.

*Since* $L_1 < \frac{r}{(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}} e^{(r-a)\daleth}}$, *we infer that* $\mathbb{P}$ *is a contraction on* $PC[\aleph, \mathbb{R}_F]$. *The Banach fixed point theorem guarantees the existence of a unique fixed point for* $\mathbb{P}$ *so* (1) *has a unique solution in the* (c1)*-differentiable case.* □

**Theorem 3.3.** *((c2)-differentiable case). Suppose* $\beth : [0,\daleth] \times \mathbb{R}_F \to \mathbb{R}_F$ *is continuous and satisfies (f1)-(f3):*

$(f1)$ $\exists L > 0$ *such that* $D[\beth(\iota,\psi), \beth(\iota,\phi)] \leq LD[\psi,\phi]$, $\forall \psi,\phi \in \mathbb{R}_F$ *and* $\forall \iota \in [0,\daleth]$;

$(f2)$ $\exists M > 0$ *and* $r > 0$ *such that* $D[\beth(\iota,\hat{0}),\hat{0}] \leq Me^{r\iota}$ *for all* $\iota \in [0,\daleth]$;

$(f3)$ $\exists K > 0$ *such that* $D[\beth_k, 0] \leq K$ *for* $k \in \{1, ...., m\}$.

*Then* (1) *has a unique solution provided that* $L < \frac{r-2a}{e^{(r-2a)\daleth}(m+1)\hat{c}^m}$, *where* $a < 0$, $\hat{c} = \max\{\mid 1 + c_k \mid ; k = 1,2,\cdots,m\}$ *and assuming the Hukuhara difference in* (3) *exists.*

*Proof.* Define an operator $\mathbb{P}$ on $PC[\aleph, \mathbb{R}_F]$ by

$$(\mathbb{P}\nu)(\iota) = \begin{cases} e^{a\iota}\nu_0 \ominus \int_0^\iota (-1)e^{a(\iota-s)}\beth(s,\nu(s))ds, \ \iota \in [0,\iota_1], \\ e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-\iota_0^+)}\nu_0 \ominus \Big\{ \int_0^{\iota_1} (-1)e^{a(\iota-\iota_1^+)}(1+c_1)e^{a(\iota_1-s)}\beth(s,\nu(s))ds \\ + \int_{\iota_1}^{\iota}(-1)e^{a(\iota-s)}\beth(s,\nu(s))ds \Big\} + e^{a(\iota-\iota_1^+)}\beth_1, \iota \in (\iota_1,\iota_2], \\ \cdots \\ e^{a(\iota-\iota_k^+)} \prod_{j=k}^{1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\nu_0 \ominus \Big\{ \sum_{i=1}^{k-1} \int_{\iota_{i-1}}^{\iota_i} (-1)e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)} \\ \times(1+c_i)e^{a(\iota_i-s)}\beth(s,\nu(s))ds + \int_{\iota_{k-1}}^{\iota_k}(-1)e^{a(\iota-\iota_k^+)}(1+c_k)e^{a(\iota_k-s)}\beth(s,\nu(s))ds \\ + \int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\beth(s,\nu(s))ds \Big\} + \sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}\beth_i + e^{a(\iota-\iota_k^+)}\beth_k, \\ \iota \in (\iota_k,\iota_{k+1}]. \end{cases}$$

Let $\nu \in PC[\aleph, \mathbb{R}_F]$. Then with $\rho = D_{ra}[\nu,\hat{0}]$ we have $D[\nu(\iota),\hat{0}] \leq \rho e^{(r-a)\iota}$ for any $\iota \in (\iota_k,\iota_{k+1}]$, $k \in \mathbb{M}_0$. Note $\mathbb{P}\nu \in PC[\aleph, \mathbb{R}_F]$ and for $\iota \in [0,\iota_1]$ we have

$$\begin{aligned} D[(\mathbb{P}\nu)(\iota),\hat{0}] &= D[e^{a\iota}\nu_0 \ominus \int_0^\iota (-1)e^{a(\iota-s)}\beth(s,\nu(s))ds, \hat{0}] \\ &\leq D[e^{a\iota}\nu_0,\hat{0}] + D[\int_0^\iota (-1)e^{a(\iota-s)}\beth(s,\nu(s))ds,\hat{0}] \\ &\leq D[e^{a\iota}\nu_0,\hat{0}] + \int_0^\iota e^{a(\iota-s)}D[\beth(s,\nu(s)),\hat{0}]ds \\ &\leq D[e^{a\iota}\nu_0,\hat{0}] + \int_0^\iota e^{a(\iota-s)}\Big[D[\beth(s,\nu(s)),\beth(s,\hat{0})] + D[\beth(s,\hat{0}),\hat{0}]\Big]ds \\ &\leq D[e^{a\iota}\nu_0,\hat{0}] + \frac{L\rho e^{a\iota}}{r-2a}(e^{(r-2a)\iota}-1) + \frac{Me^{a\iota}}{r-a}(e^{(r-a)\iota}-1) \\ &\leq D[e^{a\iota}\nu_0,\hat{0}] + \frac{L\rho e^{a\iota}}{r-2a}e^{(r-2a)\iota} + \frac{Me^{a\iota}}{r-a}e^{(r-a)\iota}, \end{aligned}$$

and for $\iota \in (\iota_k, \iota_{k+1}]$, $k \in \aleph$, we get

$$
\begin{aligned}
& D[(\mathbb{P}\nu)(\iota), \hat{0}] \\
\leq\ & D\Big[e^{a(\iota-\iota_k^+)} \prod_{j=k}^{1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\nu_0, \hat{0}\Big] \\
& + L\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\hat{c}\frac{\rho}{r-2a} e^{a\iota_i}\Big(e^{(r-2a)\iota_i} - e^{(r-2a)\iota_{i-1}}\Big) \\
& + \frac{M}{r-a}\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\hat{c}e^{a\iota_i}\Big(e^{(r-a)\iota_i} - e^{(r-a)\iota_{i-1}}\Big) \\
& + \frac{\rho L}{r-2a}\hat{c}e^{a\iota}\Big(e^{(r-2a)\iota_k} - e^{(r-2a)\iota_{k-1}}\Big) + \frac{M\hat{c}e^{a\iota}}{r-a}\Big(e^{(r-a)\iota_k} - e^{(r-a)\iota_{k-1}}\Big) \\
& + \frac{L\rho e^{a\iota}}{r-2a}\Big(e^{(r-2a)\iota} - e^{(r-2a)\iota_k}\Big) + \frac{Me^{a\iota}}{r-a}\Big(e^{(r-a)\iota} - e^{(r-a)\iota_k}\Big) \\
& + D\Big[\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\beth_i, \hat{0}\Big] + D[e^{a(\iota-\iota_k^+)}\beth_k, \hat{0}],
\end{aligned}
$$

so

$$
\begin{aligned}
\sup_{\iota\in[0,\iota_1]} D[(\mathbb{P}\nu)(\iota), \hat{0}]e^{-(r-a)\iota} &= \sup_{\iota\in[0,\iota_1]} \Big\{D[e^{a\iota}\nu_0, \hat{0}] + \frac{L\rho e^{a\iota}}{r-2a}e^{(r-2a)\iota} + \frac{Me^{a\iota}}{r-a}e^{(r-a)\iota}\Big\}e^{-(r-a)\iota} \\
&\leq D[\nu_0, \hat{0}] + \frac{L\rho}{r-2a} + \frac{M}{r-a},
\end{aligned}
$$

and for $k \in \aleph$, we get

$$
\begin{aligned}
& \sup_{\iota\in(\iota_k,\iota_{k+1}]} D[(\mathbb{P}\nu)(\iota), \hat{0}]e^{-(r-a)\iota} \\
=\ & \sup_{\iota\in(\iota_k,\iota_{k+1}]} \Big\{D\Big[e^{a(\iota-\iota_k^+)} \prod_{j=k}^{1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\nu_0, \hat{0}\Big] \\
& + L\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\hat{c}\frac{\rho}{r-2a} e^{a\iota_i}\Big(e^{(r-2a)\iota_i} - e^{(r-2a)\iota_{i-1}}\Big) \\
& + \frac{M}{r-a}\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\hat{c}e^{a\iota_i}\Big(e^{(r-a)\iota_i} - e^{(r-a)\iota_{i-1}}\Big) \\
& + \frac{\rho L}{r-2a}\hat{c}e^{a\iota}\Big(e^{(r-2a)\iota_k} - e^{(r-2a)\iota_{k-1}}\Big) + \frac{M\hat{c}e^{a\iota}}{r-a}\Big(e^{(r-a)\iota_k} - e^{(r-a)\iota_{k-1}}\Big) \\
& + \frac{L\rho e^{a\iota}}{r-2a}\Big(e^{(r-2a)\iota} - e^{(r-2a)\iota_k}\Big) + \frac{Me^{a\iota}}{r-a}\Big(e^{(r-a)\iota} - e^{(r-a)\iota_k}\Big) \\
& + D\Big[\sum_{i=1}^{k-1} e^{a(\iota-\iota_k^+)} \prod_{j=k}^{i+1} [\hat{c}e^{a(\iota_j-\iota_{j-1}^+)}]\beth_i, \hat{0}\Big] + D[e^{a(\iota-\iota_k^+)}\beth_k, \hat{0}]\Big\}e^{-(r-a)\iota} \\
\leq\ & D[e^{-a\iota_m}\hat{c}^m\nu_0, \hat{0}] + \frac{\rho L}{r-2a}e^{(r-2a)\urcorner}(m+1)\hat{c}^m \\
& + \frac{M}{r-a}e^{(r-a)\urcorner}(m+1)\hat{c}^m + e^{-a\urcorner}m\hat{c}^{m-1}K.
\end{aligned}
$$

Now, let $\mathbb{P}x, \mathbb{P}\nu \in PC[\aleph, \mathbb{R}_F]$. For $\iota \in [0, \iota_1]$, we have

$$\begin{aligned} D[(\mathbb{P}\nu)(\iota), (\mathbb{P}x)(\iota)] &= D\Big[e^{a\iota}\nu_0 \ominus \int_0^\iota (-1)e^{a(\iota-s)}\gimel(s,\nu)ds, e^{a\iota}\nu_0 \ominus \int_0^\iota (-1)e^{a(\iota-s)}\gimel(s,x)ds\Big] \\ &\leq \int_0^\iota e^{a(\iota-s)}D\Big[\gimel(s,\nu), \gimel(s,x)\Big]ds \\ &\leq L\int_0^\iota D[\nu,x]e^{a(\iota-s)}ds \\ &= \frac{LD_{ra}[\nu,x]}{r-2a}e^{a\iota}(e^{(r-2a)\iota}-1), \end{aligned}$$

and thus (recall $a < 0$)

$$\begin{aligned} &\sup_{\iota\in[0,\iota_1]} D[(\mathbb{P}\nu)(\iota), (\mathbb{P}x)(\iota)]e^{-(r-a)\iota} \\ =\ &\sup_{\iota\in[0,\iota_1]} \Big\{\frac{LD_{ra}[\nu,x]}{r-2a}e^{a\iota}(e^{(r-2a)\iota}-1)\Big\}e^{-(r-a)\iota} \\ \leq\ &\frac{L}{r-2a}D_{ra}[\nu,x]. \end{aligned} \tag{10}$$

Similarly, for $\iota \in (\iota_k, \iota_{k+1}]$, $k \in \aleph$, we have

$$\begin{aligned} &D[(\mathbb{P}\nu)(\iota), (\mathbb{P}x)(\iota)] \\ \leq\ &D\Big[\sum_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}(-1)e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}(1+c_i)e^{a(\iota_i-s)}\gimel(s,\nu(s))ds, \\ &\sum_{i=1}^{k-1}\int_{\iota_{i-1}}^{\iota_i}(-1)e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(\iota_j-\iota_{j-1}^+)}(1+c_i)e^{a(\iota_i-s)}\gimel(s,x(s))ds\Big] \\ &+D\Big[\int_{\iota_{k-1}}^{\iota_k}(-1)e^{a(\iota-\iota_k^+)}(1+c_k)e^{a(\iota_k-s)}\gimel(s,\nu(s))ds, \\ &\int_{\iota_{k-1}}^{\iota_k}(-1)e^{a(\iota-\iota_k^+)}(1+c_k)e^{a(\iota_k-s)}\gimel(s,x(s))ds\Big] \\ &+D\Big[\int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,\nu(s))ds, \int_{\iota_k}^{\iota}(-1)e^{a(\iota-s)}\gimel(s,x(s))ds\Big] \\ \leq\ &\frac{L}{r-2a}\sum_{i=1}^{k-1}e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[\hat{c}e^{a(\iota_j-\iota_{j-1})}]\hat{c}(e^{(r-2a)\iota_i}-e^{(r-2a)\iota_{i-1}})e^{a\iota_i}D_{ra}[\nu,x] \\ &+\frac{Le^{a\iota_k}}{r-2a}e^{a(\iota-\iota_k^+)}\hat{c}(e^{(r-2a)\iota_k}-e^{(r-2a)\iota_{k-1}})D_{ra}[\nu,x] \\ &+\frac{Le^{a\iota}}{r-2a}(e^{(r-2a)\iota}-e^{(r-2a)\iota_k})D_{ra}[\nu,x], \end{aligned}$$

and so

$$\begin{aligned} &\sup_{\iota\in(\iota_k,\iota_{k+1}]} D[(\mathbb{P}\nu)(\iota), (\mathbb{P}x)(\iota)]e^{-(r-a)\iota} \\ \leq\ &\sup_{\iota\in(\iota_k,\iota_{k+1}]}\Big\{\frac{L}{r-2a}\sum_{i=1}^{k-1}e^{a(\iota-\iota_k^+)}\prod_{j=k}^{i+1}[\hat{c}e^{a(\iota_j-\iota_{j-1})}] \\ &\times\hat{c}(e^{(r-2a)\iota_i}-e^{(r-2a)\iota_{i-1}})e^{a\iota_i}D_{ra}[\nu,x] \end{aligned}$$

$$+\frac{Le^{a\iota_k}}{r-2a}e^{a(\iota-\iota_k^+)}\hat{c}(e^{(r-2a)\iota_k}-e^{(r-2a)\iota_{k-1}})D_{ra}[\nu,x]$$
$$+\frac{Le^{a\iota}}{r-2a}(e^{(r-2a)\iota}-e^{(r-2a)\iota_k})D_{ra}[\nu,x]\Big\}e^{-(r-a)\iota} \quad (11)$$
$$\leq \frac{L}{r-2a}e^{(r-2a)\daleth}(m+1)\hat{c}^m D_{ra}[\nu,x].$$

From (10) and (11) one has

$$D_{ra}[(\mathbb{P}\nu),(\mathbb{P}x)] \leq \frac{L}{r-2a}D_{ra}[\nu,x], \text{ for } \iota\in[0,\iota_1]$$

and

$$D_{ra}[(\mathbb{P}\nu),(\mathbb{P}x)] \leq \frac{L}{r-2a}e^{(r-2a)\daleth}(m+1)\hat{c}^m D_{ra}[\nu,x],$$

for $\iota\in(\iota_k,\iota_{k+1}]$, $k\in\aleph$. Since $L<\frac{r-2a}{e^{(r-2a)\daleth}(m+1)\hat{c}^m}$, where $a<0$, we infer that $\mathbb{P}$ is a contraction on $PC[\aleph,\mathbb{R}_F]$. The Banach fixed point theorem guarantees the existence of a unique fixed point for $\mathbb{P}$, thus (1) has a unique solution in the ($c2$)-differentiable case. □

## 4. An example

In this section we present an example to illustrate our theory.

**Example 4.1.** *Let us consider a class of impulsive fuzzy differential equations:*

$$\begin{cases}\nu'(\iota)=a\nu(\iota)+\beth(\iota,\nu(\iota)),\ \iota\in[0,\iota_3],\iota\neq\iota_k,k=1,2,\\ \Delta\nu(\iota_k)=c_k\nu(\iota_k^-)+\beth_k,\ k=1,2,\\ \nu(0)=\gamma,\end{cases} \quad (12)$$

*where* $a=-\frac{1}{5}$, $\beth(\iota,\nu(\iota))=\iota^2\sin(\nu(\iota))\gamma$, $[\gamma]^\alpha=[\alpha-1,1-\alpha]$, $\iota_1=0.1$, $\iota_2=0.2$, $\iota_3=0.3$, $c_1=-2.1$, $c_2=-2.2$, $\beth_1=3\gamma$ *and* $\beth_2=1.5\gamma$.

*Note* $\beth:[0,\daleth]\times\mathbb{R}_F\to\mathbb{R}_F$ *is continuous and for* $\psi,\varphi\in\mathbb{R}_F$,

$$\begin{aligned}D[\beth(\iota,\psi),\beth(\iota,\varphi)] &= D[\iota^2\sin(\psi)\gamma,\iota^2\sin(\varphi)\gamma]\\ &\leq \iota^2\cdot D[\psi,\varphi]\\ &\leq \begin{cases}0.01\cdot D[\psi,\varphi],\ \iota\in[0,0.1],\\ 0.04\cdot D[\psi,\varphi],\ \iota\in(0.1,0.2],\\ 0.09\cdot D[\psi,\varphi],\ \iota\in(0.2,0.3].\end{cases}\end{aligned}$$

*Let* $L_1=L=0.09$. *Note*

$$D[\beth(\iota,\hat{0}),\hat{0}]=D[\iota^2\sin(\hat{0})\gamma,\hat{0}]=0,$$

*for* $\iota\in[0,0.3]$, *so trivially (F2) and (f2) hold. For illustration we will take* $r=2$. *Note* $D(\beth_k,0)\leq 3$, *for* $k\in\{1,2\}$. *Let* $K_1=K=3$ .

*Note*

$$L_1=0.09<\frac{r}{(m+1)\hat{c}^m e^{-a(m-1)\hat{\iota}}e^{(r-a)\daleth}}=0.2345,$$

$$L=0.09<\frac{r-2a}{e^{(r-2a)\daleth}(m+1)\hat{c}^m}=0.2704.$$

*Therefore,* (12) *has a unique solution on* $[0,0.3]$.

*In fact, the solutions of* (12) *can be expressed in the following explicit forms:*

$(c1)$-*differentiable solution:*

$$\underline{\nu}(\iota) = \begin{cases} -\Big\{\cosh(-\frac{1}{5}\iota) - \sinh(-\frac{1}{5}\iota) \\ -\int_0^\iota[\cosh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s)) - \sinh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s))]ds\Big\},\ \iota\in[0,0.1], \\ -\Big\{(-1.1)\sinh(\frac{1}{5}\iota) - (-1.1)\cosh(\frac{1}{5}\iota) \\ -\int_0^{0.1}[(-1.1)\sinh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s)) - (-1.1)\cosh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s))]ds \\ -\int_{0.1}^\iota[\cosh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s)) + \sinh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s))]ds \\ +3\times\cosh(-\frac{1}{5}(\iota-0.1)) + 3\times\sinh(-\frac{1}{5}(\iota-0.1))\Big\},\iota\in(0.1,0.2], \\ -\Big\{\Big[\cosh(-\frac{1}{5}(\iota-0.2))\frac{[(-1.2)\sinh(-0.02)+(-1.2)\cosh(-0.02)][(-1.1)\sinh(-0.02)+(-1.1)\cosh(-0.02)]}{2} \\ +\sinh(-\frac{1}{5}(\iota-0.2))\frac{[(-1.2)\sinh(-0.02)-(-1.2)\cosh(-0.02)][(-1.1)\sinh(-0.02)-(-1.1)\cosh(-0.02)]}{2}\Big] \\ -\int_0^{0.1}\Big\{(-1.1)\sinh(-\frac{1}{5}(\iota-s-0.1)\times(-2.4)\sinh(-0.02)s^2\sin(\underline{\nu}(s)) \\ -(-1.1)\cosh(-\frac{1}{5}(\iota-s-0.1)\times(-2.4)\sinh(-0.02)s^2\sin(\underline{\nu}(s))\Big\}ds \\ -\int_{0.1}^{0.2}\Big\{\Big[(-1.2)\sinh(-\frac{1}{5}(\iota-0.2))\cosh(-\frac{1}{5}(0.2-s)) \\ +(-1.2)\cosh(-\frac{1}{5}(\iota-0.2))\sinh(-\frac{1}{5}(0.2-s))\Big]s^2\sin(\underline{\nu}(s)) \\ -\Big[(-1.2)\sinh(-\frac{1}{5}(\iota-0.2))\sinh(-\frac{1}{5}(0.2-s)) \\ +(-1.2)\cosh(-\frac{1}{5}(\iota-0.2))\cosh(-\frac{1}{5}(0.2-s))\Big]s^2\sin(\underline{\nu}(s))\Big\}ds \\ -\int_{0.2}^\iota\Big[\cosh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s)) - \sinh(-\frac{1}{5}(\iota-s))s^2\sin(\underline{\nu}(s))\Big]ds \\ -3\times\Big[\cosh(-\frac{1}{5}(\iota-0.2))(-1.1)\sinh(-0.02) + \sinh(-\frac{1}{5}(\iota-0.2))(-1.1)\cosh(-0.02)\Big] \\ -3\times\Big[\cosh(-\frac{1}{5}(\iota-0.2))(-1.1)\cosh(-0.02) + \sinh(-\frac{1}{5}(\iota-0.2))(-1.1)\sinh(-0.02)\Big] \\ +1.5\cosh(-\frac{1}{5}(\iota-0.2)) - 1.5\sinh(-\frac{1}{5}(\iota-0.2))\Big\},\iota\in(0.2,0.3], \end{cases}$$

*and*

$$\overline{\nu}(\iota) = -\underline{\nu}(\iota).$$

$(c2)$-*differentiable solution:*

$$\underline{\nu}(\iota) = \begin{cases} -\Big\{e^{-\frac{1}{5}\iota} - \int_0^\iota(-1)e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds\Big\},\ \iota\in[0,0.1], \\ -\Big\{e^{-\frac{1}{5}(\iota-0.1)}(1.1)e^{-0.02} + \Big\{\int_0^{0.1}1.1e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds \\ +\int_{0.1}^\iota(-1)e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds\Big\} + 3e^{-\frac{1}{5}(\iota-0.1)}\Big\},\iota\in(0.1,0.2], \\ -\Big\{1.32e^{-\frac{1}{5}(\iota-0.2)}e^{-0.04} - \Big\{\int_0^{0.1}(-1.32)e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds \\ -\int_{0.1}^{0.2}1.2e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds + \int_{0.2}^\iota(-1)e^{-\frac{1}{5}(\iota-s)}s^2\sin(\underline{\nu}(s))ds\Big\} \\ +3.6e^{-\frac{1}{5}(\iota-0.2)}e^{-0.02} + 1.5e^{-\frac{1}{5}(\iota-0.2)}\Big\},\iota\in(0.2,0.3], \end{cases}$$

*and*

$$\overline{v}(\iota) = -\underline{v}(\iota).$$

*Finally, (c1)-solution (when $\alpha = 0$) is shown in Figure 1; (c2)-solution (when $\alpha = 0$) is shown in Figure 2.*

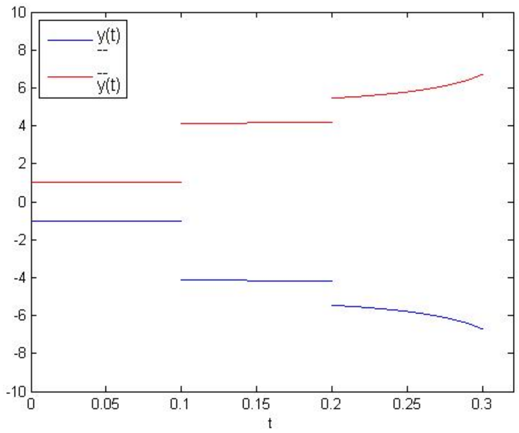


Figure 1: (c1)-solution (when $\alpha = 0$) of Example 4.1.

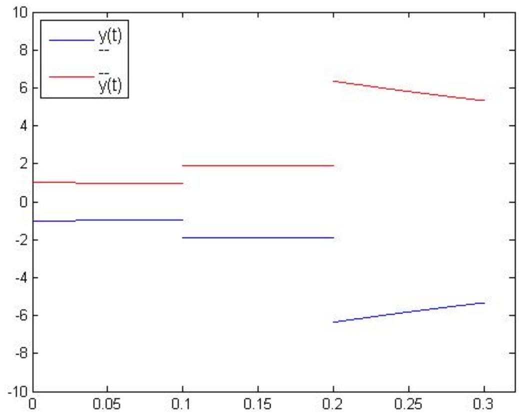


Figure 2: (c2)-solution (when $\alpha = 0$) of Example 4.1.

## 5. Conclusions

The purpose of this paper is establish the existence and uniqueness of ($c1$)-differentiable and ($c2$)-differentiable solutions to first-order nonlinear impulsive fuzzy differential equations under generalized Hukuhara differentiability using the contraction mappings principle. An example is given to prove our results.

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