Filomat 37:4 (2023), 1155–1166
https://doi.org/10.2298/FIL2304155C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some basic inequalities on golden Riemannian product manifolds with constant curvatures

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Abstract. This article is devoted to prove the basic Chen's inequalities for slant submanifolds in Riemannian space forms equipped with Golden structure. The equality case and some particular cases of derived inequalities are discussed.

1. Introduction

The concept of polynomial structures on a manifold came to discussion in [12] and it paved the foundation of Golden structure [8]. The study of invariant submanifolds for their different properties in a Riemannian manifold equipped with Golden structure appeared in [13] and results related to integrability in the same ambient manifold were proved in [11]. On the other side, the concept of Golden maps was proposed and harmonicity was established for such maps due to Sahin and Akyol in [18]. As far as warped product structures are concerned on Golden Riemannian manifold, their study was carried out in [2]. The Golden structure on Semi-Riemannian manifolds has also been investigated in recent years ([16],[17]).

The theory of slant submanifolds came into picture due to [3] and later researched in ([20],[19]). Recently, slant submanifolds were took into investigation due to Bahadir and Uddin [1] in Riemannian manifolds with Golden structure.

On the other hand, in 1993, Chen considered submanifolds of real space form [4] and introduced the basic idea for the sharp relationships between the intrinsic and extrinsic invariants. Later on, Chen-like inequalities were also studied in many other ambient spaces [5],[14],[15],[9],[10] and the references therein.

Inspired by all the above developments in the field, we establish sharp inequalities for golden Riemannian space forms.

2. Preliminaries

Consider any Riemannian manifold $(\tilde{M}^m, g)$ with dimension equal to m and assume M^n to be any Riemannian manifold isometrically immersing in $\tilde{M}$. Identify with the help of ∇, the covariant differentiation

2020 *Mathematics Subject Classification*. 53B05; 53B20; 53C25; 53C40
Keywords. Golden Riemannian manifolds; slant submanifolds; curvature
Received: 05 February 2022; Accepted: 09 May 2022
Communicated by Mića S. Stanković
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induced on M and with $\nabla^{\perp}$, the normal connection induced by ∇ on $TM^{\perp}$. When σ describes the second fundamental form, one can note down that

$$\tilde{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + \sigma(Y_1, Y_2), \quad \tilde{\nabla}_{Y_1}V = -A_V Y_1 + \nabla^{\perp}_{Y_1}V, \quad \forall Y_1, Y_2 \in \Gamma(TM), \quad \forall V \in \Gamma(TM^{\perp}).$$

Following link also hold

$$g(A_V Y_1, Y_2) = g(\sigma(Y_1, Y_2), V).$$

One writes Gauss equation as

$$\tilde{R}(Y_1, Y_2, Y_3, Y_4) = R(Y_1, Y_2, Y_3, Y_4) - g(\sigma(Y_1, Y_4), \sigma(Y_2, Y_3)) + g(\sigma(Y_1, Y_3), \sigma(Y_2, Y_4)). \tag{1}$$

Let us fix local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ on M. Then, one may estimate

$$H(p) = \sum_{i=1}^{n} \frac{1}{n}\sigma(e_i, e_i), \quad \sigma^s_{ij} = g(\sigma(e_i, e_j), e_s), \quad 1 \leq i, j \leq n; \quad n+1 \leq s \leq m.$$

We recall.

Lemma 2.1. *When one represents by $u_1, \ldots, u_t, v$ the $(t+1), t \geq 2$ real numbers provided [4]*

$$\Big(\sum_{k=1}^{t} u_k\Big)^2 = (t-1)\Big(\sum_{k=1}^{t} u_k^2 + v\Big),$$

then, $2u_1u_2 \geq v$ and equality holds if and only if $u_1 + u_2 = u_3 = \cdots = u_t$.

We have the following set to explain the relative null space of M in $\tilde{M}$

$$L_p = \{Y_1 \in T_pM \mid \sigma(Y_1, Y_2) = 0, \ \forall\ Y_2 \in T_pM\}, p \in M.$$

Let $\pi \subset T_pM$ represents a plane section and $K(\pi)$ be standing for the sectional curvature of M. We estimate

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_i), \ p \in M.$$

We also have

$$(\inf K)(p) = \inf\{K(\pi)|\pi \subset T_pM, \dim \pi = 2\}, \quad \delta_M(p) = \tau(p) - (\inf K)(p).$$

In our case $\delta_M(p)$ is used for Chen first invariant.

Next, we identify by L' the subspace of T_pM of dimension equal to q with $q \geq 2$ and its orthonormal basis by $\{e_1, \ldots, e_q\}$. We have

$$\tau(L') = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta),$$

L' represents q-plane section. Now, we move on by considering μ-tuples $(\lambda'_1 \ldots \lambda'_\mu)$ of integers ≥ 2 in the form of a set $S(\lambda', \mu)$ holding for the following inequality

$$\lambda'_1 < \lambda', \lambda'_1 + \cdots + \lambda'_\mu \leq \lambda',$$

for any integer $\mu \geq 0$. In addition, let us fix a λ' and consider unordered μ-tuples in the form of a set $S(\lambda')$. In this way, we note down the Riemannian invariant as

$$\delta(\lambda'_1 \ldots \lambda'_\mu)(p) = \tau(p) - S(\lambda'_1 \ldots \lambda'_\mu)(p), \quad \forall(\lambda'_1 \ldots \lambda'_\mu) \in S(\lambda'),$$

where $S(\lambda'_1 \dots \lambda'_\mu)(p) = \inf\{\tau(L'_1) + \cdots + \tau(L'_\mu)\}$, here, $L'_1 \dots L'_\mu$ varies for all μ mutually orthogonal subspaces of T_pM having $\dim L'_i = \lambda'_i, i \in \{1, \dots, \mu\}$. Set the following real constants

$$d(\lambda'_1, \dots, \lambda'_\mu) = \frac{1}{2}\frac{(\lambda' + \mu - 1 - \sum_{i=1}^{\mu} \lambda'_i)}{(\lambda' + \mu - \sum_{i=1}^{\mu} \lambda'_i)}\lambda'^2$$

and

$$b(\lambda'_1, \dots, \lambda'_\mu) = \frac{1}{2}[\lambda'(\lambda' - 1) - \sum_{i=1}^{k} \lambda'_i(\lambda'_i - 1)].$$

3. Golden Riemannian manifolds

Assume any $(1, 1)$-tensor field $\mathcal{X}$ on any Riemannian manifold $\tilde{M}^m$. Then, $\mathcal{X}$ produces a polynomial structure on $\tilde{M}$ if [1, 8, 12]

$$\mathcal{P}(Y) = Y^n + a_nY^{n-1} + \dots + a_2Y + a_1I = 0,$$

here, I is taken for identity $(1, 1)$-tensor field and at $p \in \tilde{M}, I, \mathcal{X}^{n-1}(p), \mathcal{X}^{n-2}(p), \dots, \mathcal{X}(p)$ are linearly independent. In present case, $\mathcal{P}(Y)$ is known as structure polynomial.

Any $(1, 1)$-tensor field φ produces structure of Golden type on $(\tilde{M}^m, g)$ provided [1, 7, 12]

$$\varphi^2 - \varphi - I = 0,$$

I, in the present situation is used for identity transformation. Furthermore, $\forall Y_1, Y_2 \in \Gamma(T\tilde{M})$, φ-compatible Riemannian metric g satisfies

$$g(\varphi Y_1, Y_2) = g(Y_1, \varphi Y_2). \tag{2}$$

$(\tilde{M}, g)$ equipped with Golden structure φ is termed as Golden Riemannian manifold [1, 8] provided Riemannian metric g satisfies (2). Substituting φY_1 in place of Y_1 in (2), creates the following

$$g(\varphi Y_1, \varphi Y_2) = g(\varphi^2 Y_1, Y_2) = g(\varphi Y_1, Y_2) + g(Y_1, Y_2).$$

Any $(1, 1)$-tensor field $\mathcal{X}$ produces an almost product structure on any differentiable manifold, provided [1]

$$\mathcal{X}^2 = I, \ \mathcal{X} \neq \pm I,$$

in this case, I is allocated for identity transformation. Additionally, if $\mathcal{X}$ also supports the following relation

$$g(\mathcal{X}Y_1, Y_2) = g(Y_1, \mathcal{X}Y_2),$$

$(\tilde{M}, g)$ turns to be almost product Riemannian manifold .

In case, φ is a structure of Golden type, it produces an almost product structure [8]

$$\mathcal{X} = \frac{1}{\sqrt{5}}(2\varphi - I), \tag{3}$$

and $\mathcal{X}$ produces a structure of Golden type

$$\varphi = \frac{1}{2}(I + \sqrt{5}\mathcal{X}). \tag{4}$$

We identify a submanifold M as

- totally umbilical provided

$$\sigma(Y_1, Y_2) = g(Y_1, Y_2)H,$$

in this situation, $\forall Y_1, Y_2 \in \Gamma(TM)$,

- totally umbilical submanifold becomes totally geodesic provided the second fundamental form vanishes identically.

Let $(\tilde{M}, g, \varphi)$ stands for Golden Riemannian manifold and (M, g) be any Riemannian manifold.

One calls M as slant submanifold of $\tilde{M}$ provided slant angle $\theta(Y)$ between TM and φY is independent of $p \in M$ and a nonzero vector Y tangent to M at p.

A slant submanifold becomes

- φ-invariant with $\theta = 0$;
- φ-anti-invariant when $\theta = \frac{\pi}{2}$;
- proper θ-slant (if neither invariant nor anti-invariant).

Also, $\forall Y \in \Gamma(TM)$, one might express

$$\varphi Y = TY + QY,$$

here, TY stands for tangent component and QY for normal component of φY.

Lemma 3.1. *Any submanifold (M, g) of $(\tilde{M}, g, \varphi)$, is recognized as slant if and only if $\exists\ \lambda \in [0, 1]$ and the following relation is satisfied [1]*

$$T^2 = \lambda(\varphi + I).$$

Additionally, when θ is used to denote slant angle of M, one observes that

$$\lambda = \cos^2 \theta.$$

Lemma 3.2. *For slant submanifold (M, g) of $(\tilde{M}, g, \varphi)$, notice that [1]*

$$g(TY_1, TY_2) = \cos^2 \theta(g(Y_1, Y_2) + g(Y_1, TY_2)).$$

Example 3.3. Assume that $\mathbb{E}^4$ denotes an Euclidean 4-space with standard coordinates (a_1, a_2, a_3, a_4) and φ be $(1, 1)$-tensor field [1]

$$\varphi(a_1, a_2, a_3, a_4) = ((1 - \psi)a_1, \psi a_2, (1 - \psi)a_3, \psi a_4)$$

$\forall (a_1, a_2, a_3, a_4) \in \mathbb{E}^4$, in this case $\psi = \frac{1+\sqrt{5}}{2}$ and $1 - \psi = \frac{1-\sqrt{5}}{2}$ represent roots of $t^2 = t + 1$. This implies

$$\begin{aligned} \varphi^2(a_1, a_2, a_3, a_4) &= ((1 - \psi)^2 a_1, \psi^2 a_2, (1 - \psi)^2 a_3, \psi^2 a_4) \\ &= ((1 - \psi)a_1, \psi a_2, (1 - \psi)a_3, \psi a_4) + (a_1, a_2, a_3, a_4) \end{aligned}$$

implying $\varphi^2 = \varphi + I$. In addition, we have

$$< \varphi(a_1, a_2, a_3, a_4), (a_1', a_2', a_3', a_4') >=< (a_1, a_2, a_3, a_4), \varphi(a_1', a_2', a_3', a_4') >$$

for each vector field (a_1, a_2, a_3, a_4) and (a_1', a_2', a_3', a_4') in $\mathbb{E}^4$, here $<, >$ is used for standard metric on $\mathbb{E}^4$. This shows that $(\mathbb{E}^4, <, >, \varphi)$ is a Golden Riemannian manifold. Further, assume any submanifold M in $\mathbb{E}^4$ satisfying

$$t(z_1, z_2) = ((1 - \psi)z_1, p\psi z_2, (1 - \psi)z_1, p\psi z_2)$$

for $p \neq 0, 1$. Now, we see $E_1 = (1 - \psi, p\psi, 0, 0)$, $E_2 = (0, 0, 1 - \psi, p\psi)$ and $\varphi E_1 = (-1, -p, 0, 0)$, $\varphi E_2 = (0, 0, -1, -p)$ obtaining

$$< \varphi E_1, E_1 >=< \varphi E_2, E_2 >= (-p^2 + 1)\psi - 1 \quad \text{and} \quad < \varphi E_1, E_2 >= 0.$$

When we denote the slant angle of M by θ, its value is given by $cos^{-1}\Big(\frac{-1+\psi-p^2\psi}{\sqrt{p^2+1}}\Big)$ and M becomes a slant submanifold.

Let us identify by M_p a real-space form having sectional curvature equals to a constant c_p and by M_q another real-space form having sectional curvature equals to constant c_q. Hence for a locally Golden product space form $\tilde{M}(= M_p(c_p) \times M_q(c_q), g, \varphi)$, one writes [6]:

$$
\begin{aligned}
R(Y_1, Y_2)Y_3 = {} & \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2] \\
& + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10}[g(\varphi Y_2, Y_3)Y_1 - g(\varphi Y_1, Y_3)Y_2 + g(Y_2, Y_3)\varphi Y_1 - g(Y_1, Y_3)\varphi Y_2] \\
& + \frac{c_p + c_q}{5}[g(\varphi Y_2, Y_3)\varphi Y_1 - g(\varphi Y_1, Y_3)\varphi Y_2]. \qquad (5)
\end{aligned}
$$

4. Chen-type Inequalities on Golden Riemannian manifolds

Now, we establish the following δ-invariant inequalities.

Theorem 4.1. *Any proper θ-slant submanifold M^n isometrically immersed in locally Golden product manifold $\tilde{M}^m$ holds following inequality*

$$
\begin{aligned}
\delta_M(p) \le {} & \frac{(n-2)}{2}\Big[\frac{n^2}{(n-1)}\|H\|^2 + \frac{1}{10}(c_p + c_q)\{3(n+1) - 2Trace(\varphi)\}\Big] \\
& + \frac{1}{10}(c_p + c_q)\Big[(Trace(T) + (4-n))cos^2\theta - Trace^2(\varphi)\Big] \\
& + \frac{1}{4\sqrt{5}}(c_p - c_q)(n-2)\Big[2\mathrm{Trace}(\varphi) - (\mathrm{n}+1)\Big], \quad \mathrm{p} \in \mathrm{M}. \qquad (6)
\end{aligned}
$$

Proof. Thanks to (1), we get

$$
\begin{aligned}
R(Y_1, Y_2, Y_3, Y_4) = {} & \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[g(Y_2, Y_3)g(Y_1, Y_4) - g(Y_1, Y_3)g(Y_2, Y_4)] \\
& + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10}[g(\varphi Y_2, Y_3)g(Y_1, Y_4) - g(\varphi Y_1, Y_3)g(Y_2, Y_4) \\
& + g(Y_2, Y_3)g(\varphi Y_1, Y_4) - g(Y_1, Y_3)g(\varphi Y_2, Y_4)] \qquad (7) \\
& + \frac{c_p + c_q}{5}[g(\varphi Y_2, Y_3)g(\varphi Y_1, Y_4) - g(\varphi Y_1, Y_3)g(\varphi Y_2, Y_4)] \\
& + g(\sigma(Y_1, Y_4), \sigma(Y_2, Y_3)) - g(\sigma(Y_1, Y_3), \sigma(Y_2, Y_4)),
\end{aligned}
$$

$\forall Y_i \in \Gamma(TM), i = 1, 2, 3, 4$. Consider $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ and let $\pi = \mathrm{Span}\{e_1, e_2\}$ and for any $p \in M$, e_{n+1} is parallel to $H(p)$. Then, in the light of (1), we can obtain the scalar curvature τ as follows

$$
\begin{aligned}
2\tau(p) = {} & \frac{1}{4}(c_p + c_q)\frac{n(n-1)}{5}\Big\{6 - \frac{4}{n}\mathrm{Trace}(\varphi) + \frac{4}{\mathrm{n(n-1)}}[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T) + n})\cos^2\theta]\Big\} \\
& + \frac{1}{4}(c_p - c_q)\frac{(n-1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi) - 2\mathrm{n}) + n^2\|H\|^2 - \|\sigma\|^2, \qquad (8)
\end{aligned}
$$

where we have used Lemma 3.2. Taking

$$
\begin{aligned}
\varepsilon = {} & 2\tau(p) - \|H\|^2\frac{1}{n-1}(n^3 - 2n^2) - \frac{1}{20}(c_p + c_q)(n^2 - n)\Big\{6 - \frac{4}{n}\mathrm{Trace}(\varphi) \\
& + 4[(\mathrm{Trace}(\varphi))^2 + \mathrm{n} - (\mathrm{Trace(T)})\cos^2\theta]\frac{1}{(\mathrm{n}^2 - \mathrm{n})}\Big\} - \frac{1}{4}(\mathrm{c_p} - \mathrm{c_q})\frac{(\mathrm{n}-1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi) - 2\mathrm{n}), \qquad (9)
\end{aligned}
$$

then, equations (8) and (9) will result

$$\varepsilon + \|\sigma\|^2 = \frac{n^2\|H\|^2}{n-1}. \tag{10}$$

This can be simplified to

$$\Big(\sum_{j=1}^{n} \sigma_{jj}^{n+1}\Big)^2 = (n-1)\Big\{\varepsilon + \sum_{j=1}^{n}(\sigma_{jj}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{s})^2\Big\}. \tag{11}$$

Taking

$$a_1 = \sigma_{11}^{n+1}, a_2 = \sigma_{22}^{n+1}, \dots, a_n = \sigma_{nn}^{n+1}, \;\; b = \varepsilon + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{s})^2,$$

we get

$$\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geq \frac{1}{2}[\varepsilon + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{s})^2], \tag{12}$$

in above calculations Lemma 2.1 has been applied. Also, in the light of (1) and (5), we have

$$\begin{aligned} K(\pi) &= \frac{1}{20}(c_p + c_q)\Big\{6 - 2\mathrm{Trace}(\varphi) + 4[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T)} + 2)\cos^2\theta]\Big\} \\ &+ \frac{1}{4\sqrt{5}}(c_p - c_q)\,(2\mathrm{Trace}(\varphi) - 2) + \sum_{s=n+1}^{m}[\sigma_{11}^{s}\sigma_{22}^{s} - (\sigma_{12}^{s})^2]. \end{aligned} \tag{13}$$

Hence, in view of (12) and (13), we obtain

$$\begin{aligned} K(\pi) &\geq \frac{1}{20}(c_p + c_q)\Big\{6 - 2\mathrm{Trace}(\varphi) + 4[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T)} + 2)\cos^2\theta]\Big\} + \frac{1}{2}\sum_{\mathrm{i}\neq\mathrm{j}}(\sigma_{\mathrm{ij}}^{\mathrm{n+1}})^2 + \sum_{\mathrm{s=n+2}}^{\mathrm{m}}\sigma_{11}^{\mathrm{s}}\sigma_{22}^{\mathrm{s}} \\ &- \sum_{s=n+1}^{m}(\sigma_{12}^{s})^2 + \frac{1}{2}\sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma_{ij}^{s})^2 + \frac{1}{2}\varepsilon + \frac{1}{4\sqrt{5}}(c_p - c_q)\,(2\mathrm{Trace}(\varphi) - 2) \\ &= \frac{1}{20}(c_p + c_q)\Big\{6 - 2\mathrm{Trace}(\varphi) + 4[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T)} + 2)\cos^2\theta]\Big\} + \frac{1}{4\sqrt{5}}(\mathrm{c_p} - \mathrm{c_q})\,(2\mathrm{Trace}(\varphi) - 2) \\ &+ \frac{1}{2}\varepsilon + \frac{1}{2}\sum_{i\neq j>2}(\sigma_{ij}^{n+1})^2 + \frac{1}{2}\sum_{s=n+2}^{m}\sum_{i,j>2}(\sigma_{ij}^{s})^2 + \sum_{s=n+1}^{m}\sum_{i>2}[(\sigma_{1i}^{s})^2 + (\sigma_{2i}^{s})^2] + \frac{1}{2}\sum_{s=n+2}^{m}(\sigma_{11}^{s} + \sigma_{22}^{s})^2, \end{aligned} \tag{14}$$

i.e., we have

$$\begin{aligned} K(\pi) &\geq \frac{1}{20}(c_p + c_q)\Big\{6 - 2\mathrm{Trace}(\varphi) + 4[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T)} + 2)\cos^2\theta]\Big\} \\ &+ \frac{1}{4\sqrt{5}}(c_p - c_q)\,(2\mathrm{Trace}(\varphi) - 2) + \frac{1}{2}\varepsilon. \end{aligned} \tag{15}$$

whereby proving the required result. □

For the equality case, we write.

Theorem 4.2. *When all considerations for above Theorem 4.1 hold, equality is satisfied in (6) at $p \in M$ if and only if for $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, A has following form:*

$$A_{n+1} = \begin{pmatrix} c & 0 & 0 & \ldots & 0 \\ 0 & d & 0 & \ldots & 0 \\ 0 & 0 & c+d & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & c+d \end{pmatrix}, \quad A_s = \begin{pmatrix} c_s & d_s & 0 & \ldots & 0 \\ d_s & -c_s & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad n+2 \le s \le m. \tag{16}$$

Proof. The equality is satisfied in (6) if and only if equality holds in each and every previous inequality and in Lemma 2.1:

$$\begin{aligned} \sigma_{ij}^{n+1} &= 0, i \neq j > 2, \\ \sigma_{1i}^{s} = \sigma_{2i}^{s} = \sigma_{ij}^{s} &= 0, s \ge n+2, i, j > 2, \\ \sigma_{1i}^{n+1} = \sigma_{2i}^{n+1} &= 0, i > 2, \\ \sigma_{11}^{s} + \sigma_{22}^{s} &= 0, s \ge n+2, \\ \sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} &= \cdots = \sigma_{nn}^{n+1}. \end{aligned}$$

Finally, the shape operators $A_s, s \in \{n+1, \ldots, m\}$ appear to be like in (16) as one can opt $\{e_1, e_2\}$ fulfilling $\sigma_{12}^{n+1} = 0$. □

Next, we derive inequality involving $\delta(n_1, \ldots, n_\mu)$.

Theorem 4.3. *For proper θ-slant submanifold M^n immersed in $\tilde{M}$, the following inequality holds for any μ-tuples $(n_1, \ldots, n_\mu) \in S(n)$,*

$$\begin{aligned} \delta(n_1, \ldots, n_\mu) \le T_3 &- \frac{1}{10}(c_p + c_q)\{cos^2\theta + Trace(\varphi)\}(n - \sum_{j=1}^{k} n_j) \\ &- \frac{1}{4\sqrt{5}}(c_p - c_q)\{(n + \sum_{j=1}^{k} n_j) - 2Trace(\varphi) - 1\}(n - \sum_{j=1}^{k} n_j). \end{aligned} \tag{17}$$

Here, $T_3 = d(n_1, \ldots, n_\mu)\|H\|^2 + \frac{3}{10}(c_p + c_q)b(n_1, \ldots, n_k)$. Additionally, equality in (17) at $p \in M \iff \exists$ $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ and A appear as follows:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}, \quad A_s = \begin{pmatrix} B_1^s & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & B_\mu^s & 0 & \ldots & 0 \\ 0 & \ldots & 0 & c_s & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & c_s \end{pmatrix}, \qquad s \in \{n+2, \ldots, m\}, \tag{18}$$

and $a_1, \ldots, a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1 + \ldots n_{\mu-1}+1} + \cdots + a_{n_1+\ldots n_\mu} = a_{n_1+\ldots n_\mu+1} = \cdots = a_n$$

where B_i^s is a symmetric $n_i \times n_i$ submatrix that satisfies

$$\mathrm{Trace}(\mathrm{B}_1^{\mathrm{s}}) = \cdots = \mathrm{Trace}(\mathrm{B}_\mu^{\mathrm{s}}) = \mathrm{c_s}.$$

Proof. For $p \in M$, fix e_{n+1} parallel to $H(p)$. Also, opt μ mutually orthogonal subspaces of T_pM represented by $L_1, \dots, L_\mu$ and assume $\dim L_i = n_i, \forall i \in \{1, \dots, \mu\}$ so that

$$L_1 = \mathrm{Span}\{e_1, \dots, e_{n+1}\}, \quad L_2 = \mathrm{Span}\{e_{n_1+1}, \dots, e_{n_1+n_2}\}, \dots, \quad L_\mu = \mathrm{Span}\{e_{n_1+\cdots+n_{\mu-1}+1}, \dots, e_{n_1+\cdots+n_\mu}\}.$$

Then, in view of Gauss equation, we obtain

$$\tau(L_i) = \frac{1}{40}(c_p + c_q)n_i(n_i - 1)\Big\{6 - \frac{4}{n_i}\mathrm{Trace}(\varphi) + \frac{4}{\mathrm{n_i(n_i - 1)}}[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T) + n_i})\cos^2\theta]\Big\}$$
$$+ \frac{1}{8}(c_p - c_q)\frac{(n_i - 1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi) - 2\mathrm{n_i}) + \sum_{s=n+1}^{m}\sum_{\alpha_i<\beta_i}[\sigma^s_{\alpha_i\alpha_i}\sigma^s_{\beta_i\beta_i} - (\sigma_{\alpha_i\beta_i})^2]. \tag{19}$$

Let us put

$$\eta = 2\tau(p) - 2d(n_1, \dots, n_\mu)\|H\|^2 - \frac{1}{4}(c_p + c_q)\frac{n(n-1)}{5}\Big\{6 - \frac{4}{n}\mathrm{Trace}(\varphi) + \frac{4}{\mathrm{n(n-1)}}[(\mathrm{Trace}(\varphi))^2$$
$$- (\mathrm{Trace(T) + n})\cos^2\theta]\Big\} - \frac{1}{4}(\mathrm{c_p - c_q})\frac{(\mathrm{n} - 1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi) - 2\mathrm{n}) \tag{20}$$

and

$$\vartheta = n + \mu - \sum_{i=1}^{\mu} n_i. \tag{21}$$

Then, we have

$$\eta + \|\sigma\|^2 = \frac{n^2\|H\|^2}{\vartheta}, \tag{22}$$

and hence, one obtains

$$\Big(\sum_{j=1}^{n}\sigma^{n+1}_{jj}\Big)^2 = \vartheta\Big\{\eta + \sum_{j=1}^{n}(\sigma^{n+1}_{jj})^2 + \sum_{i\neq j}(\sigma^{n+1}_{ij})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma^s_{ij})^2\Big\}, \tag{23}$$

that reduces to

$$\Big(\sum_{j=1}^{\vartheta+1} b_j\Big)^2 = \vartheta\Big\{\eta + \sum_{j=1}^{\vartheta+1}(b_j)^2 + \sum_{i\neq j}(\sigma^{n+1}_{ij})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma^s_{ij})^2 - 2\sum_{i=1}^{\mu}\sum_{\alpha_i<\beta_i} a_{\alpha_i}a_{\beta_i}\Big\}, \tag{24}$$

where

$a_j = \sigma^{n+1}_{jj}, \forall j \in \{1, \dots, n\}$,
$b_1 = a_1, b_2 = a_2 + \cdots + a_{n_1}, b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2,\dots,}\ \ b_{\mu+1} = a_{n_1+\cdots+n_{\mu-1}+1} + \cdots + a_{n_1+n_2\cdots+n_\mu}$,
$b_{\mu+2} = a_{n_1+\cdots+n_\mu+1}, \dots, b_{\vartheta+1} = a_n$.

Hence, we have

$$\sum_{i=1}^{\mu}\sum_{\alpha_i<\beta_i} a_{\alpha_i}a_{\beta_i} \geq \frac{1}{2}[\eta + \sum_{i\neq j}(\sigma^{n+1}_{ij})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\sigma^s_{ij})^2], \tag{25}$$

in above discussions, Lemma 2.1 was used.

Further, suppose that $e_1, \dots, e_\mu, e$ be the sets

$$e_1 = \{1, \dots, n_1\}, e_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \dots, \quad e_\mu = \{n_1 + \cdots + n_{\mu-1} + 1, \dots, n_1 + \cdots + n_\mu\},$$
$$e^2 = (e_1 \times e_1) \cup \cdots \cup (e_\mu \times e_\mu),$$

so, we arrive at

$$\sum_{i=1}^{\mu} \sum_{s=n+1}^{m} \sum_{\alpha_i<\beta_i} [\sigma^s_{\alpha_i\alpha_i}\sigma^s_{\beta_i\beta_i} - (\sigma^s_{\alpha_i\beta_i})^2] \geq \frac{1}{2}\eta + \frac{1}{2}\sum_{s=n+1}^{m} \sum_{(\alpha,\beta)\notin e^2} (\sigma^s_{\alpha\beta})^2 + \sum_{s=n+2}^{m} \sum_{\alpha_i\in e_i} (\sigma^s_{\alpha_i\alpha_i})^2, \tag{26}$$

that produces

$$\sum_{i=1}^{\mu} \sum_{s=n+1}^{m} \sum_{\alpha_i<\beta_i} [\sigma^s_{\alpha_i\alpha_i}\sigma^s_{\beta_i\beta_i} - (\sigma^s_{\alpha_i\beta_i})^2] \geq \frac{1}{2}\eta, \tag{27}$$

and hence in view of (19), we obtain

$$\tau(L_i) \geq \sum_{i=1}^{\mu} \frac{1}{8}(c_p + c_q)\frac{n_i(n_i - 1)}{5}\Big\{6 - \frac{4}{n_i}\mathrm{Trace}(\varphi) + \frac{4}{\mathrm{n_i(n_i - 1)}}[(\mathrm{Trace}(\varphi))^2 - (\mathrm{Trace(T) + n_i})\cos^2\theta]\Big\}$$
$$+ \sum_{i=1}^{\mu} \frac{1}{8}(c_p - c_q)\frac{(n_i - 1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi) - 2\mathrm{n_i}) + \frac{1}{2}\eta. \tag{28}$$

Finally, taking into account (20) and (28) we have the required inequality. Additionally, (17) at $p \in M$ is valid for equality if and only if there exists equality sign in each and every previous inequality and in Lemma 2.1. Further, the shape operators $A_s, s \in \{n + 1, \dots, m\}$ reduce to be like in (18). □

As a special case of Theorems 4.1 and 4.3, we write.

Corollary 4.4. *For φ-invariant submanifold M^n immersed in $\tilde{M}$, the following inequality holds*

$$\delta_M(p) \leq \frac{(n-2)}{2}\Big[\frac{n^2}{(n-1)}\|H\|^2 + \frac{1}{10}(c_p + c_q)\{3(n + 1) - 2Trace(\varphi)\}\Big] + \frac{1}{10}(c_p + c_q)\Big[(Trace(T) + (4 - n))$$
$$- Trace^2(\varphi)\Big] + \frac{1}{4\sqrt{5}}(c_p - c_q)(n - 2)\Big[2\mathrm{Trace}(\varphi) - (\mathrm{n} + 1)\Big], \quad \mathrm{p \in M}. \tag{29}$$

Additionally, (29) holds for equality at $p \in M \Longleftrightarrow$ for orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$, A can be written like (16).

Corollary 4.5. *For φ-anti-invariant submanifold M^n immersed in $\tilde{M}$, the following inequality holds*

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{20}(n - 2)(c_p + c_q)\Big[3(n + 1) - 2Trace(\varphi) - \frac{2}{(n-2)}Trace^2(\varphi)\Big]$$
$$+ \frac{1}{4\sqrt{5}}(c_p - c_q)(n - 2)\Big[2\mathrm{Trace}(\varphi) - (\mathrm{n} + 1)\Big], \quad \mathrm{p \in M}. \tag{30}$$

Additionally, equality in (30) $\Longleftrightarrow \exists \{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ and A appear to be like (16).

Next, we have

Corollary 4.6. *For φ-invariant submanifold M^n immersed in $\tilde{M}$, the following inequality holds for any μ-tuples $(n_1,\ldots,n_\mu)\in S(n)$,*

$$\delta(n_1,\ldots,n_\mu)\leq T_3-\frac{1}{10}(c_p+c_q)\{1+Trace(\varphi)\}(n-\sum_{j=1}^{k}n_j)$$
$$-\frac{1}{4\sqrt{5}}(c_p-c_q)\{(n+\sum_{j=1}^{k}n_j)-2Trace(\varphi)-1\}(n-\sum_{j=1}^{k}n_j). \tag{31}$$

Additionally, if for some orthonormal frame $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_m\}$, A appear like (18) $\Longleftrightarrow$ equality holds in (31) at $p\in M$.

Corollary 4.7. *For φ-anti-invariant submanifold M^n immersed in $\tilde{M}$, the following inequality holds for any μ-tuples $(n_1,\ldots,n_\mu)\in S(n)$,*

$$\delta(n_1,\ldots,n_\mu)\leq T_3-\frac{1}{10}(c_p+c_q)(n-\sum_{j=1}^{k}n_j)Trace(\varphi)$$
$$-\frac{1}{4\sqrt{5}}(c_p-c_q)\{(n+\sum_{j=1}^{k}n_j)-2Trace(\varphi)-1\}(n-\sum_{j=1}^{k}n_j). \tag{32}$$

Additionally, equality in (32) $\Longleftrightarrow$ for $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_m\}$, A appear to be like (18).

5. Inequalities for Ricci curvature tensor

Consider proper θ-slant submanifold M^n immersed in $\tilde{M}^m$, we fix unit tangent vector $Y\in T_tM, \forall t\in M$ and on M, identify local orthonormal frame with the help of $\{e_1,\ldots,e_n\}$ so that $e_1=Y$. Taking into use (8), we get

$$2\tau(t)=\frac{1}{4}(c_p+c_q)\frac{n(n-1)}{5}\Big\{6-\frac{4}{n}\mathrm{Trace}(\varphi)+\frac{4}{\mathrm{n(n-1)}}[(\mathrm{Trace}(\varphi))^2-(\mathrm{Trace(T)+n})\cos^2\theta]\Big\}$$
$$+\frac{1}{4}(c_p-c_q)\frac{(n-1)}{\sqrt{5}}(4\mathrm{Trace}(\varphi)-2\mathrm{n})+n^2\|H\|^2-\frac{1}{2}\sum_{s=n+1}^{m}[\sum_{j=1}^{n}(\sigma_{jj}^s)^2+(\sigma_{11}^s-\sum_{j=2}^{n}\sigma_{jj}^s)^2]$$
$$-2\sum_{s=n+1}^{m}[\sum_{i<j}(\sigma_{ij}^s)^2-\sum_{2\leq j<i\leq n}\sigma_{jj}^s\sigma_{ii}^s], \tag{33}$$

where we have taken help of lemma 3.1 and lemma 3.2.

Also, the Gauss equation produces

$$\sum_{2\leq j<i\leq n}K(e_j\wedge e_i)=\frac{1}{8}(c_p+c_q)\frac{(n-1)(n-2)}{5}\Big\{6-\frac{4}{n-1}\mathrm{Trace}(\varphi)+\frac{4}{\mathrm{(n-1)(n-2)}}[(\mathrm{Trace}(\varphi))^2$$
$$-(\mathrm{Trace(T)+n-1})\cos^2\theta]\Big\}+\frac{1}{8}(\mathrm{c_p-c_q})\frac{\mathrm{(n-2)}}{\sqrt{5}}(4\mathrm{Trace}(\varphi)-2(\mathrm{n}-1))$$
$$+\sum_{s=n+1}^{m}\sum_{2\leq i<j\leq n}[\sigma_{jj}^s\sigma_{ii}^s-(\sigma_{ij}^s)^2]. \tag{34}$$

(33), (34) deliver the following

$$\frac{1}{2}n^2\|H\|^2 \geq 2\tau(t) + 2\sum_{sn+1}^{m}\sum_{i=2}^{n}(\sigma_{1i}^s)^2 - 2\sum_{2\leq j<i\leq n} K(e_j \wedge e_i) + \frac{1}{5}(c_p + c_q)[\mathrm{Trace}(\varphi) - 3(\mathrm{n}-1) + \cos^2\theta]$$
$$+ \frac{1}{\sqrt{5}}(c_p - c_q)[(n-1) - \mathrm{Trace}(\varphi)], \tag{35}$$

whereby proving

$$Ric(Y) \leq \frac{n^2}{4}\|H\|^2 - T_1 - \frac{1}{10}(c_p + c_q)cos^2\theta, \tag{36}$$

where

$$T_1 = \frac{1}{2\sqrt{5}}(c_p - c_q)[(n-1) - \mathrm{Trace}(\varphi)] + \frac{1}{10}(\mathrm{c_p} + \mathrm{c_q})\{\mathrm{Trace}(\varphi) - 3(\mathrm{n}-1)\}.$$

Moreover, with $H(t) = 0$, equality in (36)

$$\iff \sigma_{1i}^s = 0, \quad i \in \{2, \ldots, n\}, \quad \sigma_{11}^s = \sum_{j=2}^{n}\sigma_{jj}^s, \quad s \in \{n+1, \ldots, m\} \tag{37}$$

implying that Y is a member of relative null space L_t. Additionally, equality in (36)

$$\iff \sigma_{ij}^s = 0, \quad n+1 \leq s \leq m, i \neq j, \quad \sum_{j=1}^{n}\sigma_{jj}^s = 2\sigma_{ii}^s, \quad s \in \{n+1, \ldots, m\}, i \in \{1, \ldots, n\}. \tag{38}$$

Concluding point t to be

- totally geodesic provided $n \neq 2$
- totally umbilical if $n = 2$.

One can observe that the converse part is obvious.

Hence, one may summarize it as

Theorem 5.1. *For any proper θ-slant submanifold of M^n of $\tilde{M}^m$ $(= M_p(c_p) \times M_q(c_q), g, \varphi)$,*

$$Ric(Y) \leq \frac{n^2}{4}\|H\|^2 - T_1 - \frac{1}{10}(c_p + c_q)cos^2\theta, \tag{39}$$

here Y is used for unit tangent vector on M.

In view of Theorem 5.1, we get

Corollary 5.2. *When M^n represents φ-invariant submanifold of $\tilde{M}^m$, we get*

$$Ric(Y) \leq \frac{n^2}{4}\|H\|^2 - T_1 - \frac{1}{10}(c_p + c_q), \tag{40}$$

where $Y \in T_tM, \forall t \in M$ is a unit vector.

Corollary 5.3. *The φ-anti-invariant submanifold M^n isometrically immersed in $\tilde{M}^m$ has relation:*

$$Ric(Y) \leq \frac{n^2}{4}\|H\|^2 - T_1. \tag{41}$$

In this case, $Y \in T_tM, t \in M$ represents a unit vector.

Remark 5.4. When $H(t) = 0$, (39),(40) and (41) hold for equality if and only if Y is a member of relative null space L_t. Moreover, (39),(40) and (41) satisfy equality $\Longleftrightarrow$ t be totally geodesic point in M or $n = 2$ and with totally umbilical point t.

Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G-1284-130-1440. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

We express our sincere thanks to the editor and the anonymous referees for their valuable suggestions and comments that help greatly to improve the article.

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