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Existence and convergence results for a class of non-expansive type mappings in Banach spaces

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Abstract. In this article, we concentrate on common fixed points of a pair of generalized non-expansive mappings, viz., generalized α-Reich-Suzuki non-expansive mappings. In this sequel, we introduce the three step Abbas-Nazir iterative algorithm for a pair of mappings. Then we obtain some results related to weak and strong convergence of sequences, satisfying this iterative algorithm, to obtain the common fixed points of two generalized α-Reich-Suzuki non-expansive mappings. Finally, we compare the convergence rate of our iteration technique to that of some well-known iteration techniques by some constructive numerical examples.

1. Introduction and preliminaries

Throughout the last few decades, existence of fixed point of different types of non-expansive mappings and convergence of iteration algorithms to the fixed point and their convergence rate (i.e., speed of the iteration) have drawn attentions of many mathematicians. We first recollect two important classes of non-expansive mappings.

Definition 1.1. *Suppose that $(E, \|.\|)$ is a Banach space, and U be a non-empty subset of E. A mapping $T : U \to U$ is said to be non-expansive if for all $x, y \in U$,*

$$\left\|Tx - Ty\right\| \leq \left\|x - y\right\|.$$

Further, it is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - w\| \leq \|x - w\|,$$

for all $x \in U$ and for all $w \in F(T)$.

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If the Banach space E is uniformly convex and if U is closed, bounded, convex, then it is well-known that the set of fixed points $F(T)$ of a non-expansive mapping $T : U \to U$ is non-empty. Many authors have studied the existence of fixed points of various non-expansive mappings and obtained a variety of interesting results, see [1, 4, 6, 12, 16, 18, 21, 26, 28, 34, 35].

In an attempt to weaken the notion of non-expansive mappings, in 2008, Suzuki [33] introduced a generalization of non-expansive mappings, usually known as mappings satisfying condition (C).

Definition 1.2. *[33] Let T be a self-mapping defined on a subset U of a Banach space E. Then T is said to satisfy Condition (C) if*

$$\frac{1}{2}\left\|x - Ty\right\| \le \left\|x - y\right\| \Rightarrow \left\|Tx - Ty\right\| \le \left\|x - y\right\|$$

holds for all $x, y \in U$. Also, if any mapping satisfies the Condition (C), then it is known as a Suzuki generalized non-expansive mapping.

In their research article, Karapınar and Taş [17] generalized the notion of Condition (C) and obtained some novel results. Thereupon, Dhompongsa et al. [8], Khan and Suzuki [13] obtained some fixed points and weak convergence results for Suzuki generalized mappings. After this, many mathematicians have generalized non-expansive mappings in different directions, some remarkable ones of these generalizations are α-non-expansive mappings due to Aoyama and Kohsaka [5], generalized α-non-expansive mappings due to Pant and Shukla [27], mappings satisfying condition (D_a) due to Donghan et al. [9], Reich-Suzuki non-expansive mappings due to Pandey et al. [24], Reich and Chatterjea type non-expansive mappings due to Som et al. [30]. After all such generalizations, Pandey et al. [25] proposed a novel notion of extended family of non-expansive mappings which properly contains both those of Reich-Suzuki non-expansive mappings and generalized α-non-expansive mappings, and investigated for several interesting properties involving these maps.

Definition 1.3. *[25] A mapping $T : U \to U$ is said to be a generalized α-Reich-Suzuki non-expansive mapping if there exists an $\alpha \in [0, 1)$ such that for each $x, y \in U$,*

$$\frac{1}{2}\|x - Tx\| \le \left\|x - y\right\| \implies \left\|Tx - Ty\right\| \le \max\{P(x, y), Q(x, y)\}$$

where

$$P(x, y) = \alpha\|Tx - x\| + \alpha\left\|Ty - y\right\| + (1 - 2\alpha)\left\|x - y\right\|$$

and

$$Q(x, y) = \alpha\left\|Tx - y\right\| + \alpha\left\|Ty - x\right\| + (1 - 2\alpha)\left\|x - y\right\|.$$

Similar to [26, Proposition 3.5], we can show that a generalized α-Reich-Suzuki non-expansive mapping is quasi-nonexpansive also. If we go through the literature of all the above kinds of non-expansive mappings, then we can observe that most of the above generalizations are concerned with fixed points of a single mapping, and there are a few results concerning common fixed points of two or more mappings. Motivated by these facts, in this article, our main objective is to enrich the theory of common fixed points of two or more mappings of non-expansive mappings by investigating common fixed points and related results of two or more non-expansive (type) mappings. To be precise, here we inquire into the existence of common fixed points of two generalized α-Reich-Suzuki non-expansive mappings defined on a Banach space.

On the other hand, it is also well-known that in general the Picard iteration does not converge to the fixed points of a non-expansive mapping even if it owns a fixed point. So in order to get the fixed points of such mappings by convergence of iterations, many iterative algorithm have been originated. Some impressive iterative algorithms are due to Mann [20], Ishikawa [15], Xu [36], Noor [22], Agarwal et al. [3], Liu et al. [19], Abbas and Nazir [2], Hussain et al. [10, 11] and many others. Among all these, Abbas and

Nazir iterative algorithm (three step algorithm) converges faster than that of Mann, Ishikawa, Xu, Noor and also, S-iteration methods numerically. Here we recall Abbas and Nazir iterative algorithm:

$$\begin{aligned} x_1 &= x \in U \\ x_{n+1} &= (1-a_n)Ty_n + a_nTz_n, \\ y_n &= (1-b_n)Tx_n + b_nTz_n, \\ z_n &= (1-c_n)x_n + c_nTx_n, \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$, where (a_n), (b_n) and (c_n) are sequences in $(0,1)$.

It is to be mentioned that all the above algorithms are related to fixed points of a single mapping only, and there are few iterative algorithms that are concerned with fixed points of two or more mappings. Among such few algorithms, the commonly utilized one is Liu et al.'s iterative algorithm [19] where the iterative sequence (x_n) is generated from $x_1 \in U$, and is defined as

$$\begin{aligned} x_{n+1} &= (1-a_n)T_1x_n + a_nT_2y_n, \\ y_n &= (1-b_n)T_1x_n + b_nT_2x_n, \end{aligned} \tag{1.2}$$

for all $n \in \mathbb{N}$, where (a_n) and (b_n) are sequences in $(0,1)$. The other objective of this article is to approximate the common fixed points of generalized α-Reich-Suzuki non-expansive mappings by some other iterative algorithm which has a better rate of convergence than that of Liu et al. To fulfil such objective, we extend Abbas-Nazir iterative process for single mapping to Abbas-Nazir iteration technique for two mappings. We make use of such three-step iteration process to secure some weak and strong convergence results. Finally, we furnish with some simulation (using MATLAB 2017*a*) results to authenticate our findings by constructing a couple of numerical examples. In particular, we carry out the comparative study of convergence behaviour of a few well-known iterative schemes with tabular and figurative demonstrations which suggests that the newly proposed iterative scheme has a faster convergence rate.

2. Preliminaries

Throughout the article, we use the notation $\mathbb{N}$ to mean the set of positive integers and $\mathbb{R}$ for the set of real numbers. Also, $F(T)$ stands for the set of all fixed points of a mapping T. In the following discourse, we talk over a couple of requisite definitions, terminologies, notations and results. Firstly, a Banach space E is said to be smooth if the subsequent limit

$$\lim_{t\to 0} \frac{\|u+tv\| - \|u\|}{t} \tag{2.1}$$

exists for all $u, v \in S_E$, where $S_E = \{x \in E : \|x\| = 1\}$ is the unit sphere in the corresponding Banach space. The norm of E is Fréchet differentiable if for each $u \in E$, the limit (2.1) is achieved uniformly for $v \in S_E$. In this case

$$\frac{1}{2}\|x\|^2 + \langle u, J(x)\rangle \le \frac{1}{2}\|x+u\|^2 \le \frac{1}{2}\|x\|^2 + \langle u, J(x)\rangle + g(\|u\|) \tag{2.2}$$

for all $x, u \in E$, where $J(x)$ is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in E$ and g is an increasing function defined on $[0,\infty)$ such that $\lim\limits_{t\to 0}\dfrac{g(t)}{t} = 0$. Besides, for each ϵ satisfying $0 \le \epsilon \le 2$, the modulus $\delta_E(\epsilon)$ of convexity of a Banach space E is defined as

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|u+v\|}{2} : \|u\| \le 1, \|v\| \le 1, \|u-v\| \ge \epsilon\right\}.$$

Also, E is uniformly convex if for every $\epsilon \ge 0$, $\delta_E(\epsilon) \ge 0$, and is strictly convex if $\frac{\|u+v\|}{2} < 1$ for every $u, v \in S_E$ with $u \neq v$.

Definition 2.1. *Let U be a non-empty subset of a Banach space E and also suppose that $T_1, T_2, \cdots, T_m : U \to U$ are self-mappings. An element $x \in U$ is said to be a common fixed point of these mappings if $T_i x = x$ for all $i = 1, 2, \cdots, m$.*

Definition 2.2. *[23] A Banach space E satisfies the Opial's property if for each weakly convergent sequence (x_n) in E with weak limit x,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \left\|x_n - y\right\|$$

holds for all $y \in E$ where $x \neq y$.

Definition 2.3. *Suppose that U is a non-empty closed and convex subset of a Banach space E and also assume that (x_n) is a bounded sequence in E. Then for any $x \in E$, we define the following:*

(a) asymptotic center of (x_n) at x as

$$r(x, (x_n)) = \limsup_{n\to\infty} \|x_n - x\|.$$

(b) asymptotic radius of (x_n) relative to U as

$$r(U, (x_n)) = \inf\{r(x, x_n) : x \in U\}.$$

(c) asymptotic centre of (x_n) relative to U as

$$A(U, (x_n)) = \{x \in U : r(x, (x_n)) = r(U, (x_n))\}.$$

Here we must note that, for any uniformly convex Banach space, the set $A(U, (x_n))$ consists exactly one point. However, the ensuing lemma is playing a pivotal role in this article.

Lemma 2.4. *[31] Let E be a uniformly convex Banach space and (u_n) is a sequence such that $0 < p \leq u_n \leq q < 1$ for all $n \in \mathbb{N}$. Consider (a_n) and (b_n) are two sequences such that $\limsup_{n\to\infty} \|a_n\| \leq t$, $\limsup_{n\to\infty} \|b_n\| \leq t$ and $\lim_{n\to\infty} \|u_n a_n + (1 - u_n) b_n\| = t$ for any $t \geq 0$. Then,*

$$\lim_{n\to\infty} \|a_n - b_n\| = 0.$$

Lemma 2.5. *[25] Let U be a non-empty subset of a Banach space E and $T : U \to U$ be a generalized α-Reich-Suzuki non-expansive mapping. Then for each $x, y \in U$ we have,*

$$\left\|x - Ty\right\| \leq \left(\frac{3 + \alpha}{1 - \alpha}\right) \|x - Tx\| + \left\|x - y\right\|.$$

3. Convergence results for two generalized α-Reich-Suzuki non-expansive mappings

In this section, we introduce the three-step iterative algorithm due to Abbas-Nazir for two mappings $T_1, T_2 : U \to U$, U being a non-empty subset of a Banach space E, which is as follows:

$$\begin{aligned} x_1 &= x \in U, \\ x_{n+1} &= (1 - a_n) T_1 y_n + a_n T_2 z_n, \\ y_n &= (1 - b_n) T_1 x_n + b_n T_2 z_n, \\ z_n &= (1 - c_n) x_n + c_n T_1 x_n \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$, where (a_n), (b_n) and (c_n) are sequences in $(0, 1)$. At first, we prove the following lemma regarding the above iterative algorithm for two generalized α-Reich-Suzuki non-expansive mappings.

Lemma 3.1. *Let U be a non-empty closed and convex subset of a Banach space E and $T_1, T_2 : U \to U$ be two generalized α-Reich-Suzuki non-expansive mappings with $F(T) = F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $q \in F(T)$ and (x_n) is a sequence defined by* (3.1). *Then $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(T)$.*

Proof. Since $T_1, T_2 : U \to U$ are two generalized α-Reich-Suzuki non-expansive mappings, we have

$$\|T_1 x - q\| \leq \|x - q\| \text{ and } \|T_2 x - q\| \leq \|x - q\|,$$

for each $q \in F(T)$. Using the iterative scheme (3.1), we obtain,

$$\begin{aligned}
\|x_{n+1} - q\| =& \|(1 - a_n)T_1 y_n + a_n T_2 z_n - q\| \\
=& \|(1 - a_n)[T_1 y_n - q] + a_n [T_2 z_n - q]\| \\
\leq& (1 - a_n)\|T_1 y_n - q\| + a_n \|T_2 z_n - q\| \\
\leq& (1 - a_n)\|y_n - q\| + a_n \|z_n - q\| \\
=& (1 - a_n)\|(1 - b_n)T_1 x_n + b_n T_2 z_n - q\| + a_n \|(1 - c_n)x_n + c_n T_1 x_n - q\| \\
=& (1 - a_n)\|(1 - b_n)[T_1 x_n - q] + b_n [T_2 z_n - q]\| + a_n \|(1 - c_n)[x_n - q] + c_n [T_1 x_n - q]\| \\
=& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \|z_n - q\|\right] + a_n \left[(1 - c_n)\|x_n - q\| + c_n \|x_n - q\|\right] \\
=& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \|z_n - q\|\right] + a_n \|x_n - q\| \\
=& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \|(1 - c_n)x_n + c_n T_1 x_n - q\|\right] + a_n \|x_n - q\| \\
=& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \|(1 - c_n)[x_n - q] + c_n [T_1 x_n - q]\|\right] + a_n \|x_n - q\| \\
\leq& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \left\{(1 - c_n)\|x_n - q\| + c_n \|T_1 x_n - q\|\right\}\right] + a_n \|x_n - q\| \\
\leq& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \left\{(1 - c_n)\|x_n - q\| + c_n \|x_n - q\|\right\}\right] + a_n \|x_n - q\| \\
=& (1 - a_n)\left[(1 - b_n)\|x_n - q\| + b_n \|x_n - q\|\right] + a_n \|x_n - q\| \\
=& (1 - a_n)\|x_n - q\| + a_n \|x_n - q\| \\
=& \|x_n - q\|.
\end{aligned} \tag{3.2}$$

The above implication infers that $(\|x_n - q\|)$ is a non-increasing sequence and also, bounded below for all $q \in F(T)$. Therefore, $\lim_{n\to\infty} \|x_n - q\|$ exists. □

Using the above lemma, we now derive a necessary and sufficient condition for the existence of common fixed points of two generalized α-Reich-Suzuki non-expansive mappings.

Theorem 3.2. *Let E be a uniformly convex Banach space and U be a non-empty closed convex subset of E. Consider $T_1, T_2 : U \to U$ be two generalized α-Reich-Suzuki non-expansive mappings. For any $x_1 \in U$, we define the sequence (x_n) as the iteration scheme given by* (3.1). *Then $F(T) = F(T_1) \cap F(T_2) \neq \emptyset$ if and only if $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|$.*

Proof. Let $q \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists and suppose

$$\lim_{n\to\infty} \|x_n - q\| = t. \tag{3.3}$$

Also, we have,

$$\begin{aligned} &\left\|T_1x_n - q\right\| \leq \left\|x_n - q\right\| \\ \Rightarrow &\limsup_{n\to\infty} \left\|T_1x_n - q\right\| \leq \limsup_{n\to\infty} \left\|x_n - q\right\| \leq t, \end{aligned} \tag{3.4}$$

and similarly,

$$\limsup_{n\to\infty} \left\|T_2x_n - q\right\| \leq \limsup_{n\to\infty} \left\|x_n - q\right\| \leq t.$$

Next, from (3.2), we have,

$$\begin{aligned} t = \lim_{n\to\infty} \left\|x_{n+1} - q\right\| &= \lim_{n\to\infty} \left\|(1 - a_n)T_1y_n + a_nT_2z_n - q\right\| \\ &\leq \lim_{n\to\infty} \left\|x_n - q\right\| \\ &\leq t. \end{aligned} \tag{3.5}$$

Then from (3.5), we get

$$\lim_{n\to\infty} \left\|(1 - a_n)T_1y_n + a_nT_2z_n - q\right\| = t. \tag{3.6}$$

This leads to

$$\limsup_{n\to\infty} \left\|(1 - a_n)(T_1y_n - q) + a_n(T_2z_n - q)\right\| = t.$$

Also, making use of (3.1), we get

$$\begin{aligned} \left\|z_n - q\right\| &= \left\|(1 - c_n)x_n + c_nT_1x_n - q\right\| \\ &= \left\|(1 - c_n)(x_n - q) + c_n(T_1x_n - q)\right\| \\ &\leq (1 - c_n)\left\|x_n - q\right\| + c_n\left\|T_1x_n - q\right\| \\ &\leq (1 - c_n)\left\|x_n - q\right\| + c_n\left\|x_n - q\right\| \\ &= \left\|x_n - q\right\|. \end{aligned} \tag{3.7}$$

As T_2 is a generalized α-Reich-Suzuki non-expansive mapping, we have

$$\begin{aligned} &\left\|T_2z_n - q\right\| \leq \left\|z_n - q\right\| \\ &\qquad\qquad\quad \leq \left\|x_n - q\right\| \\ \Rightarrow &\limsup_{n\to\infty} \left\|T_2z_n - q\right\| \leq \limsup_{n\to\infty} \left\|x_n - q\right\| = t. \end{aligned} \tag{3.8}$$

Again, using the fact that T_1 is a generalized α-Reich-Suzuki non-expansive mapping and also using (3.8), we get,

$$\begin{aligned} \left\|T_1y_n - q\right\| &= \left\|T_1\left((1 - b_n)T_1x_n + b_nT_2z_n\right) - q\right\| \\ &\leq \left\|(1 - b_n)T_1x_n + b_nT_2z_n - q\right\| \\ &\leq (1 - b_n)\left\|T_1x_n - q\right\| + b_n\left\|T_2z_n - q\right\| \\ &\leq (1 - b_n)\left\|x_n - q\right\| + b_n\left\|x_n - q\right\| \\ &= \left\|x_n - q\right\| \\ \Rightarrow \limsup_{n\to\infty} \left\|T_1y_n - q\right\| &\leq \limsup_{n\to\infty} \left\|x_n - q\right\| = t. \end{aligned} \tag{3.9}$$

Using (3.8), (3.9) and (3.6) on Lemma 2.4, we obtain,

$$\lim_{n\to\infty} \left\|T_1 y_n - T_2 z_n\right\| = 0. \tag{3.10}$$

Again using Abbas-Nazir iterative scheme (3.1) and (3.10),

$$\begin{aligned}
\lim_{n\to\infty} \|x_{n+1} - T_2 z_n\| &= \lim_{n\to\infty} \left\|(1-a_n)T_1 y_n + a_n T_2 z_n - T_2 z_n\right\| \\
&= \lim_{n\to\infty} \left\|(1-a_n)T_1 y_n - (1-a_n)T_2 z_n\right\| \\
&= \lim_{n\to\infty} (1-a_n)\left\|T_1 y_n - T_2 z_n\right\| \\
&= 0
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \|x_{n+1} - T_2 z_n\| = 0. \tag{3.11}$$

Using triangle inequality and the fact that T_2 is quasi-nonexpansive, we have

$$\begin{aligned}
\left\|x_{n+1} - q\right\| &\le \|x_{n+1} - T_2 z_n\| + \left\|T_2 z_n - q\right\| \\
&\le \|x_{n+1} - T_2 z_n\| + \left\|z_n - q\right\|.
\end{aligned}$$

From (3.3) and (3.11), we obtain

$$t \le \liminf_{n\to\infty} \left\|z_n - q\right\|. \tag{3.12}$$

Taking lim sup as $n \to \infty$ on (3.7),

$$\limsup_{n\to\infty} \left\|z_n - q\right\| \le t. \tag{3.13}$$

Taking (3.12) and (3.13) together,

$$\lim_{n\to\infty} \left\|z_n - q\right\| = t. \tag{3.14}$$

Further letting $n \to \infty$ on (3.7) and using (3.14), we deduce

$$\begin{aligned}
t = \lim_{n\to\infty} \left\|z_n - q\right\| &= \lim_{n\to\infty} \left\|(1-c_n)(x_n - q) + c_n(T_1 x_n - q)\right\| \\
&\le \lim_{n\to\infty} \left\|x_n - q\right\| \\
&\le t,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|(1-c_n)(x_n - q) + c_n(T_1 x_n - q)\right\| = t. \tag{3.15}$$

Employing Lemma 2.4 and (3.4) in (3.15), we get

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{3.16}$$

Now using Abbas-Nazir iteration (3.1) and (3.10), we derive

$$\begin{aligned} x_{n+1} &= (1-a_n)T_1y_n + a_nT_2z_n \\ &= T_1y_n + a_n(T_2z_n - T_1y_n) \\ \Rightarrow \left\|x_{n+1} - T_1y_n\right\| &= |a_n|\left\|T_2z_n - T_1y_n\right\| \\ \Rightarrow \lim_{n\to\infty}\left\|x_{n+1} - T_1y_n\right\| &= \lim_{n\to\infty} a_n\left\|T_2z_n - T_1y_n\right\| \\ \Rightarrow \lim_{n\to\infty}\left\|x_n - T_1y_n\right\| &= 0. \end{aligned} \tag{3.17}$$

Also, using triangle inequality and (3.1), we get

$$\begin{aligned} \|T_2x_n - x_n\| &\le \|T_2x_n - T_2z_n\| + \left\|T_2z_n - T_1y_n\right\| + \left\|T_1y_n - x_n\right\| \\ &\le \|x_n - z_n\| + \left\|T_2z_n - T_1y_n\right\| + \left\|T_1y_n - x_n\right\| \\ &= \|x_n - (1-c_n)x_n - c_nT_1x_n\| + \left\|T_2z_n - T_1y_n\right\| + \left\|T_1y_n - x_n\right\| \\ &= c_n\|x_n - T_1x_n\| + \left\|T_2z_n - T_1y_n\right\| + \left\|T_1y_n - x_n\right\|. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$ and employing (3.10), (3.16) and (3.17) on (3.18), this implies that

$$\lim_{n\to\infty}\|T_2x_n - x_n\| = 0.$$

Hence we are done.

Conversely, let $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0 = \lim_{n\to\infty}\|x_n - T_2x_n\|$. Now, by triangle inequality we have,

$$\left\|T_1q - x_n\right\| \le \left\|T_1q - T_1x_n\right\| + \|T_1x_n - x_n\|.$$

Since $\frac{1}{2}\|T_1x_n - x_n\| = 0 \le \left\|x_n - q\right\|$, then

$$\left\|T_1q - T_1x_n\right\| \le \max\{P(q,x_n), Q(q,x_n)\},$$

where

$$P(q,x_n) = \alpha\|T_1x_n - x_n\| + \alpha\left\|T_1q - q\right\| + (1-2\alpha)\left\|x_n - q\right\|$$

and

$$Q(q,x_n) = \alpha\left\|T_1x_n - q\right\| + \alpha\left\|T_1q - x_n\right\| + (1-2\alpha)\left\|x_n - q\right\|.$$

Then, employing Lemma 2.5 and putting $x = x_n$ and $y = q$ for the mapping T_1, we have,

$$\left\|x_n - T_1q\right\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\|x_n - T_1x_n\| + \left\|x_n - q\right\|. \tag{3.19}$$

Considering lim sup as $n \to \infty$ on (3.19) and using Definition 2.3, we have

$$r(T_1q,(x_n)) = \limsup_{n\to\infty}\left\|x_n - q\right\| = r(q,(x_n)) = r(U,(x_n)),$$

which implies that $T_1q \in A(U,(x_n))$. Since E is a uniformly convex Banach space, $A(U,(x_n))$ has exactly one element and therefore, $T_1q = q$. In a similar manner, we can prove that $T_2q = q$. This leads to the conclusion that $F(T)$ is non-empty and the converse is also proved. □

The subsequent lemma deals with the demiclosedness principle of generalized α-Reich-Suzuki non-expansive mappings at zero.

Lemma 3.3. *Let E be a Banach space with the Opial's property and U be a non-empty closed convex subset of E. Let $S : U \to U$ be a generalized α-Reich-Suzuki non-expansive mapping. Then $I - S$ is demiclosed at zero.*

Proof. Let (x_n) be a sequence in U such that (x_n) converges weakly to $l \in U$ and $(I - S)x_n \to \theta$ as $n \to \infty$. Then by Lemma 2.5, we have

$$\left\|x_n - Ty\right\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\|x_n - Tx_n\| + \left\|x_n - y\right\|.$$

Letting lim inf as $n \to \infty$ on both sides of the above inequality, we get

$$\liminf_{n\to\infty} \|x_n - Tl\| \le \liminf_{n\to\infty} \|x_n - l\|.$$

If $l \ne Tl$, then by the Opial's property of E, we have

$$\liminf_{n\to\infty} \|x_n - l\| < \liminf_{n\to\infty} \|x_n - Tl\|,$$

which is a contradiction. So we must have $l = Tl$, i.e., $(I - S)l = \theta$. This concludes that $I - S$ is demiclosed at zero. □

Our upcoming two results deal with the weak convergence of the iterative scheme (3.1) to obtain some common fixed points of T_1 and T_2.

Theorem 3.4. *Let E be a uniformly convex Banach space and U be any non-empty closed convex subset of E. Suppose that E satisfies the Opial's property. Let $T_1, T_2 : U \to U$ be two generalized α-Reich-Suzuki non-expansive mappings with $F(T) = F(T_1) \cap F(T_2) \ne \emptyset$. For any $x_1 \in U$, we define the sequence (x_n) as the iteration scheme given by* (3.1). *Then (x_n) converges weakly to an element of $F(T)$.*

Proof. Since $F(T) \ne \emptyset$ and $T_1, T_2 : U \to U$ are two generalized α-Reich-Suzuki non-expansive mappings, then by Theorem 3.2 we have,

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|.$$

Consider that E satisfies Opial's property and let l_1 and l_2 be two weak subsequential limits of (x_n). Suppose that (x_{n_i}) weakly converges to l_1 and (x_{n_j}) weakly converges to l_2. We want to show that $l_1, l_2 \in F(T)$. By Lemma 3.3, we have $I - T_1$ is demiclosed at zero. Then we have $(I - T_1)l_1 = 0$. This implies that $T_1 l_1 = l_1$. Following similar arguments, we can conclude that $T_2 l_1 = l_1$. Therefore $l_1 \in F(T)$. Similarly one can prove that, $l_2 \in F(T)$. Now we want to show $l_1 = l_2$. Let if possible, $l_1 \ne l_2$. Then we obtain,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - l_1\| &= \liminf_{i\to\infty} \left\|x_{n_i} - l_1\right\| \\
&< \liminf_{i\to\infty} \left\|x_{n_i} - l_2\right\| \\
&= \lim_{n\to\infty} \|x_n - l_2\| \\
&= \liminf_{j\to\infty} \left\|x_{n_j} - l_2\right\| \\
&< \liminf_{j\to\infty} \left\|x_{n_j} - l_1\right\| \\
&= \lim_{n\to\infty} \|x_n - l_1\|
\end{aligned}$$

which leads to a contradiction. Therefore $l_1 = l_2$ and this implies that (x_n) converges weakly to an element of $F(T)$. □

One can note that, there are classes of uniformly convex Banach spaces where the Opial's property does not hold. Therefore the previously discussed result is not true for such structures. As an alternative way of proof, in the next result, we assume the existence of Fréchet differential norm instead of the Opial's property.

Theorem 3.5. *In Theorem 3.4, we replace the Opial's property by the assumption that E has a Fréchet differential norm, and also consider that* $\lim_{n\to\infty}\|tx_n + (1-t)u - v\|$ *exists for all* $u, v \in F(T)$. *Additionally, suppose that* $I - T_1$ *and* $I - T_2$ *are demiclosed at zero. Then* (x_n) *converges weakly to a common fixed point of* T_1 *and* T_2.

Proof. Our aim is to show that (x_n) has exactly one limit point. Suppose that (x_{n_i}) weakly converges to l_1 and (x_{n_j}) weakly converges to l_2. Since $F(T) \neq \emptyset$, from Theorem 3.2 we get, $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0 = \lim_{n\to\infty}\|x_n - T_2x_n\|$. Also, since $I - T_1$ and $I - T_2$ are demiclosed at zero, this further leads to the fact that $l_1, l_2 \in F(T)$. Putting $u - v$ and $t(x_n - u)$ instead of x and u respectively in (2.2), we have

$$\begin{aligned}\frac{1}{2}\|u-v\|^2 + t\langle x_n - u, J(u-v)\rangle \leq& \frac{1}{2}\|tx_n + (1-t)u - v\|^2\\ \leq& \frac{1}{2}\|u-v\|^2 + t\langle x_n - u, J(u-v)\rangle + g(t\|x_n - u\|).\end{aligned}$$

Using the given condition, we get

$$\begin{aligned}\frac{1}{2}\|u-v\|^2 + t\limsup_{n\to\infty}\langle x_n - u, J(u-v)\rangle \leq& \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)u - v\|^2\\ \leq& \frac{1}{2}\|u-v\|^2 + t\liminf_{n\to\infty}\langle x_n - u, J(u-v)\rangle + O(t).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\langle x_n - u, J(u-v)\rangle \leq \liminf_{n\to\infty}\langle x_n - u, J(u-v)\rangle + \frac{O(t)}{t}.$$

Taking $t \to 0^+$, we get $\lim_{n\to\infty}\langle x_n - u, J(u-v)\rangle$ exists. Now, we have $\langle l_1 - u, J(u-v)\rangle = d$ (say) and also, $\langle l_2 - u, J(u-v)\rangle = d$. So, $\langle l_1 - l_2, J(u-v)\rangle = 0$ for all $u, v \in F(T)$. From this we obtain,

$$\|l_1 - l_2\|^2 = \langle l_1 - l_2, J(l_1 - l_2)\rangle = 0$$

which is impossible, unless $l_1 = l_2$. Then (x_n) converges weakly to a common fixed point of T_1 and T_2. □

Next, we recall the concept of Condition (A') originally brought about by Chidume and Ali [7]. Two mappings $T_1, T_2 : U \to U$, where U is a non-empty subset of a Banach space E, are said to satisfy Condition (A') if there is any non-decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|T_1x - x\| \geq f(d(x,F))$ or $\|T_2x - x\| \geq f(d(x,F))$ for each $x \in U$. Here, we must mention that this criteria is the extension of Condition (A) which is applicable for a single mapping and introduced by Senter and Dotson [32]. Now, we modify the previous assumption slightly and come up with the following notion which is prerequisite for the imminent strong convergence result.

The mappings $T_1, T_2 : U \to U$ with $F(T) \neq \emptyset$ satisfy Condition (B) if there exists a non-decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in U$, $\max\{\|T_1x - x\|, \|T_2x - x\|\} \geq f(d(x_n, F))$, where $d(x_n, F) = \inf_{z\in F}\{\|x - z\|\}$.

Next, we state the following simple result as a lemma.

Lemma 3.6. *Suppose that (a_n) and (b_n) are two sequences of non-negative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_n b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

Using the above lemma and the notion of Condition (B), we now prove the following strong convergence result of the iterative scheme (3.1).

Theorem 3.7. *Let E be a uniformly convex Banach space and U be any non-empty closed convex subset of E. Let $T_1, T_2 : U \to U$ be two generalized α-Reich-Suzuki non-expansive mappings with $F(T) = F(T_1) \cap F(T_2) \neq \emptyset$. For any $x_1 \in U$, we define the sequence (x_n) as Abbas-Nazir iteration scheme* (3.1). *Suppose T_1, T_2 satisfy the Condition (B). Then (x_n) converges strongly to some common fixed point of T_1 and T_2.*

Proof. Let $q \in F(T)$. Then by the Theorem 3.1, $\lim_{n\to\infty} \left\|x_n - q\right\|$ exists for all $q \in F(T)$. Also

$$\left\|x_{n+1} - q\right\| \leq \left\|x_n - q\right\|$$

for all $n \in \mathbb{N}$. Then, $d(x_{n+1}, F) \leq d(x_n, F)$. Therefore by Lemma 3.6, $\lim_{n\to\infty} d(x_n, F)$ exists. Again from Theorem 3.2 we have, $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. Since T_1 and T_2 satisfy the Condition (B), we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$ which implies that $\lim_{n\to\infty} d(x_n, F) = 0$.

Then, we can choose a subsequence (x_{n_k}) of (x_n) and some sequence (p_k) in $F(T)$ such that $\left\|x_{n_k} - p_k\right\| < \frac{\epsilon}{2}$ for all $k \in \mathbb{N}$. Next, we want to show that (x_n) is a Cauchy sequence. Now, for all $n, m \geq k$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \left\|x_{n+m} - p_k\right\| + \left\|x_n - p_k\right\| \\
&\leq \left\|x_{n+m-1} - p_k\right\| + \left\|x_n - p_k\right\| \\
&\leq \left\|x_{n+m-2} - p_k\right\| + \left\|x_n - p_k\right\| \\
&\ \ \vdots \\
&\leq 2\left\|x_n - p_k\right\| \\
&\leq 2\left\|x_{n_k} - p_k\right\| \\
&< \epsilon.
\end{aligned}$$

Thus (x_n) is a Cauchy sequence in U. Since U is a closed subset of E, $\lim_{n\to\infty} x_n = x$ for some $x \in U$. Now we have to show that $x \in F(T)$. Since $F(T)$ is closed and $\lim_{n\to\infty} d(x_n, F) = 0$, we can conclude that $x \in F(T)$. Hence the theorem follows. □

Remark 3.8. *Our findings extend, complement and unify those of Khatoon et al. [29], Pandey et al. [25] and Khatoon et al. [14] which deal with the convergence results and a few fixed point results concerning a generalized α-Reich Suzuki non-expansive mapping.*

4. Numerical Examples

In this section, we illustrate a couple of numerical examples to endorse our achieved results. Additionally, we vindicate the conclusions drawn out from these examples with the help of effective tabular and figurative documentations. Here, we can note that the newly introduced algorithm gives better convergence rate than those of Mann, Ishikawa, Agarwal iterations and we confirm this by changing the parameters. For this simulation, we make use of MATLAB 2017a software. Throughout this section, we use the notation F for the set $F(T)$.

Example 4.1. *Let $E = \mathbb{R}^2$ and $C = \{x = (x_1, x_2) | (x_1, x_2) \in [0,\infty) \times [0,\infty)\}$ be a subset of E equipped with the norm $\|x\| = \left|\frac{x_1}{3}\right| + |x_2|$. Define two self-mappings T_1, T_2 on C by*

$$T_1(x_1,x_2) = \begin{cases} \left(\frac{5+x_1}{16}, \frac{5+x_2}{16}\right), & \text{when } (x_1,x_2) \in [0,\frac{1}{3}] \times [0,\infty); \\ \left(\frac{1+x_1}{4}, \frac{1+x_2}{4}\right), & \text{when } (x_1,x_2) \in (\frac{1}{3},\infty) \times [0,\infty), \end{cases}$$

and

$$T_2(x_1,x_2) = \begin{cases} \left(\frac{3+x_1}{10}, \frac{3+x_2}{10}\right), & \text{when } (x_1,x_2) \in [0,\frac{1}{3}] \times [0,\infty); \\ \left(\frac{2+x_1}{7}, \frac{2+x_2}{7}\right), & \text{when } (x_1,x_2) \in (\frac{1}{3},\infty) \times [0,\infty). \end{cases}$$

Here, we can verify that T_1 and T_2 are not non-expansive but generalized α-Reich Suzuki non-expansive mappings. Take $x = (\frac{1}{3}, 0)$ and $y = (\frac{1001}{3000}, 0)$, then $T_1(x) = (\frac{1}{3}, \frac{5}{16})$ and $T_1(y) = (\frac{4001}{12000}, \frac{1}{4})$. Now, $\left\|T_1(x) - T_1(y)\right\| = \frac{9004}{144000} > \frac{1}{9000} = \left\|x - y\right\|$.

Again, consider $x = (\frac{1}{3}, 0)$ and $y = (\frac{1001}{3000}, 0)$, then $T_2(x) = (\frac{1}{3}, \frac{3}{10})$, $T_2(y) = (\frac{7001}{21000}, \frac{2}{7})$. Now

$$\left\|T_2(x) - T_2(y)\right\| = \frac{901}{63000} > \frac{1}{9000} = \left\|x - y\right\|.$$

Therefore, our claim that T_1 and T_2 are not non-expansive is validated. Now we assume that $x, y \in C$ be arbitrary. Then we have the following cases.

Case-I: *When $x, y \in [0, \frac{1}{3}] \times [0,\infty)$. Then,*

$$\begin{aligned} \left\|T_1(x) - T_1(y)\right\| &= \left\|\left(\frac{x_1 - y_1}{16}, \frac{x_2 - y_2}{16}\right)\right\| \\ &= \frac{1}{16}\left\{\left|\frac{x_1 - y_1}{3}\right| + |x_2 - y_2|\right\} \\ &\leq \frac{1}{2}\left\|x - y\right\| \\ &\leq \frac{1}{2}\left\|x - y\right\| + \frac{1}{4}\|x - T_1(x)\| + \frac{1}{4}\left\|y - T_1(y)\right\| \\ &= \alpha\|x - T_1(x)\| + \alpha\left\|y - T_1(y)\right\| + (1 - 2\alpha)\left\|x - y\right\| \\ &\leq \max\{P(x,y), Q(x,y)\}, \end{aligned}$$

with $\alpha = \frac{1}{4}$.

Case-II: *When* $x, y \in (\frac{1}{3}, \infty) \times [0, \infty)$. *Then,*

$$\begin{aligned}
\left\|T_1(x) - T_1(y)\right\| &= \left\|\left(\frac{x_1 - y_1}{4}, \frac{x_2 - y_2}{4}\right)\right\| \\
&= \frac{1}{4}\left\{\left|\frac{x_1 - y_1}{3}\right| + |x_2 - y_2|\right\} \\
&\leq \frac{1}{2}\left\|x - y\right\| \\
&\leq \frac{1}{2}\left\|x - y\right\| + \frac{1}{4}\|x - T_1(x)\| + \frac{1}{4}\left\|y - T_1(y)\right\| \\
&= \alpha\|x - T_1(x)\| + \alpha\left\|y - T_1(y)\right\| + (1 - 2\alpha)\left\|x - y\right\| \\
&\leq \max\{P(x, y), Q(x, y)\},
\end{aligned}$$

with $\alpha = \frac{1}{4}$.

Case-III: *When* $x \in [0, \frac{1}{3}] \times [0, \infty)$ *and* $y \in (\frac{1}{3}, \infty) \times [0, \infty)$. *Then,*

$$\begin{aligned}
\left\|T_1(x) - T_1(y)\right\| &= \left\|\left(\frac{x_1 - 4y_1 + 1}{16}, \frac{x_2 - 4y_2 + 1}{16}\right)\right\| \\
&= \frac{1}{16}\left\{\left|\frac{x_1 - 4y_1 + 1}{3}\right| + |x_2 - 4y_2 + 1|\right\} \\
&\leq \frac{1}{4}\left|\frac{x_1 - y_1}{3}\right| + \frac{1}{16}\left|\frac{3x_1 - 1}{3}\right| + \frac{1}{16}|x_2 - y_2| + \frac{1}{16}|3y_2 - 1| \\
&\leq \frac{1}{4}\left|\frac{x_1 - y_1}{3}\right| + \frac{1}{4}|x_2 - y_2| + \frac{1}{16}\left|\frac{3x_1 - 1}{3}\right| + \frac{1}{16}|3x_2 - 1| \\
&\quad + \frac{1}{16}\left|\frac{3y_2 - 1}{3}\right| + \frac{1}{16}|3y_2 - 1| \\
&= \frac{1}{4}\left\|x - y\right\| + \frac{1}{16}\|x - T_1(x)\| + \frac{1}{16}\left\|y - T_1(y)\right\| \\
&\leq \frac{1}{2}\left\|x - y\right\| + \frac{1}{4}\|x - T_1(x)\| + \frac{1}{4}\left\|y - T_1(y)\right\| \\
&= \alpha\|x - T_1(x)\| + \alpha\left\|y - T_1(y)\right\| + (1 - 2\alpha)\left\|x - y\right\| \\
&\leq \max\{P(x, y), Q(x, y)\},
\end{aligned}$$

where $\alpha = \frac{1}{4}$.

Case-IV: *When* $x, y \in [0, \frac{1}{3}] \times [0, \infty)$. *Then,*

$$\begin{aligned}
\left\|T_2(x) - T_2(y)\right\| &= \left\|\left(\frac{x_1 - y_1}{10}, \frac{x_2 - y_2}{10}\right)\right\| \\
&= \frac{1}{10}\left\{\left|\frac{x_1 - y_1}{3}\right| + |x_2 - y_2|\right\} \\
&\leq \frac{1}{2}\left\|x - y\right\| \\
&\leq \frac{1}{2}\left\|x - y\right\| + \frac{1}{4}\|x - T_2(x)\| + \frac{1}{4}\left\|y - T_2(y)\right\| \\
&= \alpha\|x - T_2(x)\| + \alpha\left\|y - T_2(y)\right\| + (1 - 2\alpha)\left\|x - y\right\| \\
&\leq \max\{P(x, y), Q(x, y)\},
\end{aligned}$$

where $\alpha = \frac{1}{4}$.

Case-V: *When* $x, y \in (\frac{1}{3}, \infty) \times [0, \infty)$. *Then,*

$$\begin{aligned}
\|T_2(x) - T_2(y)\| &= \left\|\left(\frac{x_1 - y_1}{7}, \frac{x_2 - y_2}{7}\right)\right\| \\
&= \frac{1}{7}\left\{\left|\frac{x_1 - y_1}{3}\right| + |x_2 - y_2|\right\} \\
&\leq \frac{1}{2}\|x - y\| \\
&\leq \frac{1}{2}\|x - y\| + \frac{1}{4}\|x - T_2(x)\| + \frac{1}{4}\|y - T_2(y)\| \\
&= \alpha\|x - T_2(x)\| + \alpha\|y - T_2(y)\| + (1 - 2\alpha)\|x - y\| \\
&\leq \max\{P(x, y), Q(x, y)\},
\end{aligned}$$

where $\alpha = \frac{1}{4}$.

Case-VI: *When* $x \in [0, \frac{1}{3}] \times [0, \infty)$ *and* $y \in (\frac{1}{3}, \infty) \times [0, \infty)$. *Then,*

$$\begin{aligned}
\|T_2(x) - T_2(y)\| &= \left\|\left(\frac{7x_1 - 10y_1 + 1}{70}, \frac{7x_2 - 10y_2 + 1}{70}\right)\right\| \\
&= \frac{1}{70}\left\{\left|\frac{7x_1 - 10y_1 + 1}{3}\right| + |7x_2 - 10y_2 + 1|\right\} \\
&\leq \frac{1}{70}\left|\frac{x_1 - y_1}{3}\right| + \frac{1}{70}\left|\frac{3y_1 - 1}{3}\right| + \frac{1}{70}|x_2 - y_2| + \frac{1}{70}|3y_2 - 1| \\
&\leq \frac{1}{70}\|x - y\| + \frac{1}{16}\|y - T_2(y)\| + \frac{1}{70}\|x - T_2(x)\| \\
&\leq \frac{1}{2}\|x - y\| + \frac{1}{4}\|y - T_2(y)\| + \frac{1}{4}\|x - T_2(x)\| \\
&= \alpha\|x - T_2(x)\| + \alpha\|y - T_2(y)\| + (1 - 2\alpha)\|x - y\|,
\end{aligned}$$

where $\alpha = \frac{1}{4}$. *Then we can conclude that* T_1 *and* T_2 *both are generalized* α*-Reich Suzuki non-expansive mappings for* $\alpha = \frac{1}{4}$. *Again we have* $F(T_1) = \{(\frac{1}{3}, \frac{1}{3})\}$ *and* $F(T_2) = \{(\frac{1}{3}, \frac{1}{3})\}$ *and this implies that* $F(T) \neq \emptyset$. *We consider a non-decreasing map* $f(x) = x$ *which satisfies* $f(0) = 0$ *and* $f(r) > 0$ *when* $r \in (0, \infty)$. *Now, we discuss the following two cases.*

Case-I: *When* $x \in [0, \frac{1}{3}] \times [0, \infty)$. *Then we derive that,*

$$\begin{aligned}
d(x_n, F) &= \inf_{z \in F} \|x - z\| \\
&= \inf \left\|(x_1, x_2) - \left(\frac{1}{3}, \frac{1}{3}\right)\right\| \\
&= \inf \left\|\left(x_1 - \frac{1}{3}, x_2 - \frac{1}{3}\right)\right\| \\
&= \inf \left\{\left|\frac{3x_1 - 1}{9}\right|, \left|\frac{3x_2 - 1}{3}\right|\right\} \\
&= \frac{1}{3} \inf \left\{\left|\frac{3x_1 - 1}{3}\right|, |3x_2 - 1|\right\} \\
&= \frac{5}{48} \inf \|T_1 x - x\|.
\end{aligned}$$

Also we have,

$$\begin{aligned}
\|T_1x - x\| &= \left\|\left(\frac{5+x_1}{16} - x_1, \frac{5+x_2}{16} - x_2\right)\right\| \\
&= \left\|\left(\frac{5-15x_1}{16}, \frac{5-15x_2}{16}\right)\right\| \\
&= \left|\frac{5-15x_1}{48}\right| + \left|\frac{5-15x_2}{16}\right| \\
&= \frac{5}{16}\left\{\left|\frac{3x_1-1}{3}\right| + |3x_2 - 1|\right\}.
\end{aligned}$$

Then it is very much obvious that,

$$\max\{\|T_1x - x\|, \|T_2x - x\|\} \geq f(d(x_n, F)).$$

Case-II: *When $x \in (\frac{1}{3}, \infty) \times [0, \infty)$. Then we have*

$$\begin{aligned}
\|T_1x - x\| &= \left\|\left(\frac{1+x_1}{4} - x_1, \frac{1+x_2}{4} - x_2\right)\right\| \\
&= \left\|\left(\frac{1-3x_1}{4}, \frac{1-3x_2}{4}\right)\right\| \\
&= \left|\frac{3x_1-1}{12}\right| + \left|\frac{3x_2-1}{4}\right| \\
&= \frac{1}{4}\left\{\left|\frac{3x_1-1}{3}\right| + |3x_2 - 1|\right\}.
\end{aligned}$$

From here also, one can easily affirm that,

$$\max\{\|T_1x - x\|, \|T_2x - x\|\} \geq f(d(x_n, F)).$$

So, both the mappings T_1 and T_2 satisfy Condition (B) and therefore all the assumptions of Theorem 3.7. Hence, they own a common fixed point which is $(\frac{1}{3}, \frac{1}{3})$.

Example 4.2. *Let us consider the Banach space $E = \mathbb{R}$ with the usual norm and $C = [-3, \infty)$ be a non-empty closed convex subset of E. We define the following two self-mappings*

$$T_1(x) = \begin{cases} \frac{x}{5}, & \text{when } x \in [-3, \frac{1}{3}]; \\ \frac{x}{6}, & \text{elsewhere,} \end{cases}$$

and

$$T_2(x) = \begin{cases} \frac{x}{6}, & \text{when } x \in [-3, \frac{1}{3}]; \\ \frac{x}{9}, & \text{elsewhere.} \end{cases}$$

Here both of the mappings T_1 and T_2 are not non-expansive. This is clear from the fact that T_1 and T_2 both are discontinuous at $x = \frac{1}{3}$. Also by proceeding in a similar manner to that of the previous examples, one can easily check that the mappings T_1 and T_2 are generalized α-Reich-Suzuki non-expansive mappings with $\alpha = \frac{2}{5}$.

Iteration No.	$x_1 = 5$	$x_1 = 1.96$	$x_1 = 0.1$	$x_1 = -0.96$	$x_1 = -2.5$
1	5.000000	1.960000	0.100000	-0.960000	-2.500000
2	0.125526	0.059365	0.004062	-0.038997	-0.101555
3	0.005969	0.002823	0.000193	-0.001854	-0.004830
⋮	⋮	⋮	⋮	⋮	⋮
5	0.000017	0.000008	0.000000	-0.000005	-0.000013
6	0.000001	0.000000	⋮	0.000000	-0.000001
7	0.000000	⋮	⋮	⋮	0.000000

Table 1: The values of (x_n) for different initial values

Initial values	Mann iteration	Ishikawa iteration	Agarwal iteration	Liu iteration	Abbas iteration
-0.8	35	170	25	21	**13**
-0.6	35	169	25	21	**13**
-0.3	34	165	24	20	**12**
0.1	33	160	23	20	**12**
0.4	34	167	24	20	**12**
0.9	35	171	25	21	**13**

Table 2: Influence of initial points on different iteration schemes and number of iterations required to attain the common fixed point

Further, we have, $F(T_1) = \{0\}$ and $F(T_2) = \{0\}$ and this implies that $F(T) \neq \emptyset$. We consider a non-decreasing map $f(x) = \frac{x}{2}$ which satisfies $f(0) = 0$ and $f(r) > 0$ when $r \in (0, \infty)$. Now,

$$\begin{aligned} d(x_n, F) &= \inf_{z \in F} \|x - z\| \\ &= \inf \|x - 0\| \\ &= \inf \|x\| \\ &= \begin{cases} 0, & \text{if } x \in [-3, \frac{1}{3}]; \\ \frac{1}{3}, & \text{elsewhere.} \end{cases} \end{aligned}$$

From the previous calculation, we have

$$f(d(x_n, F)) = \begin{cases} 0, & \text{if } x \in [-3, \frac{1}{3}]; \\ \frac{1}{6}, & \text{elsewhere.} \end{cases}$$

Now, we discuss the subsequent cases.

Case-I: *When $x \in [-3, \frac{1}{3}]$. Then we have,*

$$\|T_1 x - x\| = \left\| \frac{x}{5} - x \right\| = \left| \frac{4x}{5} \right|,$$

and also

$$\|T_2 x - x\| = \left\| \frac{x}{6} - x \right\| = \left| \frac{5x}{6} \right|.$$

$a_n = \frac{n-1}{3n+1},\ b_n = \frac{n}{3n+1},\ c_n = \frac{n+2}{3n+1}$

Iterations	Initial values					
	-0.8	-0.6	-0.3	0.1	0.4	0.9
Mann	42	42	41	40	41	42
Ishikawa	112	111	109	106	110	112
Agarwal	31	31	30	29	31	31
Liu	20	20	20	19	20	20
Abbas	**13**	**13**	**13**	**13**	**13**	**13**

$a_n = \frac{50n+49}{101n+100},\ b_n = \frac{21n+30}{101n+100},\ c_n = \frac{30n+21}{101n+100}$

Iterations	Initial values					
	-0.8	-0.6	-0.3	0.1	0.4	0.9
Mann	67	66	65	63	66	67
Ishikawa	67	67	65	63	66	67
Agarwal	56	55	54	53	55	56
Liu	18	18	17	17	17	18
Abbas	**16**	**16**	**16**	**15**	**16**	**16**

$a_n = \frac{99n-100}{200n-199},\ b_n = \frac{51n-50}{200n-199},\ c_n = \frac{50n-49}{200n-199}$

Iterations	Initial values					
	-0.8	-0.6	-0.3	0.1	0.4	0.9
Mann	67	67	65	63	66	67
Ishikawa	66	66	65	63	65	67
Agarwal	56	56	55	53	55	56
Liu	18	18	17	17	17	18
Abbas	**16**	**16**	**16**	**15**	**16**	**16**

Table 3: Influence of initial points: comparison of various iteration schemes

Then obviously one can verify that for this case,

$$\max\{\|T_1x - x\|, \|T_2x - x\|\} \geq f(d(x_n, F)).$$

Case-II: *When $x \in (\frac{1}{3}, \infty)$. Here we have,*

$$\|T_1x - x\| = \left\|\frac{x}{6} - x\right\| = \left|\frac{5x}{6}\right|,$$

and again

$$\|T_2x - x\| = \left\|\frac{x}{9} - x\right\| = \left|\frac{8x}{9}\right|.$$

Now in this case also,

$$\max\{\|T_1x - x\|, \|T_2x - x\|\} \geq f(d(x_n, F)).$$

Hence, both the mappings T_1 and T_2 satisfy Condition (B) and therefore all the hypotheses of Theorem 3.7. So, they possess a common fixed point which is 0.

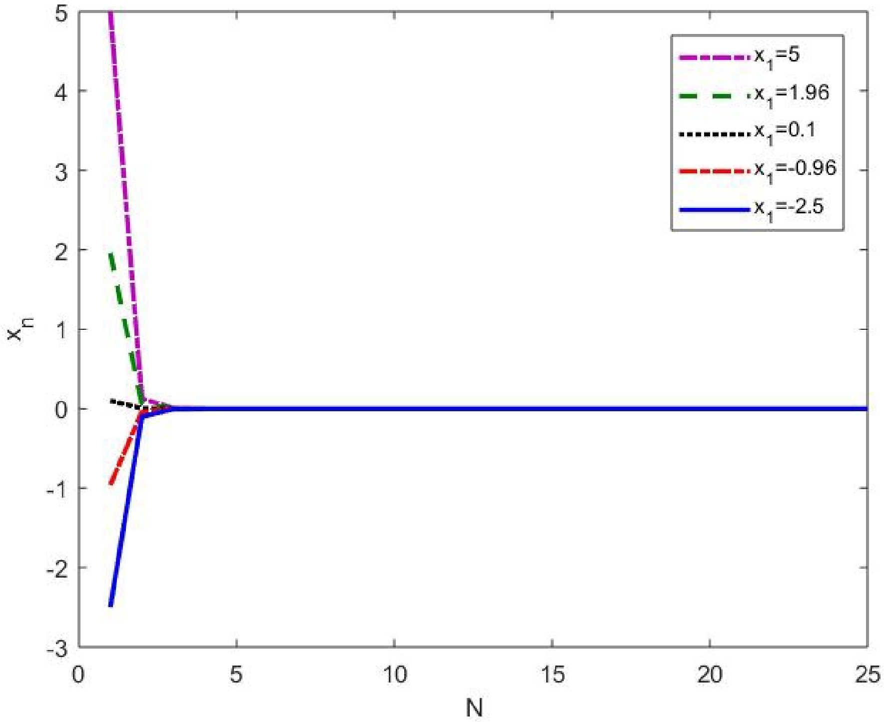

Figure 1: Convergence behaviour of Abbas-Nazir iterative scheme (3.1) for different initial values, where N= number of iterations and x_n= initial values

By means of MATLAB 2017*a software, we can verify that the sequence generated by Abbas-Nazir iterative scheme* (3.1) *converges to* 0*, which is exhibited in Table 1 and Figure 1. For that, we pick the control sequences* (a_n), (b_n) *and* (c_n) *given by* $a_n = \frac{n-1}{3n+1}$, $b_n = \frac{n}{3n+1}$ *and* $c_n = \frac{n+1}{3n+1}$ *from* $(0,1)$ *and initial values as* $x_1 = 5, 1.96, 0.1, -0.96, -2.5$.

Additionally, in Table 2, we enquire for the impact of initial guesses on the Mann, Ishikawa, Agarwal, Liu and Abbas-Nazir iteration schemes using the control sequences (a_n), (b_n) *and* (c_n) *given by* $a_n = \frac{2n-3}{9n-10}$, $b_n = \frac{3n-4}{9n-10}$ *and* $c_n = \frac{4n-3}{9n-10}$ *from* $(0,1)$.

Moreover, in Table 3, we correlate the convergence behaviour of aforementioned notable iterative algorithms for Example 4.2. We choose distinct sequences (a_n), (b_n) *and* (c_n) *from* $(0,1)$ *and fix* $\|x_n - x\| < 10^{-15}$ *as our rounding off criteria, where x is a common fixed point of the considered mappings. However, in Figure 1, we show the convergence behaviour of the newly proposed iterative method* (3.1) *for a particular choice of control sequence set and different choice of initial guesses.*

Remark 4.3. *From the above discussion and tables, we take a note of the fact that for distinct selection of parameters and initial values, the three-step Abbas-Nazir iterative scheme involving two mappings satisfying Condition* (B) *converges faster than the other comparable iterative algorithms.*

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References

[1] M. Aliyari, M. Gabeleh and E. Karapınar. Mann and Ishikawa iterative processes for cyclic relatively nonexpansive mappings in uniformly convex Banach spaces. *J. Nonlinear Convex Anal.*, 22(4):699–713, 2021.

[2] M. Abbas and T. Nazir. A new faster iteration process applied to constrained minimization and feasibility problems. *Mat. Vesnik*, 66(2):223–234, 2014.

[3] R.P. Agarwal, D. O'Regan, and D.R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *J. Nonlinear Convex Anal.*, 8(1):61–79, 2007.

[4] J. Ali and F. Ali. Approximation of common fixed points and the solution of image recovery problem. *Results Math.*, 74(4), 2019. Article No. 130.

[5] K. Aoyama and F. Kohsaka. Fixed point theorem for α-nonexpansive mappings in Banach spaces. *Nonlinear Anal.*, 74(13):4387–4391, 2011.

[6] A. Bera, A. Chanda, L.K. Dey, J. Ali. Iterative approximation of fixed points of a general class of non-expansive mappings in hyperbolic metric spaces. *J. Appl. Math. Comput.*, 68(3):1817-1839, 2022.

[7] C.E. Chidume and B. Ali. Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.*, 330(1):377–387, 2007.

[8] S. Dhompongsa, W. Inthakon, and A. Kaewkhao. Edelstein's method and fixed point theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.*, 350(1):12–17, 2009.

[9] C. Donghan, S. Jie, and L. Weiyi. Fixed point theorems and convergence theorems for a new generalized nonexpansive mapping. *Numer. Funct. Anal. Optim.*, 39(16):1742–1754, 2018.

[10] A. Hussain, D. Ali and E. Karapınar. Stability data dependency and errors estimation for a general iteration method. *Alexandria Engg. J.*, 60(1):703-710, 2021.

[11] A. Hussain, K. Ullah and M. Arshad. Fixed point approximation of Suzuki generalized nonexpansive mappings via new faster iteration process. *J. Nonlinear Convex Anal.*, 19(8):1383-1393, 2018.

[12] G.F. Hardy and T.D. Rogers. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.*, 16(2):201–206, 1973.

[13] S.H. Khan and T. Suzuki. A Reich-type convergence theorem for generalized nonexpansive mappings in uniformly convex Banach spaces. *Nonlinear Anal.*, 80:211–215, 2013.

[14] S. Khatoon, I. Uddin and V. Colao. Approximating fixed points of generalized α-Reich-Suzuki nonexpansive mapping in CAT(0) space. *J. Nonlinear Convex Anal.*, 21(9):2139–2150, 2020.

[15] S. Ishikawa. Fixed points by a new iteration method. *Proc. Amer. Math. Soc.*, 44(1):147–150, 1974.

[16] R. Kannan. Fixed point theorems in reflexive Banach spaces. *Proc. Amer. Math. Soc.*, 38(1):111–118, 1973.

[17] E. Karapınar and K. Taş. Generalized (C)-conditions and related fixed point theorems. *Comput. Math. Appl.*, 61(11):3370-3380, 2011.

[18] S.H. Khan and H. Fukher-ud-din. Weak and strong convergence of a scheme with errors for two nonexpansive mappings. *Nonlinear Anal.*, 61(8):1295–1301, 2005.

[19] Z. Liu, C. Feng, J.S. Ume, and S.M. Kang. Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings. *Taiwansese J. Math.*, 11(1):27–42, 2007.

[20] W.R. Mann. Mean value methods in iteration. *Proc. Amer. Math. Soc.*, 4:506–510, 1953.

[21] G. Marino, B. Scardamaglia and E. Karapınar . Strong convergence theorem for strict pseudo-contractions in Hilbert spaces. *J. Inequal. Appl.*, 2016, 2016. Article number: 134.

[22] M.A. Noor. New approximation schemes for general variational inequalities. *J. Math. Anal. Appl.*, 251(1):217–229, 2000.

[23] Z. Opial. Weak convergence of the sequence of successive approximations for non-expansive mappings. *Bull. Amer. Math. Soc.*, 73(4):595–597, 1967.

[24] R. Pandey, R. Pant, and A. Al-Rawashdeh. Fixed point results for a class of monotone nonexpansive type mappings in hyperbolic spaces. *J. Funct. Spaces*, 2018, 2018. Article ID 5850181.

[25] R. Pandey, R. Pant, V. Rakočević, and R. Shukla. Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications. *Results Math.*, 74(1), 2019. Article No. 7.

[26] R. Pant and R. Pandey. Fixed point results for a class of monotone nonexpansive type mappings in hyperbolic spaces. *Appl. Gen. Topol.*, 20(1):281–295, 2019.

[27] R. Pant and R. Shukla. Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces. *Numer. Funct. Anal. Optim.*, 38(2):248–266, 2017.

[28] S. Reich. Kannan's fixed point theorem. *Boll. Un. Mat. Ital.*, 4(4):1–11, 1971.

[29] S. Khatoon, I. Uddin and D. Baleanu. Approximation of fixed point and its application to fractional differential equation. *J. Appl. Math. Comput.*, 66(1-2):507-525, 2021.

[30] H. Garai, S. Som, A. Petruşel and L.K. Dey. Some characterizations of Reich and Chatterjea type nonexpansive mappings. *J. Fixed Point Theory Appl.*, 21(4), 2019. Article No: 94.

[31] J. Schu. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bull. Austral. Math. Soc.*, 43(1):153–159, 1991.

[32] H.F. Senter and Jr. W.G. Dotson. Approximating fixed points of nonexpansive mappings. *Proc. Amer. Math. Soc.*, 44(2):375–380, 1974.

[33] T. Suzuki. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.*, 340(2):1088–1095, 2008.

[34] W. Takahashi and T. Tamura. Limit theorems of operators by convex combinations of nonexpansive retractions in Banach spaces. *J. Approx. Theory*, 91(3):386–397, 1997.

[35] I. Uddin, J. Ali, and J.J. Nieto. An iteration scheme for a family of multivalued mappings in CAT(0) spaces with an application to image recovery. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 112(2):373–384, 2018.

[36] Y. Xu. Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations. *J. Math. Anal. Appl.*, 224(1):91–101, 1998.