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On generalized Suzuki-Proinov type $\left(\alpha, \mathcal{Z}_{\mathsf{E}}^{*}\right)$−contractions in modular b−metric spaces

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Abstract. This paper's objective is to put forward a new kind of E−type contraction, which includes rational expression, by considering Proinov type functions and $C_{\mathcal{G}}$−simulation functions. This type of contraction is termed as a Suzuki-Proinov type generalized $\left(\alpha, \mathcal{Z}_{\mathsf{E}}^{*}\right)$ − contraction mapping. Further, some common fixed point theorems using these new mappings, which are triangular α−admissible pairs, are demonstrated in the setting of modular b−metric space. Besides, the given example indicates the applicability and validity of the outcomes of this study.

1. Introduction

Throughout the study, the symbol $\mathbb{N}$ represents the set of all positive natural numbers, and $\mathbb{R}_{+}$ is used to represent the set of all non-negative real numbers.

Let $\mathcal{X}$ be a non-void set and $S, T : \mathcal{X} \to \mathcal{X}$ be self-mappings. Thereby, the following ones represent the set of fixed points of S and the set of common fixed points of S and T, respectively:

- $Fix\,(S) = \{\chi \in \mathcal{X} : \; S\chi = \chi\}$
- $C_{Fix}\,(S, T) = \{\chi \in \mathcal{X} : \; S\chi = T\chi = \chi\}$.

The studies [2]-[5] by Chistyakov constitute the basis of the studies on modular metrics, which is a very new and attractive concept.

Let $\mathcal{X}$ be a non-empty set and $\rho : (0, \infty) \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ be a function. For simplicity, we will write:

$$\rho_{\mu}\left(\chi, y\right) = \rho\left(\mu, \chi, y\right)$$

for all $\mu > 0$ and $\chi, y \in \mathcal{X}$.

Definition 1.1. *[3] Let $\mathcal{X}$ be a non-empty set. A function $\rho : (0, \infty)\times\mathcal{X}\times\mathcal{X} \to [0, \infty]$ is said to be a* metric modular *on $\mathcal{X}$ if the following axioms are satisfied, for all $\chi, y, z \in \mathcal{X}$,*

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(ρ_1) $\rho_\mu(\chi, y) = 0$ *for all* $\mu > 0$ *if and only if* $\chi = y$,

(ρ_2) $\rho_\mu(\chi, y) = \rho_\mu(y, \chi)$ *for all* $\mu > 0$,

(ρ_3) $\rho_{\mu+\lambda}(\chi, y) \leq \rho_\mu(\chi, z) + \rho_\lambda(z, y)$ *for all* $\mu, \lambda > 0$.

If instead of (ρ_1), *we have only the condition*

(ρ_1') $\rho_\mu(\chi, \chi) = 0$ *for all* $\mu > 0$, *then* ρ *is said to be a* (metric) pseudomodular *on* $\mathcal{X}$.

One of the most important and popular generalizations of the metric function is the *b*-metric function, which first appeared in Bakhtin's work [7] in 1989, but attracted the attention of researchers with Czerwik's studies ([8],[9]) in 1993 and 1998.

Definition 1.2. *[8] Let* $\mathcal{X}$ *be a non−empty set and* $\mathsf{s} \geq 1$ *be a given real number. A function* $\mathsf{b} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ *is a b−metric on* $\mathcal{X}$ *if, for all* $\chi, y, z \in \mathcal{X}$, *the following conditions hold:*

(b_1) $\mathsf{b}(\chi, y) = 0 \Leftrightarrow \chi = y$,

(b_2) $\mathsf{b}(\chi, y) = \mathsf{b}(y, \chi)$,

(b_3) $\mathsf{b}(\chi, y) \leq \mathsf{s}\left[\mathsf{b}(\chi, z) + \mathsf{b}(z, y)\right]$.

In this case, the pair $(\mathcal{X}, \mathsf{b})$ *is called a b−metric space.*

If we consider $\mathsf{s} = 1$, then the definitions of *b*−metric and ordinary metric coincide.

Furthermore, *b*−metric is not always a continuous function of its variables, unlike the metric. Therefore, the following lemma is of considerable importance for the *b*−metric.

Lemma 1.3. [10] *Let* $(\mathcal{X}, \mathsf{b})$ *be a b−metric space with* $\mathsf{s} \geq 1$ *and* $\{\chi_n\}$, $\{y_n\}$ *be two convergent sequences, to* χ *and* y, *respectively. Then*

$$\frac{1}{\mathsf{s}^2}\mathsf{b}(\chi, y) \leq \liminf_{n\to\infty} \mathsf{b}(\chi_n, y_n) \leq \limsup_{n\to\infty} \mathsf{b}(\chi_n, y_n) \leq \mathsf{s}^2\mathsf{b}(\chi, y).$$

Especially, if $\chi = y$, *then* $\lim_{n\to\infty} \mathsf{b}(\chi_n, y_n) = 0$. *Also, for* $z \in \mathcal{X}$, *we have*

$$\frac{1}{\mathsf{s}}\mathsf{b}(\chi, z) \leq \liminf_{n\to\infty} \mathsf{b}(\chi_n, z) \leq \limsup_{n\to\infty} \mathsf{b}(\chi_n, z) \leq \mathsf{s}\mathsf{b}(\chi, z).$$

In 2018, the modular *b*−metric function, by combining the above descriptions, has been acquainted by M.E. Ege and C. Alaca [14], and some fixed point results have been established in the setting of the space endowed with modular *b*−metric function.

Definition 1.4. *[14] Let* $\mathcal{X}$ *be a non-empty set and let* $\mathsf{s} \geq 1$ *be a real number. A map* $\omega : (0, \infty) \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ *is called a modular b−metric, if the following axioms are provided, for all* $\chi, y, z \in \mathcal{X}$,

(ω_1) $\omega_\mu(\chi, y) = 0$ *for all* $\mu > 0$ *if and only if* $\chi = y$,

(ω_2) $\omega_\mu(\chi, y) = \omega_\mu(y, \chi)$ *for all* $\mu > 0$,

(ω_3) $\omega_{\mu+\lambda}(\chi, y) \leq \mathsf{s}\left[\omega_\mu(\chi, z) + \omega_\lambda(z, y)\right]$ *for all* $\mu, \lambda > 0$.

Also, the pair $(\mathcal{X}, \omega)$ *is named a* modular *b*−metric space.

In the above definition, if $\mathsf{s} = 1$, then it is a natural extension of a modular metric.

If ω is a modular $b-$metric on a set $\mathcal{X}$, then a modular set is identified by

$$\mathcal{X}_\omega = \left\{ y \in \mathcal{X} : y \overset{\omega}{\sim} x \right\},$$

where $\overset{\omega}{\sim}$ is a binary relation on $\mathcal{X}$ defined by $x \sim y \Leftrightarrow \lim_{\mu \to \infty} \omega_\mu \left(x, y \right) = 0$ for $x, y \in \mathcal{X}$.

Also, note that the set

$$\mathcal{X}_\omega^* = \left\{ x \in \mathcal{X} : \exists \mu = \mu(x) > 0 \text{ such that } \omega_\mu (x, x_0) < \infty \right\} \ (x_0 \in \mathcal{X})$$

are mentioned as modular metric space (around x_0).

Now, we hold forth some examples of modular $b-$metric functions and modular $b-$metric spaces.

Example 1.5. *[14] Let us regard the space*

$$\ell_p = \left\{ \left\{ x_j \right\} \subset \mathbb{R} : \sum_{j=1}^{\infty} \left| x_j \right|^p < \infty \right\} \quad 0 < p < 1.$$

For $\mu \in (0, \infty)$ if we specify $\omega_\mu \left(x, y \right) = \frac{\mathsf{d}(x,y)}{\mu}$ such that

$$\mathsf{d}\left(x, y \right) = \left(\sum_{j=1}^{\infty} \left| x_j - y_j \right|^p \right)^{\frac{1}{p}}, \quad \left\{ x_j \right\}, \left\{ y_j \right\} \in \ell_p$$

then, it could be easily seen that $(\mathcal{X}, \omega)$ is a modular $b-$metric space.

Example 1.6. *[15] Let $(\mathcal{X}, \rho)$ be a modular metric space and let $k \geq 1$ be a real number. Take $\omega_\mu \left(x, y \right) = \left(\rho_\mu \left(x, y \right) \right)^k$. Using the convexity of the function $f(\iota) = \iota^k$ for $\iota \geq 0$, also from Jensen inequality, we achieve*

$$(\alpha + \beta)^k \leq 2^{k-1} \left(\alpha^k + \beta^k \right)$$

for $\alpha, \beta \geq 0$. Thus, $(\mathcal{X}, \omega)$ is a modular $b-$metric space with the constant $\mathsf{s} = 2^{k-1}$.

Some fundamental topological properties of a modular $b-$metric space such as $\omega-$convergence, $\omega-$Cauchy sequences, and $\omega-$completeness are characterized as below.

Definition 1.7. *Let $\mathcal{X}_\omega^*$ be a modular $b-$metric space and $\left\{ x_j \right\}_{j \in \mathbb{N}} \in \mathcal{X}_\omega^*$ be a sequence.*

(i) $\left\{ x_j \right\}_{j \in \mathbb{N}}$ is $\omega-$convergent to $x \in \mathcal{X}_\omega^*$ if and only if $\omega_\mu \left(x_j, x \right) \to 0$, as $j \to \infty$ for all $\mu > 0$ and x is called $\omega-$limit of $\left\{ x_j \right\}_{j \in \mathbb{N}}$.

(ii) If $\lim_{j,m \to \infty} \omega_\mu \left(x_j, x_m \right) = 0$, for all $\mu > 0$, the sequence $\left\{ x_j \right\}_{j \in \mathbb{N}}$ in $\mathcal{X}_\omega^*$ is named an $\omega-$Cauchy sequence.

(iii) If any $\omega-$Cauchy sequence in $\mathcal{X}_\omega^*$ is $\omega-$convergent to the point of $\mathcal{X}_\omega^*$ then $\mathcal{X}_\omega^*$ is called $\omega-$complete space.

Definition 1.8. *Let $\mathcal{X}_\omega^*$ be a modular $b-$metric space. $S : \mathcal{X}_\omega^* \to \mathcal{X}_\omega^*$ is $\omega-$continuous if $\omega_\mu \left(x_j, x \right) \to 0$, provided to $\omega_\mu \left(S x_j, S x \right) \to 0$ as $j \to \infty$.*

Khojasteh et al. [16] asserted a novel auxiliary function called simulation function in 2015.

Definition 1.9. *[16] Let $\chi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Presume that the following axioms are provided:*

(χ_1) $\chi(0,0) = 0$,

(χ_2) $\chi(\eta,\nu) < \nu - \eta$ *for all* $\eta,\nu > 0$,

(χ_3) *if* $\{\eta_j\}$, $\{\nu_j\}$ *are sequences in the interval* $(0,\infty)$ *such that* $\lim_{j\to\infty} \eta_j = \lim_{j\to\infty} \nu_j > 0$ *then,* $\limsup_{j\to\infty} \chi(\eta_j,\nu_j) < 0$.

The function χ is entitled a *simulation function*, and $\mathcal{Z}$ stands for the set of all simulation functions. By virtue of the axiom (χ_2), we have $\chi(\eta,\eta) < 0$ for all $\eta > 0$.

Definition 1.10. *[16] Let $S : \mathcal{X} \to \mathcal{X}$ be a map on metric space $(\mathcal{X}, \mathsf{d})$ and $\chi \in \mathcal{Z}$. If*

$$\chi\left(\mathsf{d}\left(S\chi, Sy\right), \mathsf{d}\left(\chi, y\right)\right) \geq 0 \text{ for all } \chi, y \in \mathcal{X},$$

is satisfied, then S is called a $\mathcal{Z}$-contraction in respect of χ.

Also, if we agree on $\chi(\eta,\nu) = \lambda\nu - \eta$ for all $\eta,\nu \in [0,\infty)$ and $\lambda \in [0,1)$, then we achieve the Banach contraction.

In 2018, A. Fulga and E. Karapınar [18] acquainted a new consequence regarding simulation function and E -type contraction.

Definition 1.11. *[18] A self-mapping S defined on a complete metric space $(\mathcal{X}, \mathsf{d})$ is a $\mathcal{Z}-$contraction of E -type with respect to χ if there exists $\chi \in \mathcal{Z}$ such that for all $\chi, y \in \mathcal{X}$*

$$\chi\left(\mathsf{d}\left(S\chi, Sy\right), E\left(\chi, y\right)\right) \geq 0,$$

where

$$E\left(\chi, y\right) = \mathsf{d}\left(\chi, y\right) + \left|\mathsf{d}\left(\chi, S\chi\right) - \mathsf{d}\left(y, Sy\right)\right|.$$

Theorem 1.12. [18] *If S is a $\mathcal{Z}-$contraction of $E-$type with respect to χ on $\mathcal{X}$, then S admits a fixed point in $\mathcal{X}$.*

Besides, in [19], A.H. Ansari demonstrated a new class of functions named $C-$class functions, as indicated below.

Definition 1.13. *[19] A function $\mathcal{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is named as a $C-$class function provided that it is continuous and it satisfies the circumstances below.*

($\mathcal{G}_1$) $\mathcal{G}(\eta,\nu) \leq \eta$;

($\mathcal{G}_2$) $\mathcal{G}(\eta,\nu) = \eta$ *implies that either* $\eta = 0$ *or* $\nu = 0$;

for all $\eta,\nu \in [0,\infty)$.

Let the set of $C-$class functions be symbolised as $\mathbf{C}$.

Considering the $C-$class functions and the simulation functions, Radenović et al. [20] introduced the concept of $C_{\mathcal{G}}-$simulation functions.

Definition 1.14. *[20] A $C_{\mathcal{G}}-$simulation function is a mapping $\varsigma : [0,\infty)^2 \to \mathbb{R}$ satisfying the following conditions:*

(ς_1) $\varsigma(\eta,\nu) < \mathcal{G}(\nu,\eta)$ *for all* $\eta,\nu > 0$, *where* $\mathcal{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is a* $C-$*class function,*

(ς_2) *if* $\{\eta_n\}$, $\{\nu_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \nu_n > 0$ *and* $\nu_n < \eta_n$, *then, there exists* $C_{\mathcal{G}} \geq 0$ *such that* $\limsup_{n\to\infty} \varsigma(\eta_n,\nu_n) < C_{\mathcal{G}}$.

The family of all $C_{\mathcal{G}}$−simulation functions is denoted by $\mathcal{Z}^*$.

Definition 1.15. *[20] A mapping $\mathcal{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ has the property $C_{\mathcal{G}}$, if there exists $C_{\mathcal{G}} \geq 0$ such that*

(*i*) $\mathcal{G}(\eta, \nu) > C_{\mathcal{G}}$ *implies* $\eta > \nu$,

(*ii*) $\mathcal{G}(\eta, \eta) \leq C_{\mathcal{G}}$ *for all* $\eta \in [0,\infty)$.

Moreover, Suzuki [21] attached a new precondition to contractive mapping, accepted as the Suzuki type contraction, and demonstrated a fixed point theorem, as indicated below.

Theorem 1.16. [21] *Let $(\mathcal{X}, \mathsf{d})$ be a compact metric space and $S : \mathcal{X} \to \mathcal{X}$ be a mapping. Presume that for all distinct $\chi, y \in \mathcal{X}$, the following statement holds:*

$$\frac{1}{2}\mathsf{d}(\chi, S\chi) < \mathsf{d}\left(\chi, y\right) \Rightarrow \mathsf{d}\left(S\chi, Sy\right) < \mathsf{d}\left(\chi, y\right).$$

Then, S has a unique fixed point in $\mathcal{X}$.

In the recent two decades, the conclusion of Proinov [22] made a significant impact on the fixed point theory. Considering some auxiliary functions, Proinov put forth exciting fixed point theorems that generalize and extend diverse comparable results in the existing literature.

Definition 1.17. *[22] Let $(\mathcal{X}, \mathsf{d})$ be a metric space. A mapping $S : \mathcal{X} \to \mathcal{X}$ is said to be a Proinov type contraction if for all $\chi, y \in \mathcal{X}$*

$$\Phi\left(\mathsf{d}\left(S\chi, Sy\right)\right) \leq \Psi\left(\mathsf{d}\left(\chi, y\right)\right),$$

where $\Phi, \Psi : (0,\infty) \to \mathbb{R}$ are two functions and $\mathsf{d}\left(S\chi, Sy\right) > 0$.

Theorem 1.18. [22] *Let $(\mathcal{X}, \mathsf{d})$ be a complete metric space and $S : \mathcal{X} \to \mathcal{X}$ be a Proinov type contraction, where the functions $\Phi, \Psi : (0,\infty) \to \mathbb{R}$ are such that the following conditions are satisfied:*

(p_1) Φ *is non-decreasing,*

(p_2) $\Psi(s) < \Phi(s)$ *for all* $s > 0$,

(p_3) $\limsup_{s \to s_0+} \Psi(s) < \Phi(s_0+)$ *for any* $s_0 > 0$.

Then S admits a unique fixed point in $\mathcal{X}$.

Definition 1.19. *Let $S, T : \mathcal{X} \to \mathcal{X}$ be two self-mappings and $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function. We contemplate the following circumstances.*

(α_1) $\alpha\left(\chi, y\right) \geq 1$ implies $\alpha\left(S\chi, Sy\right) \geq 1$;

(α_2) $\alpha(\chi, S\chi) \geq 1$ implies $\alpha\left(S\chi, S^2 y\right) \geq 1$;

(α_3) $\alpha\left(\chi, y\right) \geq 1$ and $\alpha\left(y, Sy\right) \geq 1$ implies $\alpha\left(\chi, Sy\right) \geq 1$;

(α_4) $\alpha\left(\chi, y\right) \geq 1$ implies $\alpha\left(S\chi, Ty\right) \geq 1$ and $\alpha\left(TS\chi, STy\right) \geq 1$;

(α_5) $\alpha(\chi, z) \geq 1$ and $\alpha\left(z, y\right) \geq 1$ implies $\alpha\left(\chi, y\right) \geq 1$.

Taking into account the function (α_i), we assert that

- *$i = 1$, S is an $\alpha-$admissible mapping in [23].*
- *$i = 2$, S is an $\alpha-$orbital admissible mapping [24].*
- *$i = 2, 3$, S is a triangular $\alpha-$orbital admissible mapping [24].*
- *$i = 1, 5$, S is a triangular $\alpha-$ admissible mapping in [25]*
- *$i = 4, 5$, the pair (S, T) is a triangular $\alpha-$admissible pair in [28].*

Lemma 1.20. [24] *Let $S, T : \mathcal{X} \to \mathcal{X}$ be two self-mappings and the pair (S, T) be a triangular $\alpha-$admissible pair. Assume that there exists $\chi_0 \in \mathcal{X}$ such that $\alpha(\chi_0, S\chi_0) \geq 1$. Define a sequence $\{\chi_j\}$ by $\chi_{j+1} = S\chi_j$. Then we have $\alpha\left(\chi_j, \chi_m\right) \geq 1$ for all $j, m \in \mathbb{N}$ with $j < m$.*

(For more details and examples see e.g.[29]-[34].)

2. Main Results

As a beginning, it is essential to indicate that the concept of metric modular does not have to be finite. Because of this, it is necessary to consult the following additional conditions to guarantee the existence and uniqueness of fixed points of contraction mappings on modular metric spaces and modular $b-$metric spaces.

(S_1) $\omega_\mu(\chi, S\chi) < \infty$ for all $\mu > 0$ and $\chi \in \mathcal{X}^*_\omega$,

(S_2) $\omega_\mu\left(\chi, y\right) < \infty$ for all $\mu > 0$ and $\chi, y \in \mathcal{X}^*_\omega$.

Subsequently, some fixed point theorems have been established by defining generalized Suzuki-Proinov type $\left(\alpha, \mathcal{Z}^*_{\mathsf{E}}\right)-$contractions with respect to ς and extending $E-$type contractions utilizing from rational expressions, which is represented by E in the framework of modular $b-$metric space.

Definition 2.1. *Let $\mathcal{X}^*_\omega$ be a modular $b-$metric space with constant $\mathsf{s} \geq 1$ and let $S, T : \mathcal{X}^*_\omega \to \mathcal{X}^*_\omega$ be two self-mappings, and $\alpha : \mathcal{X}^*_\omega \times \mathcal{X}^*_\omega \to \mathbb{R}$ be a function. The pair (S, T) is called a generalized Suzuki-Proinov type $\left(\alpha, \mathcal{Z}^*_{\mathsf{E}}\right)-$contraction if there exists a $C_{\mathcal{G}}-$simulation function $\varsigma \in \mathcal{Z}^*$ such that*

$$\frac{1}{2\mathsf{s}} \min\left\{\omega_\mu(\chi, S\chi), \omega_\mu(y, Ty)\right\} \leq \omega_\mu(\chi, y)$$

implies

$$\varsigma\left(\alpha\left(\chi, y\right)\Phi\left(\mathsf{s}^3\omega_\mu(S\chi, Ty)\right), \Psi\left(\mathsf{E}\left(\chi, y\right)\right)\right) \geq C_{\mathcal{G}}, \tag{1}$$

where $\Phi, \Psi : (0, \infty) \to \mathbb{R}$ are two functions satisfying

(c_1) *Φ is a lower semi-continuous and non-decreasing function;*

(c_2) *$\Psi(s) < \Phi(s)$ for all $s > 0$;*

(c_3) *$\limsup\limits_{s \to s_0+} \Psi(s) < \Phi(s_0+)$ for any $s_0 > 0$,*

and also,

$$\mathsf{E}\left(\chi, y\right) = \omega_\mu\left(\chi, y\right) + \left|\frac{\omega_\mu(\chi, S\chi) - \omega_\mu\left(y, Ty\right)}{1 + \omega_\mu\left(\chi, y\right)}\right|,$$

for all distinct $\chi, y \in \mathcal{X}^*_\chi$, $\omega_\mu(S\chi, Ty) > 0$ and for all $\mu > 0$.

Theorem 2.2. *Let $\mathcal{X}_\omega^*$ be an $\omega-$complete modular $b-$metric space with the constant $\mathsf{s} \geq 1$ and S and T be a generalized Suzuki-Proinov type $\left(\alpha, \mathcal{Z}_{\mathsf{E}}^*\right)-$contraction with respect to ς. Presume that the subsequent circumstances are provided:*

(1) *(S, T) is a triangular $\alpha-$admissible pair,*

(2) *there is a point $\varkappa_0 \in \mathcal{X}_\omega^*$ that has the property $\alpha(\varkappa_0, S\varkappa_0) \geq 1$,*

(3) *S, T are $\omega-$continuous mappings, or*

(3′) *if $\{\varkappa_n\}$ is a sequence satisfying*

i. $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ *for all n,*

ii. $\varkappa_n \to \chi^* \in \mathcal{X}_\omega^*$ *as $n \to \infty$,*

then we find a subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\alpha(\varkappa_{n_k}, \chi^*) \geq 1$,

(4) *there exist $\chi, y \in C_{Fix}(S, T)$ such that $\alpha\left(\chi, y\right) \geq 1$.*

If the condition (S_1) is satisfied, then there exists $\chi^* \in \mathcal{X}_\omega^*$ such that $\chi^* \in C_{Fix}(S, T)$. If, in addition, the condition (S_2) is satisfied, then $C_{Fix}(S, T) = \{\chi^*\}$.

Proof. Let $\varkappa_0 \in \mathcal{X}_\omega^*$ be a given arbitrary point such that $\alpha(\varkappa_0, S\varkappa_0) \geq 1$. Construct a sequence $\left\{\varkappa_j\right\}$ in $\mathcal{X}_\omega{}^*$ such that

$$\varkappa_{2j+1} = S\varkappa_{2j} \quad \text{and} \quad \varkappa_{2j+2} = T\varkappa_{2j+1}, \quad \text{for all } j \in \mathbb{N}.$$

Also, given the fact that (S, T) is a triangular $\alpha-$admissible pair, we derive

$$\alpha(\varkappa_0, \varkappa_1) = \alpha(\varkappa_0, S\varkappa_0) \geq 1 \;\Rightarrow\; \begin{cases} \alpha(S\varkappa_0, T\varkappa_1) = \alpha(\varkappa_1, \varkappa_2) \geq 1 \\ \text{and} \\ \alpha(TS\varkappa_0, ST\varkappa_1) = \alpha(T\varkappa_1, S\varkappa_2) = \alpha(\varkappa_2, \varkappa_3) \geq 1. \end{cases}$$

Likewise, we get

$$\alpha(\varkappa_2, \varkappa_3) \geq 1 \;\Rightarrow\; \begin{cases} \alpha(S\varkappa_2, T\varkappa_3) = \alpha(\varkappa_3, \varkappa_4) \geq 1 \\ \text{and} \\ \alpha(TS\varkappa_2, ST\varkappa_3) = \alpha(T\varkappa_3, S\varkappa_4) = \alpha(\varkappa_4, \varkappa_5) \geq 1. \end{cases}$$

Thereby, recursively, we conclude that

$$\alpha\left(\varkappa_{2j}, \varkappa_{2j+1}\right) \geq 1. \tag{2}$$

Next, if there exists some $j_0 \in \mathbb{N}$ such that $\chi_{j_0} = \chi_{j_0+1}$, then j_0 becomes a common fixed point of S and T. Consequently, we assume that $\varkappa_k \neq \varkappa_{k+1}$ for all $k \in \mathbb{N}$. Therefore, we have $\omega_\mu(\varkappa_k, \varkappa_{k+1}) > 0$ for all $\mu > 0$.

We presume that $k = 2j$ for some $j \in \mathbb{N}$. So, because

$$\begin{aligned} \frac{1}{2\mathsf{s}} \min\left\{\omega_\mu(\varkappa_{2j}, S\varkappa_{2j}), \omega_\mu(\varkappa_{2j+1}, T\varkappa_{2j+1})\right\} &= \frac{1}{2\mathsf{s}} \min\left\{\omega_\mu(\varkappa_{2j}, \varkappa_{2j+1}), \omega_\mu(\varkappa_{2j+1}, \varkappa_{2j+2})\right\} \\ &\leq \omega_\mu(\varkappa_{2j}, \varkappa_{2j+1}), \end{aligned}$$

from (1) and (ς_1), we have

$$\begin{aligned} C_{\mathcal{G}} &\leq \varsigma\left(\alpha\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(S\varkappa_{2j}, T\varkappa_{2j+1}\right)\right), \Psi\left(\mathsf{E}\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\right)\right) \\ &= \varsigma\left(\alpha\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right), \Psi\left(\mathsf{E}\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\right)\right) \\ &< \mathcal{G}\left(\Psi\left(\mathsf{E}\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\right), \alpha\left(\varkappa_{2j}, \varkappa_{2j+1}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right)\right), \end{aligned}$$

and by using (c_2), (2) and the properties of $C_{\mathcal{G}}$, we procure

$$\begin{aligned}\Phi\left(\mathsf{s}^3\omega_\mu\left(x_{2j+1}, x_{2j+2}\right)\right) &\leq \Phi\left(\alpha\left(x_{2j}, x_{2j+1}\right)\mathsf{s}^3\omega_\mu\left(x_{2j+1}, x_{2j+2}\right)\right)\\ &< \Psi\left(\mathsf{E}\left(x_{2j}, x_{2j+1}\right)\right) < \Phi\left(\mathsf{E}\left(x_{2j}, x_{2j+1}\right)\right),\end{aligned} \tag{3}$$

where

$$\begin{aligned}\mathsf{E}\left(x_{2j}, x_{2j+1}\right) &= \omega_\mu\left(x_{2j}, x_{2j+1}\right) + \left|\frac{\omega_\mu\left(x_{2j}, Sx_{2j}\right) - \omega_\mu\left(x_{2j+1}, Tx_{2j+1}\right)}{1+\omega_\mu\left(x_{2j}, x_{2j+1}\right)}\right|\\ &= \omega_\mu\left(x_{2j}, x_{2j+1}\right) + \left|\frac{\omega_\mu\left(x_{2j}, x_{2j+1}\right) - \omega_\mu\left(x_{2j+1}, x_{2j+2}\right)}{1+\omega_\mu\left(x_{2j}, x_{2j+1}\right)}\right|.\end{aligned}$$

Let us denote $\omega_\mu\left(x_j, x_{j+1}\right)$ by σ_j. Now, if we presume $\max\left\{\sigma_{2j}, \sigma_{2j+1}\right\} = \sigma_{2j+1}$, then, we get

$$\mathsf{E}\left(x_{2j}, x_{2j+1}\right) = \sigma_{2j} + \frac{\sigma_{2j+1} - \sigma_{2j}}{1+\sigma_{2j}} = \frac{\sigma_{2j}^2 + \sigma_{2j+1}}{1+\sigma_{2j}},$$

and considering the features of the function Φ and (3), we deduce

$$\sigma_{2j+1} < \frac{\sigma_{2j}^2 + \sigma_{2j+1}}{1+\sigma_{2j}} \;\Rightarrow\; \sigma_{2j+1} < \sigma_{2j}$$

such that this is in contradiction with our assumption. Thereby, we achieve $\max\left\{\sigma_{2j}, \sigma_{2j+1}\right\} = \sigma_{2j}$, which indicates that

$$\mathsf{E}\left(x_{2j}, x_{2j+1}\right) = \sigma_{2j} + \frac{\sigma_{2j} - \sigma_{2j+1}}{1+\sigma_{2j}}.$$

Thereupon, the inequality (3) gives

$$\Phi\left(\sigma_{2j+1}\right) \leq \Phi\left(\mathsf{s}^3\sigma_{2j+1}\right) \leq \Psi\left(\sigma_{2j} + \frac{\sigma_{2j} - \sigma_{2j+1}}{1+\sigma_{2j}}\right) < \Phi\left(\sigma_{2j} + \frac{\sigma_{2j} - \sigma_{2j+1}}{1+\sigma_{2j}}\right). \tag{4}$$

Therefore, by (c_1), we get that

$$\sigma_{2j+1} < \sigma_{2j} + \frac{\sigma_{2j} - \sigma_{2j+1}}{1+\sigma_{2j}} \quad \Rightarrow \sigma_{2j+1} < \sigma_{2j}.$$

Also, by similar steps, one concludes that $\sigma_{2j} < \sigma_{2j-1}$. So, it guarantees that $\left\{\sigma_j\right\} = \left\{\omega_\mu\left(x_j, x_{j+1}\right)\right\}$ is a non-increasing sequence. On the other hand, a similar consequence can be obtained when k is an odd number. Thus, the equality $\lim_{j\to\infty}\sigma_j = p$ is provided for $p \geq 0$. Now, we will present that $p = 0$. Conversely, we suppose that $p > 0$. Then, by (4), we have

$$\Phi(p) = \lim_{j\to\infty}\Phi\left(\sigma_{2j+1}\right) \leq \limsup_{j\to\infty}\Psi\left(\sigma_{2j} + \frac{\sigma_{2j} - \sigma_{2j+1}}{1+\sigma_{2j}}\right) < \limsup_{s\to p}\Phi(s)$$

such that this contradicts the assumption (c_3). Then, we notice that our assumption is false, that is, for all $\mu > 0$

$$\lim_{j\to\infty}\omega_\mu\left(x_j, x_{j+1}\right) = 0. \tag{5}$$

Now, we need to indicate that $\{x_j\}$ is an $\omega-$Cauchy sequence. For this, it is sufficient to verify that $\{x_{2j}\}$ is a $\omega-$Cauchy sequence. Unlike our assertion, consider that $\{x_{2j}\}$ is not a $\omega-$Cauchy sequence, then for $\varepsilon > 0$, we can constitute two subsequences $\{x_{2m_q}\}$ and $\{x_{2j_q}\}$ of positive integers satisfying $j_q > m_q > q$ such that j_q is the smallest index for which

$$\omega_\mu\left(x_{2m_q}, x_{2j_q}\right) \geq \varepsilon \quad \text{and} \quad \omega_\mu\left(x_{2m_q}, x_{2j_q-2}\right) < \varepsilon, \quad \text{for all } \mu > 0, \tag{6}$$

then we yield that by applying (5), (6) and the modular inequality,

$$\varepsilon \leq \omega_{4\mu}\left(x_{2m_q}, x_{2j_q}\right) \leq \mathsf{s}\omega_{2\mu}\left(x_{2m_q}, x_{2m_q+1}\right) + \mathsf{s}^2\omega_\mu\left(x_{2m_q+1}, x_{2j_q+2}\right)$$
$$+\mathsf{s}^3\omega_{\mu/2}\left(x_{2j_q+2}, x_{2j_q+1}\right) + \mathsf{s}^3\omega_{\mu/2}\left(x_{2j_q+1}, x_{2j_q}\right)$$

such that

$$\limsup_{q\to\infty} \omega_\mu\left(x_{2m_q+1}, x_{2j_q+2}\right) \geq \frac{\varepsilon}{\mathsf{s}^2}. \tag{7}$$

Also, we achieve

$$\omega_\mu\left(x_{2m_q}, x_{2j_q+1}\right) \leq \mathsf{s}\omega_{\mu/2}\left(x_{2m_q}, x_{2j_q-2}\right) + \mathsf{s}^2\omega_{\mu/4}\left(x_{2j_q-2}, x_{2j_q-1}\right)$$
$$+\mathsf{s}^3\omega_{\mu/8}\left(x_{2j_q-1}, x_{2j_q}\right) + \mathsf{s}^3\omega_{\mu/8}\left(x_{2j_q}, x_{2j_q+1}\right).$$

If we take the limit superior in the above expression and regard (5), we procure that

$$\limsup_{q\to\infty} \omega_\mu\left(x_{2m_q}, x_{2j_q+1}\right) \leq \mathsf{s}\varepsilon. \tag{8}$$

Besides, we suggest that for a sufficiently large $q \in \mathbb{N}$, if $j_q > m_q > q$, then

$$\frac{1}{2\mathsf{s}} \min\left\{\omega_\mu\left(x_{2j_q}, Sx_{2j_q}\right), \omega_\mu\left(x_{2m_q-1}, Tx_{2m_q-1}\right)\right\} \leq \omega_\mu\left(x_{2j_q}, x_{2m_q-1}\right). \tag{9}$$

Given the fact that, $j_q > m_q$ and $\left\{\omega_\mu\left(x_j, x_{j+1}\right)\right\}_{j\geq 1}$ is non-decreasing, we acquire

$$\omega_\mu\left(x_{2j_q}, Sx_{2j_q}\right) = \omega_\mu\left(x_{2j_q}, x_{2j_q+1}\right) \leq \omega_\mu\left(x_{2m_q}, x_{2m_q+1}\right) \leq \omega_\mu\left(x_{2m_q-1}, x_{2m_q}\right)$$
$$= \omega_\mu\left(x_{2m_q-1}, Kx_{2m_q-1}\right).$$

Hence,

$$\tfrac{1}{2\mathsf{s}} \min\left\{\omega_\mu\left(x_{2j_q}, Sx_{2j_q}\right), \omega_\mu\left(x_{2m_q-1}, Tx_{2m_q-1}\right)\right\} = \tfrac{1}{2\mathsf{s}}\omega_\mu\left(x_{2j_q}, Sx_{2j_q}\right)$$
$$= \tfrac{1}{2\mathsf{s}}\omega_\mu\left(x_{2j_q}, x_{2j_q+1}\right).$$

According to (5), there exists $q_1 \in \mathbb{N}$ such that for any $q > q_1$,

$$\omega_\mu\left(x_{2j_q}, x_{2j_q+1}\right) < \frac{\varepsilon}{2\mathsf{s}}.$$

Also, there exists $q_2 \in \mathbb{N}$ such that for any $q > q_2$,

$$\omega_\mu\left(x_{2m_q-1}, x_{2m_q}\right) < \frac{\varepsilon}{2\mathsf{s}}.$$

Therefore, for any $q > \max\{q_1, q_2\}$ and $j_q > m_q > q$, we have

$$\begin{aligned}\varepsilon \leq \omega_{2\mu}\left(\chi_{2j_q}, \chi_{2m_q}\right) &\leq \mathsf{s}\omega_{\mu}\left(\chi_{2j_q}, \chi_{2m_q-1}\right) + \mathsf{s}\omega_{\mu}\left(\chi_{2m_q-1}, \chi_{2m_q}\right) \\ &\leq \mathsf{s}\omega_{\mu}\left(\chi_{2j_q}, \chi_{2m_q-1}\right) + \mathsf{s}\tfrac{\varepsilon}{2\mathsf{s}}.\end{aligned}$$

So, one concludes that

$$\frac{\varepsilon}{2\mathsf{s}} \leq \omega_{\mu}\left(\chi_{2j_q}, \chi_{2m_q-1}\right).$$

Thus, we deduce that for any $q > \max\{q_1, q_2\}$ and $j_q > m_q > q$,

$$\omega_{\mu}\left(\chi_{2j_q}, \chi_{2j_q+1}\right) < \frac{\varepsilon}{2\mathsf{s}} \leq \omega_{\mu}\left(\chi_{2j_q}, \chi_{2m_q-1}\right)$$

that is, the expression (9) is proved. Thereupon, since the pair (S, T) is triangular $\alpha-$admissible, we get $\alpha\left(\chi_{2m_q}, \chi_{2j_q+1}\right) \geq 1$. Therefore, from (1) and (ς_1), we conclude that

$$\begin{aligned}C_{\mathcal{G}} &\leq \varsigma\left(\alpha\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\Phi\left(\mathsf{s}^3\omega_{\mu}\left(S\chi_{2m_q}, T\chi_{2j_q+1}\right)\right), \Psi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right)\right) \\ &= \varsigma\left(\alpha\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\Phi\left(\mathsf{s}^3\omega_{\mu}\left(\chi_{2m_q+1}, \chi_{2j_q+2}\right)\right), \Psi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right)\right) \\ &< \mathcal{G}\left(\Psi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right), \alpha\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\Phi\left(\mathsf{s}^3\omega_{\mu}\left(\chi_{2m_q+1}, \chi_{2j_q+2}\right)\right)\right)\end{aligned}$$

and by using definition 1.15, it turns into

$$\begin{aligned}\Phi\left(\mathsf{s}^3\omega_{\mu}\left(\chi_{2m_q+1}, \chi_{2j_q+2}\right)\right) &\leq \alpha\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\Phi\left(\mathsf{s}^3\omega_{\mu}\left(\chi_{2m_q+1}, \chi_{2j_q+2}\right)\right) \\ &< \Psi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right) < \Phi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right),\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right) &= \omega_{\mu}\left(\chi_{2m_q}, \chi_{2j_q+1}\right) + \left|\frac{\omega_{\mu}\left(\chi_{2m_q}, S\chi_{2m_q}\right) - \omega_{\mu}\left(\chi_{2j_q+1}, T\chi_{2j_q+1}\right)}{1+\omega_{\mu}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)}\right| \\ &= \omega_{\mu}\left(\chi_{2m_q}, \chi_{2j_q+1}\right) + \left|\frac{\omega_{\mu}\left(\chi_{2m_q}, \chi_{2m_q+1}\right) - \omega_{\mu}\left(\chi_{2j_q+1}, \chi_{2j_q+2}\right)}{1+\omega_{\mu}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)}\right|.\end{aligned}$$

Thereby, taking the limit superior in (10) and taking (5), (7), and (8) into account, we achieve that

$$\begin{aligned}\Phi(\mathsf{s}\varepsilon) \leq \limsup_{q\to\infty}\Phi\left(\mathsf{s}^3\omega_{\mu}\left(\chi_{2m_q+1}, \chi_{2j_q+2}\right)\right) &< \limsup_{q\to\infty}\Psi\left(\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right) \\ &< \Phi\left(\limsup_{q\to\infty}\mathsf{E}\left(\chi_{2m_q}, \chi_{2j_q+1}\right)\right) \\ &< \Phi(\mathsf{s}\varepsilon),\end{aligned}$$

which causes a contradiction. Hence, we say that the sequence $\left\{\chi_j\right\}$ is an $\omega-$Cauchy on $\omega-$complete modular $b-$metric space, which assures that there exists a point χ^* in $\mathcal{X}^*_{\omega}$ such that

$$\lim_{j\to\infty}\chi_j = \chi^*. \tag{11}$$

Next, our purpose is to confirm that the mappings S and T have the point $\varkappa^*$ as a common fixed point. But, initially, we state that for all $j \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2\mathsf{s}}\omega_\mu\left(\varkappa_{2j}, \varkappa_{2j+1}\right) \leq \omega_\mu\left(\varkappa_{2j}, \varkappa^*\right), \tag{12}$$

or

$$\frac{1}{2\mathsf{s}}\omega_\mu\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right) \leq \omega_\mu\left(\varkappa_{2j}, \varkappa^*\right). \tag{13}$$

Unlike, if for some $j_0 \geq 0$, both of them are not provided. Hence, we say that

$$\begin{aligned}\omega_\mu\left(\varkappa_{2j_0}, \varkappa_{2j_0+1}\right) &\leq \mathsf{s}\omega_\mu\left(\varkappa_{2j_0}, \varkappa^*\right) + \mathsf{s}\omega_\mu\left(\varkappa^*, \varkappa_{2j_0+1}\right)\\ &< \tfrac{1}{2}\omega_\mu\left(\varkappa_{2j_0}, \varkappa_{2j_0+1}\right) + \tfrac{1}{2}\omega_\mu\left(\varkappa_{2j_0+1}, \varkappa_{2j_0+2}\right)\\ &< \tfrac{1}{2}\omega_\mu\left(\varkappa_{2j_0}, \varkappa_{2j_0+1}\right) + \tfrac{1}{2}\omega_\mu\left(\varkappa_{2j_0}, \varkappa_{2j_0+1}\right) = \omega_\mu\left(\varkappa_{2j_0}, \varkappa_{2j_0+1}\right),\end{aligned}$$

such that it is a contradiction. That is why the assertion is true. From this point, one can discuss the following two subcases.

Subcase (3.1): The inequality (12) holds for infinitely many $j \geq 0$. In this case, from (3), since the mappings S, T are $\omega-$continuous, we get

$$\begin{aligned}S\varkappa^* &= S\left(\lim_{j\to\infty}\varkappa_{2j}\right) = \lim_{j\to\infty}S\varkappa_{2j} = \lim_{j\to\infty}\varkappa_{2j+1} = \varkappa^*\\ &= \lim_{j\to\infty}\varkappa_{2j+2} = \lim_{j\to\infty}T\varkappa_{2j+1}\\ &= T\left(\lim_{j\to\infty}\varkappa_{2j+1}\right) = T\varkappa^*.\end{aligned}$$

So, we achieve the desired results. Nevertheless, now let us consider the conditions (3') and (2). Then, for $k = 2j$, we can notice a subsequence $\left\{\varkappa_{2j_k}\right\}$ of $\left\{\varkappa_{2j}\right\}$ having the features $\alpha\left(\varkappa_{2j_k}, \varkappa^*\right) \geq 1$, for all k. Thereupon, we presume that $\varkappa^* \neq T\varkappa^*$, that is, $\omega_\mu\left(\varkappa^*, T\varkappa^*\right) > 0$. Then, if we consider (1) and (ς_1), we possess that

$$\begin{aligned}C_{\mathcal{G}} &\leq \varsigma\left(\alpha\left(\varkappa_{2j_k}, \varkappa^*\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(S\varkappa_{2j_k}, T\varkappa^*\right)\right), \Psi\left(\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right)\right)\right)\\ &= \varsigma\left(\alpha\left(\varkappa_{2j_k}, \varkappa^*\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\varkappa_{2j_k+1}, T\varkappa^*\right)\right), \Psi\left(\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right)\right)\right)\\ &< \mathcal{G}\left(\Psi\left(\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right)\right), \alpha\left(\varkappa_{2j_k}, \varkappa^*\right)\Phi\left(\mathsf{E}^3\omega_\mu\left(\varkappa_{2j_k+1}, T\varkappa^*\right)\right)\right).\end{aligned}$$

This can be abbreviated by the following provided that considering the condition (3') and definition 1.15

$$\begin{aligned}\Phi\left(\mathsf{s}^3\omega_\mu\left(\varkappa_{2j_k+1}, T\varkappa^*\right)\right) &\leq \alpha\left(\varkappa_{2j_k}, \varkappa^*\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\varkappa_{2j_k+1}, T\varkappa^*\right)\right)\\ &< \Psi\left(\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right)\right) < \Phi\left(\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right)\right),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\mathsf{E}\left(\varkappa_{2j_k}, \varkappa^*\right) &= \omega_\mu\left(\varkappa_{2j_k}, \varkappa^*\right) + \left|\frac{\omega_\mu\left(\varkappa_{2j_k}, S\varkappa_{2j_k}\right) - \omega_\mu\left(\varkappa^*, T\varkappa^*\right)}{1+\omega_\mu\left(\varkappa_{2j_k}, \varkappa^*\right)}\right|\\ &= \omega_\mu\left(\varkappa_{2j_k}, \varkappa^*\right) + \left|\frac{\omega_\mu\left(\varkappa_{2j_k}, \varkappa_{2j_k+1}\right) - \omega_\mu\left(\varkappa^*, T\varkappa^*\right)}{1+\omega_\mu\left(\varkappa_{2j_k}, \varkappa^*\right)}\right|.\end{aligned}$$

In the above relation, letting $(k \to \infty)$, we get $\liminf_{k\to\infty} \mathsf{E}\left(\varkappa_{2j_k}, \chi^*\right) = \omega_\mu\left(\chi^*, T\chi^*\right)$. Hence, keeping in mind the lower-semicontinuity of Φ and $\mathsf{s} \geq 1$, the inequality (14) becomes

$$\liminf_{s\to\omega_\mu(\chi^*,T\chi^*)} \Phi(s) \leq \lim_{k\to\infty} \Phi\left(\omega_\mu\left(S\varkappa_{2j_k}, T\chi^*\right)\right) \leq \Psi\left(\omega_\mu\left(\chi^*, T\chi^*\right)\right)$$

$$< \Phi\left(\omega_\mu\left(\chi^*, T\chi^*\right)\right) < \liminf_{s\to\omega_\mu(\chi^*,T\chi^*)} \Phi(s),$$

which arises a contradiction. Thereby, we gain that $\chi^* = T\chi^*$. Likewise, one can procure that χ^* is the fixed point of the mapping S, as well.

Subcase (3.2): The inequality (12) merely satisfies for finitely many $j \geq 0$.

In consequence, we can find $j_0 \geq 0$ such that (13) holds for any $j \geq j_0$. In the same way, as in Subcase (3.1), it follows that (13) also causes a contradiction unless $\chi^* = S\chi^*$ or $\chi^* = T\chi^*$.

As a result, it is seen that in both sub-cases, χ^* is the common fixed point for the mappings S and T.

Ultimately, for the uniqueness, there exists a point $\tilde{\chi}$, which is $\tilde{\chi} = S\tilde{\chi} = T\tilde{\chi}$ such that $\tilde{\chi} \neq \chi^*$. So, by dealing with the expression (4), we deduce that $\alpha(\chi^*, \tilde{\chi}) \geq 1$. Therewith, as

$$0 = \frac{1}{2\mathsf{s}} \min\left\{\omega_\mu\left(\chi^*, S\chi^*\right), \omega_\mu\left(\tilde{\chi}, T\tilde{\chi}\right)\right\} \leq \omega_\mu\left(\chi^*, \tilde{\chi}\right),$$

considering (1) and (ς_1), we get

$$\begin{aligned} C_{\mathcal{G}} &\leq \varsigma\left(\alpha\left(\chi^*, \tilde{\chi}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(S\chi^*, T\tilde{\chi}\right)\right), \Psi\left(\mathsf{E}\left(\chi^*, \tilde{\chi}\right)\right)\right) \\ &= \varsigma\left(\alpha\left(\chi^*, \tilde{\chi}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\chi^*, \tilde{\chi}\right)\right), \Psi\left(\mathsf{E}\left(\chi^*, \tilde{\chi}\right)\right)\right) \\ &< \mathcal{G}\left(\Psi\left(\mathsf{E}\left(\chi^*, \tilde{\chi}\right)\right), \alpha\left(\chi^*, \tilde{\chi}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\chi^*, \tilde{\chi}\right)\right)\right), \end{aligned}$$

and this means that

$$\begin{aligned} \Phi\left(\mathsf{s}^3\omega_\mu\left(\chi^*, \tilde{\chi}\right)\right) &\leq \alpha\left(\chi^*, \tilde{\chi}\right)\Phi\left(\mathsf{s}^3\omega_\mu\left(\chi^*, \tilde{\chi}\right)\right) \\ &< \Psi\left(\mathsf{E}\left(\chi^*, \tilde{\chi}\right)\right) < \Phi\left(\mathsf{E}\left(\chi^*, \tilde{\chi}\right)\right) \\ &= \Phi\left(\omega_\mu\left(\chi^*, \tilde{\chi}\right) + \left|\frac{\omega_\mu(\chi^*, S\chi^*) - \omega_\mu(\tilde{\chi}, T\tilde{\chi})}{1+\omega_\mu(\chi^*, \tilde{\chi})}\right|\right) \\ &= \Phi\left(\omega_\mu\left(\chi^*, \tilde{\chi}\right)\right). \end{aligned}$$

Thence, we achieve a contradiction. In turn, we deduce that χ^* is a unique common fixed point. This concludes the proof. □

Now, we furnish some consequences regarding the primary outcomes.

Corollary 2.3. *Let $\mathcal{X}_\omega^*$ be an $\omega-$complete modular $b-$metric space with a constant $\mathsf{s} \geq 1$, $\alpha : \mathcal{X}_\omega^* \times \mathcal{X}_\omega^* \to \mathbb{R}$ be a function and $S : \mathcal{X}_\omega^* \to \mathcal{X}_\omega^*$ be a self-mapping. Presume that the following statements provide:*

(1) there exists $\varsigma \in \mathcal{Z}^*$ such that

$$\frac{1}{2\mathsf{s}}\omega_\mu(\chi, S\chi) \leq \omega_\mu(\chi, y)$$

implies

$$\varsigma\left(\alpha\left(\chi, y\right)\Phi\left(\mathsf{s}^3\omega_\mu(S\chi, Sy)\right), \Psi\left(\mathsf{E}\left(\chi, y\right)\right)\right) \geq C_{\mathcal{G}},$$

where $\Phi, \Psi : (0, \infty) \to T$ *are two functions as defined in definition 2.1 and also,*

$$\mathsf{E}\left(x, y\right) = \omega_\mu\left(x, y\right) + \left|\frac{\omega_\mu\left(x, Sx\right) - \omega_\mu\left(y, Sy\right)}{1 + \omega_\mu\left(x, y\right)}\right|,$$

for all distinct $x, y \in \mathcal{X}_\omega^*$, $\omega_\mu(Sx, Sy) > 0$ *and for all* $\mu > 0$;

(2) *S is a triangular* $\alpha-$*orbital admissible mapping, and there exists* $x_0 \in \mathcal{X}_\omega^*$ *such that* $\alpha\left(x_0, Sx_0\right) \geq 1$,

(3) *S is* $\omega-$*continuous, or*

(3') *if* $\{x_n\}$ *is a sequence satisfying*

 i. $\alpha\left(x_n, x_{n+1}\right) \geq 1$, *for all n;*

 ii. $x_n \to x^* \in \mathcal{X}_\omega^*$ *as* $n \to \infty$,

then we find a subsequence $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha\left(x_{n_k}, x^*\right) \geq 1$,

(4) *there exist* $x, y \in Fix(S)$ *such that* $\alpha\left(x, y\right) \geq 1$.

Thereby, under the conditions (S_1) *and* (S_2), *the mapping S owns a unique fixed point in* $\mathcal{X}_\omega^*$.

Proof. If we take into consideration that $S = T$ and Lemma 1.20, then we achieve the intended result. □

Definition 2.4. *Let* $\mathcal{X}_\omega^*$ *be a modular b−metric space with the constant* $\mathsf{s} \geq 1$, $\alpha : \mathcal{X}_\omega^* \times \mathcal{X}_\omega^* \to \mathbb{R}$ *be a function and* $S, T : \mathcal{X}_\omega^* \to \mathcal{X}_\omega^*$ *be two self-mappings. The pair* (S, T) *is called Suzuki* $(\alpha, \Phi, \Psi)-$*rational* E *type contraction if there exist* $\Phi, \Psi : (0, \infty) \to \mathbb{R}$ *that satisfy the condition* $(c_1) - (c_3)$ *in definition 2.1 such that*

$$\frac{1}{2\mathsf{s}} \min\left\{\omega_\mu(x, Sx), \omega_\mu(y, Ty)\right\} \leq \omega_\mu(x, y)$$

implies

$$\alpha\left(x, y\right) \Phi\left(\mathsf{s}^3 \omega_\mu(Sx, Ty)\right) \leq \Psi\left(\mathsf{E}\left(x, y\right)\right), \tag{15}$$

where $\mathsf{E}\left(x, y\right)$ *is as defined in definition 2.1, for all distinct* $x, y \in \mathcal{X}_x^*$, $\omega_\mu(Sx, Ty) > 0$ *and for all* $\mu > 0$.

Theorem 2.5. *Let* $\mathcal{X}_\omega^*$ *be an* $\omega-$*complete modular b−metric space with constant* $\mathsf{s} \geq 1$, *the mappings S and T are a Suzuki* $(\alpha, \Phi, \Psi)-$*rational* E *type contraction. Presume that the following circumstances are provided:*

(1) *the pair* (S, T) *is triangular* $\alpha-$*admissible, and there exists* $x_0 \in \mathcal{X}_\omega^*$ *such that* $\alpha\left(x_0, Sx_0\right) \geq 1$,

(2) S, T *are* $\omega-$*continuous, or*

(2') *if* $\{x_n\}$ *is a sequence satisfying*

 i. $\alpha\left(x_n, x_{n+1}\right) \geq 1$, *for all n;*

 ii. $x_n \to x^* \in \mathcal{X}_\omega^*$ *as* $n \to \infty$,

then we find a subsequence $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha\left(x_{n_k}, x^*\right) \geq 1$,

(3) *there exist* $x, y \in C_{Fix}(S, T)$ *such that* $\alpha\left(x, y\right) \geq 1$.

Thereupon, together with (S_1) *and* (S_2), *there exists* $x^* \in \mathcal{X}_\omega^*$ *such that* $C_{Fix}\left(S, T\right) = \{x^*\}$.

Proof. We achieve the desired result if we evaluate the $C_{\mathcal{G}}-$simulation function $\varsigma \in \mathcal{Z}^*$ with the properties $C_{\mathcal{G}}$ in definition 1.15. □

Remark 2.6. *Note that if we consider the simulation function with respect to $\chi \in \mathcal{Z}$ instead of $C_{\mathcal{G}}-$simulation function $\varsigma \in \mathcal{Z}^*$, we can achieve the same consequences.*

Example 2.7. *Let $\mathcal{X}^*_\omega = \left\{0, \frac{1}{2}, \frac{3}{2}\right\}$ and $\omega : (0,\infty) \times \mathcal{X} \times \mathcal{X} \to [0,\infty]$ be a mapping fulfilling the following terms: for all distinct $\chi, y \in \mathcal{X}^*_\chi$ and for all $\mu > 0$,*

1. $\omega_\mu(\chi, y) = 0,$ where $\chi = y$,
2. $\omega_\mu(\chi, y) = \omega_\mu(y, \chi)$,
3. $\omega_\mu(0, \frac{1}{2}) = \frac{1}{4}$, $\omega_\mu(0, \frac{3}{2}) = \frac{1}{8}$ *and* $\omega_\mu(\frac{1}{2}, \frac{3}{2}) = \frac{1}{2}$.

It is clear that, $(\mathcal{X}^*_\omega, \omega)$ is an $\omega-$complete modular $b-$metric space with the constant $\mathsf{s} = \frac{4}{3}$. Also, let the mappings $S, T : \mathcal{X}^*_\omega \to \mathcal{X}^*_\omega$, $\alpha : \mathcal{X}^*_\omega \times \mathcal{X}^*_\omega \to \mathbb{R}$ and $\Phi, \Psi : (0,\infty) \to T$ be respectively defined by

$$S(0) = 0,\ S\left(\tfrac{1}{2}\right) = 0 \text{ and } S\left(\tfrac{3}{2}\right) = 0$$

$$T(0) = 0,\ T\left(\tfrac{1}{2}\right) = \tfrac{3}{2} \text{ and } T\left(\tfrac{3}{2}\right) = 0,$$

$$\alpha\left(\chi, y\right) = \begin{cases} 1, & \chi.y \in [0,1) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\Phi(\eta) = \frac{4}{5}\eta \text{ and } \Psi(\eta) = \frac{3}{4}\eta.$$

Now, we will verify the contractivity conditions for all distinct $\chi, y \in \mathcal{X}^*_\omega$, $\omega_\mu(S\chi, Ty) > 0$ and for all $\mu > 0$;

$$\frac{1}{2\mathsf{s}} \min\left\{\omega_\mu(\chi, S\chi), \omega_\mu(y, Ty)\right\} \le \omega_\mu(\chi, y)$$

implies

$$\varsigma\left(\alpha\left(\chi, y\right)\Phi\left(\mathsf{s}^3\omega_\mu(S\chi, Ty)\right), \Psi\left(\mathsf{E}\left(\chi, y\right)\right)\right) \ge 0,$$

via simulation function $\varsigma \in \mathcal{Z}$ and taking $\varsigma(\eta, \nu) < \frac{3}{4}\nu - \eta$ for all $\eta, \nu > 0$.

Notice that, ς belongs to the class of $\mathcal{Z}$, and similarly, if we consider the properties $C_{\mathcal{G}}$, a similar case arises.

At this point, we propose the subsequent cases to be considered:

Case (1): $\chi = 0,\ y = \frac{1}{2}$. *In this case, we achieve that*

$$0 = \frac{3}{8}\min\left\{\omega_\mu(0, S0), \omega_\mu(\frac{1}{2}, T\frac{1}{2})\right\} \le \omega_\mu\left(0,\ \frac{1}{2}\right) = \frac{1}{4},$$

which implies that

$$\varsigma\left(\alpha\left(0,\ \frac{1}{2}\right)\Phi\left(\mathsf{s}^3\omega_\mu(S0, T\frac{1}{2})\right), \Psi\left(\mathsf{E}\left(0, \frac{1}{2}\right)\right)\right) \ge 0,$$

and by $\varsigma \in \mathcal{Z}$, we get

$$\begin{aligned}
\alpha\left(0,\ \tfrac{1}{2}\right)\Phi\left(\mathsf{s}^3\omega_\mu(S0, T\tfrac{1}{2})\right) = \Phi\left(\left(\tfrac{4}{3}\right)^3\omega_\mu(0, \tfrac{3}{2})\right) = \Phi\left(\tfrac{8}{27}\right) = \tfrac{32}{135} \\
\le \tfrac{3}{4}\Psi\left(\mathsf{E}\left(0, \tfrac{1}{2}\right)\right) \\
\le \tfrac{3}{4}\Psi\left(\omega_\mu\left(0, \tfrac{1}{2}\right) + \left|\tfrac{\omega_\mu(0,S0) - \omega_\mu\left(\frac{1}{2}, T\frac{1}{2}\right)}{1+\omega_\mu\left(0,\frac{1}{2}\right)}\right|\right) \\
\le \tfrac{3}{4}\Psi\left(\tfrac{13}{20}\right) = \tfrac{117}{320}
\end{aligned}$$

such that all conditions of Theorem 2.2 are satisfied.

Case (2): $\chi = \frac{3}{2},\ y = \frac{1}{2}$. *Then, we get*

$$\frac{3}{64} = \frac{3}{8}\min\left\{\omega_\mu(\frac{3}{2}, S\frac{3}{2}), \omega_\mu(\frac{1}{2}, T\frac{1}{2})\right\} \leq \omega_\mu\left(\frac{3}{2},\ \frac{1}{2}\right) = \frac{1}{2}$$

and

$$\varsigma\left(\alpha\left(\frac{3}{2},\ \frac{1}{2}\right)\Phi\left(\mathsf{s}^3\omega_\mu(S\frac{3}{2}, T\frac{1}{2})\right), \Psi\left(\mathsf{E}\left(\frac{1}{2}, \frac{3}{2}\right)\right)\right) \geq 0.$$

By using $\varsigma \in \mathcal{Z}$ *with* $\varsigma(\eta, \nu) < \frac{3}{4}\nu - \eta$, *we deduce that*

$$\begin{aligned}\alpha\left(\tfrac{3}{2},\ \tfrac{1}{2}\right)\Phi\left(\left(\tfrac{4}{3}\right)^3\omega_\mu(S\tfrac{3}{2}, T\tfrac{1}{2})\right) = \Phi\left(\left(\tfrac{4}{3}\right)^3\omega_\mu(0, \tfrac{3}{2})\right) = \Phi\left(\tfrac{8}{27}\right) &= \tfrac{32}{135}\\ &\leq \tfrac{3}{4}\Psi\left(\mathsf{E}\left(\tfrac{3}{2}, \tfrac{1}{2}\right)\right)\\ &\leq \tfrac{3}{4}\Psi\left(\omega_\mu\left(\tfrac{3}{2}, \tfrac{1}{2}\right) + \left|\tfrac{\omega_\mu\left(\frac{3}{2}, S\frac{3}{2}\right) - \omega_\mu\left(\frac{1}{2}, S\frac{1}{2}\right)}{1 + \omega_\mu\left(\frac{3}{2}, \frac{1}{2}\right)}\right|\right)\\ &\leq \tfrac{3}{4}\Psi\left(\tfrac{3}{4}\right) = \tfrac{27}{64}.\end{aligned}$$

Thereby, this case is provided, too. Other cases are ignored on account of $\omega_\mu(S\chi, Ty) = 0$.
Accordingly, the mappings S and T provide the hypotheses of Theorem 2.2, and it is certain that $C_{Fix}(S, T) = \{0\}$.

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