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Line graphics for visualization of surfaces and curves on them

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Abstract. In this paper, we describe the main principles of our approach for the visualization of surfaces and curves on them in three-dimensional space by the use of *line graphics*. We also compare our approach with the standard method of polygon mesh. Mainly, we discuss the representation of surfaces, special lines on surfaces and lines of intersections of surfaces. Furthermore, we address the problems of visibility and finding the contour lines of surfaces. Finally we apply our method for visualizing some selected topic in mathematics.

1. Introduction

Mathematics is a very abstract science with strictly defined objects and concepts. When defining a new object, mathematicians are satisfied with the statement that the object exists, but usually do not try to visualize it. But, we know from experience that even the simplest sketch contributes to a better understanding of the object being observed.

Apart from a small number of classical mathematical objects, curves and surfaces, the general scientific community does not have a good idea of what most mathematical objects look like. Visualization greatly helps to understand the objects being explored.

Although a strict and precise definition is necessary, acquiring a visual representation of an unknown object makes it easier to become familiar with new concepts. This is as important in education as it is in scientific research. Modeling and visualizing a newly defined object can initiate and facilitate further research and in some cases even indicate its direction.

Mathematical objects are usually defined in a most abstract way, most often in n-dimensional space, and the special cases for particular values of n are considered later. Unlike in natural and technical sciences, where dimensions have certain meanings (time, temperature, sound, color, and so on), in mathematics the dimensions usually have the same meanings, and spatial dimensions are usually implied.

However, for visualization purposes, we generally use only two- and three-dimensional space. One-dimensional space is too limited to be able to display more complicated images. We have no practical experience in four-dimensional space (unless we assign a physical characteristic other than spatial for a dimension). Even the simplest images in four-dimensional space are not intelligible enough, so they cannot be used for better understanding unknown objects.

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We are used to two-dimensional images, sketching on paper or looking at a computer or smartphone screen. The two-dimensional space provides enough opportunity to display the relationships of the elements being displayed.

Three-dimensional objects can be plastically modeled in reality, but are much more often projected on a two-dimensional device. Considering our everyday real-life experience, we have no problem to reconstruct a two-dimensional image into a three-dimensional object.

In $2D$ space, we visualize points, curves, or shapes that are bordered by curves. In $3D$ space, we visualize points, curves, surfaces, or solids that are bordered by surfaces. The representation of a solid body can be achieved by representing its boundary surface, so in any case, we consider graphical representation of surfaces. Although a lot can be shown in two-dimensional space, a much more interesting case is the three-dimensional space.

2. Visualizing 3*D* Mathematical Objects

There is no problem for displaying points and lines, neither in $2D$, nor in $3D$ space. Curved lines are approximated by a series of straight line segments, the length of which can be arbitrary. Thus, the curved lines are displayed with the desired precision.

Nowadays, the generally accepted approach to surface modeling is by a *polygon mesh*, that is, a surface is approximated by the boundary of a polyhedron. This approach has a number of advantages, from a simple model, through fast rendering, to uniform handling of different types of surfaces.

The vertices of a polygon mesh are evaluated accurately and are indeed located on the surface that is modeled. The edges are straight line segments that connect two vertices and three or more connected edges form the face of a polygon mesh. As soon as a surface model is made, we do not work any more with the surface itself. All subsequent transformations are done only with the model, not with the surface itself.

If the surface that is modeled is not a plane, the edges and faces of the polygon mesh do not coincide to the appropriate part of the surface. The approximate part of the surface is distant from the surface itself.

A problem arises when two or more objects are in the same scene. If two surfaces modeled by their polygon meshes intersect, the line of intersection cannot be accurately represented. Since information about the original surfaces is not available in the model, we intersect the boundaries of the approximating polyhedra. The line of intersection consists of a series of straight lines segments that lie on the appropriate faces. The line looks broken and depends on the triangulation of the polygon meshes (Figure 1).

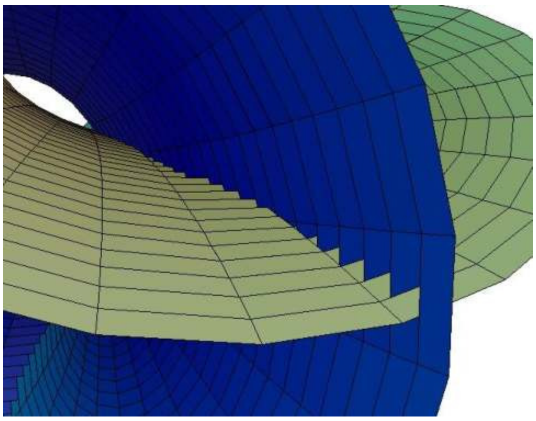

Figure 1: Intersection of surfaces in polygon mesh

Indeed, this problem occurs whenever we try to display an arbitrary line that should be on a surface approximated by a polygon mesh. Although we can accurately represent the line, the problem is its positioning on the polygon mesh. If we display the line correctly, it does not lies on the polygon mesh. If it lies on the polygon mesh, the line is not smooth. The second approach is usually used (Figure 2).

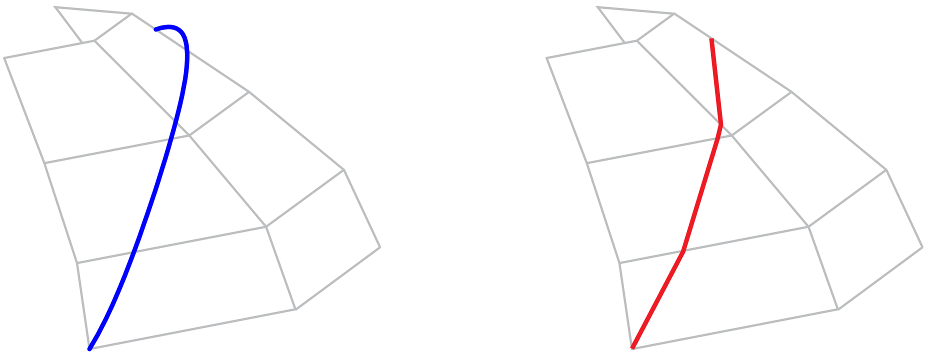

Figure 2: Left: The curve is not on the surface. Right: The curve is not smooth.

Another popular approach to visualize surfaces and solids should also be mentioned, namely *ray tracing*. It produces photo-realistic images, but requires knowledge of optics and many more mathematical calculations, including the intersection of a line and a body and the normal vector of its boundary surface.

In this paper, we describe another approach to displaying surfaces, namely *line graphics*.

3. Line Graphics

In line graphics, a surface is represented by families of lines, typically by its parameter lines (Figure 3). The lines are displayed with high precision (usually less than the display unit) since there is no polyhedron which they should belong to. In this way we get the impression of a smooth surface.

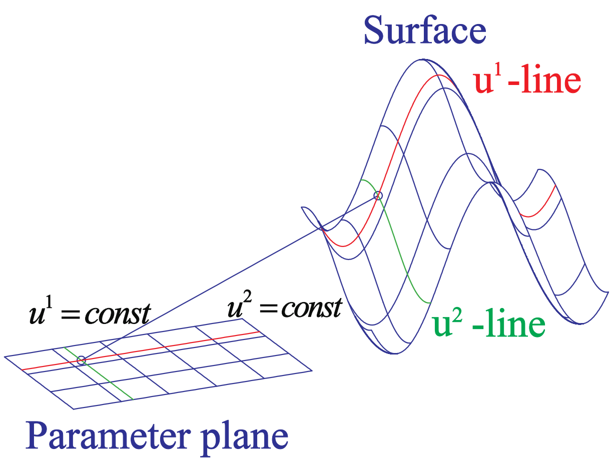

Figure 3: Parameter lines on a surface

Line graphics is of great advantage since it enables us to visualize special curves of interest on surfaces or lines of intersections of surfaces without having to develop a special strategy. For example, Figure 4 on the left side shows the line of intersection of a torus and a catenoid, and on the right side the line of intersection of Enneper's surface and a plane. Also, a surface can have lines of self-intersection. These lines for Enneper's surface are shown in Figure 5, right.

The resulting images look clear, without unnecessary details. We believe that, apart of a scientific value, they also have an artistic one. Some aesthetic aspects are discussed in the books [2, 16, 19] and the papers [7, 15, 17, 20–23].

More about line graphics can be found in the book [8].

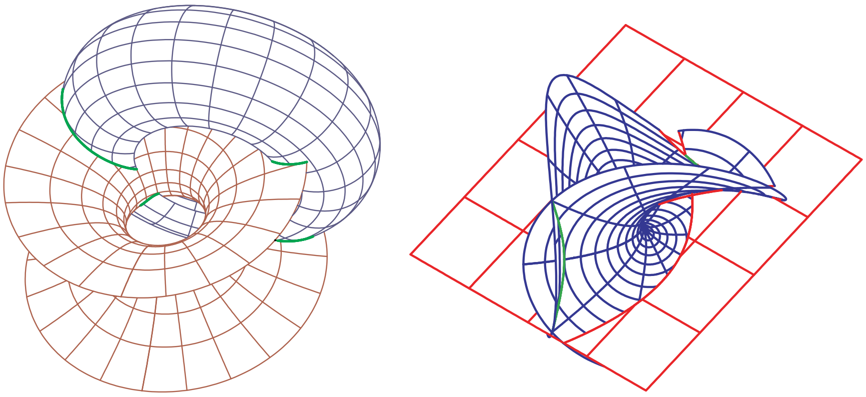

Figure 4: Left: The intersection of a torus and a catenoid. Right: The intersection of Enneper's surface and a plane.

To draw curves, we only need to calculate the coordinates of its points. However, if we do not include the fact that these curves lie on the surface, we get a simple wire model. To get the impression of a surface, the visibility of its points should be included.

3.1. *Mathematical Descriptions of a Surface*

There are several ways to mathematically describe a surface, by an equation, by a description of the process leading to a point on the surface, by a parametric representation, and so on.

If a surface is given by an equation, or equivalently, by an implicit formula of its coordinates, it is necessary to find the zeros of a real function of two real variables in its domain, as in the case of the function (5) in Subsection 3.3. Rendering an image is much faster if we find the zeros analytically. If this is not possible, some of numerical methods can be applied, like the ones described in [4, 5, 8].

If a surface is described by a process, in order to obtain each point of the surface, it is necessary to mathematically describe and follow each step of the process. This procedure is, typically, extremely slow, so it should be applied only in the absence of an alternative method. For example, the paper [13] describes Wulff's crystallization process, and models and visualizes it.

For our line graphics approach it is most convenient to use a parametric representation for visualization of surfaces; so we use it whenever it is possible.

The appearance of the surface depends a lot on the parametrization. Two different parametric representations of the same surface can give different impressions to the observers, to the extent that they may not recognize that it as the same surface. For example, the Enneper's minimum surface (Figure 5) can be given in Cartesian coordinates with

$$\vec{x}(u^i) = \left\{u^1 - \frac{(u^1)^3}{3} + u^1(u^2)^2, u^2 - \frac{(u^2)^3}{3} + u^2(u^1)^2, (u^1)^2 - (u^2)^2\right\}, \tag{1}$$

for $(u^1, u^2) \in \mathbb{R}^2$, or in polar coordinates with

$$\vec{x}(\rho, \phi) = \left\{\rho\cos\phi - \frac{\rho^3}{3}\cos(3\phi), \rho\sin\phi + \frac{\rho^3}{3}\sin(3\phi), \rho^2\cos(2\phi)\right\}, \tag{2}$$

for $(\rho, \phi) \in (0, \infty) \times (0, 2\pi)$.

Some of the papers that investigate the Enneper's surface are [1, 3, 6, 18].

3.2. *Visibility*

Some parts of the surface may hide other parts of the same or other surfaces. To give a proper impression of a surface or a solid body, hidden parts should not be visible. Sometimes surfaces can be transparent,

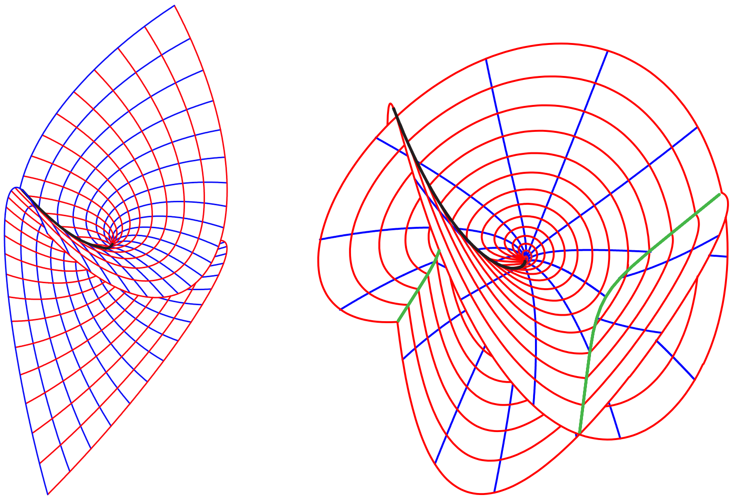

Figure 5: Enneper's surface given by (1) (left) and by (2) (right)

so the hidden objects or hidden parts of the same surface can be seen through them. In line graphics, we have decided not to show the invisible parts of the lines at all, or to dot them, as is usual in mathematical sketches.

In the polygon mesh and ray tracing approaches, the visibility check is included in the surface modeling process itself. On the contrary, we have introduced independent visibility checks in our line graphics visualization. The independence of the check procedure enables us to manipulate visibility to be able to show, if necessary, desirable but geometrically unrealistic effects, or not to use any test at all for a fast first sketch.

To illustrate manipulation of visibility we represent Dandelin's spheres in three different ways. We recall the following definition: if the intersection of a circular cone and a plane is an ellipse or a hyperbola, then, in each case, there are exactly two spheres each of which is tangent to both the plane and to the cone along a line, and if the intersection is a parabola, then there is one and only one sphere with this property. These spheres are called *Dandelin's spheres*. Furthermore, the spheres are tangent to the plane of intersection at the foci of the conic sections.

In Figure 6 we illustrate the above definition and properties, in the elliptic case. We visualize the cone, the intersecting plane, Dandelin's spheres, the line of intersection of the plane and the cone, the lines where Dandelin's spheres are tangent to the cone, and the points F_1 and F_2 where the spheres are tangent to the intersecting plane. In the geometrically realistic representation on the left, lower Dandelin's sphere and the points F_1 and F_2 are not visible at all.

In the middle we illustrate the principle by choosing the perspective such that the intersecting plane appears as a straight line and making the front part of the cone transparent, which is achieved by inverting the boolean value of realistic visibility check.

Finally on the right we manipulate the visibility to clearly show the concept. The front part of the cone is transparent, and the upper Dandelin's sphere is cut open, to clearly see the location of the point F_1. By manipulating visibility of the intersecting plane, we only show its part inside the cone which is visible with respect to the part of the upper sphere.

In line graphics, we test the visibility of a point analytically, immediately after the computation of its coordinates; thus our graphics are generated in a geometrically natural way. For this it is necessary to find the points of intersection of a projection ray and a surface. For some simple surfaces, we know such formulas from elementary mathematics, while for others it can be a demanding task.

In the general case, to determine the visibility of a point P with respect to the surface S we have to find

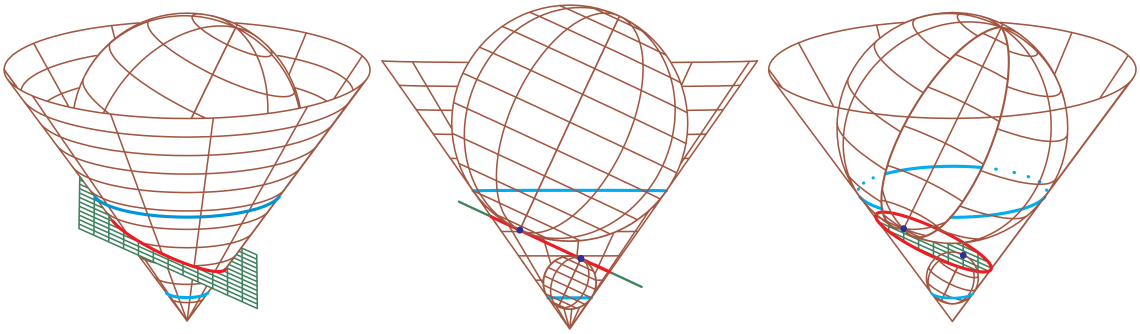

Figure 6: Dandelin's Spheres, reality (left), principle (middle) and concept (right)

the solutions of t, u_1, u_2 of the equations

$$\vec{x}(u_1, u_2) = \vec{p} + t \cdot \vec{PC} \tag{3}$$

where $\vec{x}(u_1, u_2)$ is a parametric representation of the surface S, $\vec{p}$ is the position vector of the point P, and $\vec{PC}$ is a vector in the direction of the projection ray.

Since this problem depends on the geometry of the objects to be drawn, we implemented the methods for the solution of the visibility problems for each class of the surfaces. We also develop the necessary numerical methods to solve the equations in (3).

For example, the points of intersection of Enneper's surface with a straight line were determined in [18, Section 2.1]. The calculations required for the visibility of some generalized tubular surfaces are given in [12], (Figure 7).

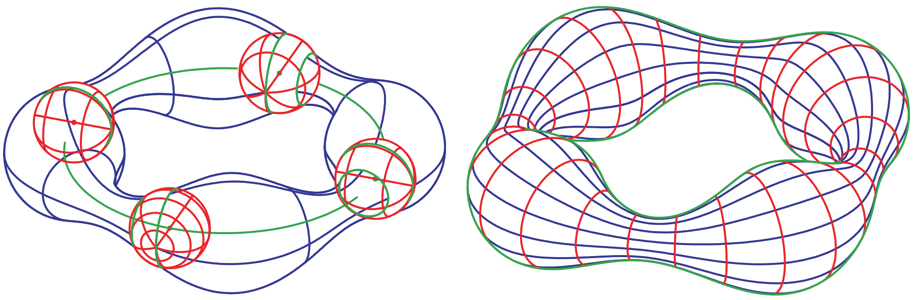

Figure 7: A tubular surface as the envelope of spheres; concept (left), reality (right).

3.3. *Contour line*

The use of line graphics has the effect that surfaces appear unfinished without their so–called contour lines. For a line that is partly visible, there is a point where the visible part of the line becomes invisible. In the picture, it looks as if the line ends at that point. No matter how many lines represent the surface and how precisely they are drawn, a surface without a contour line looks unfinished (Figure 8).

Let S be a surface with a parametric representation $\vec{x} = \vec{x}(u^i)$ and surface normal vectors $\vec{N}(u^i)$, P be a point on S and C be the center of projection. By definition, a point P is a *contour point* of a surface S if and only if the following two conditions are satisfied:

(i) The projection ray to the point P is orthogonal to the surface normal vector $\vec{N}$ at P, hence

$$\overrightarrow{CP} \bullet \vec{N} = 0. \tag{4}$$

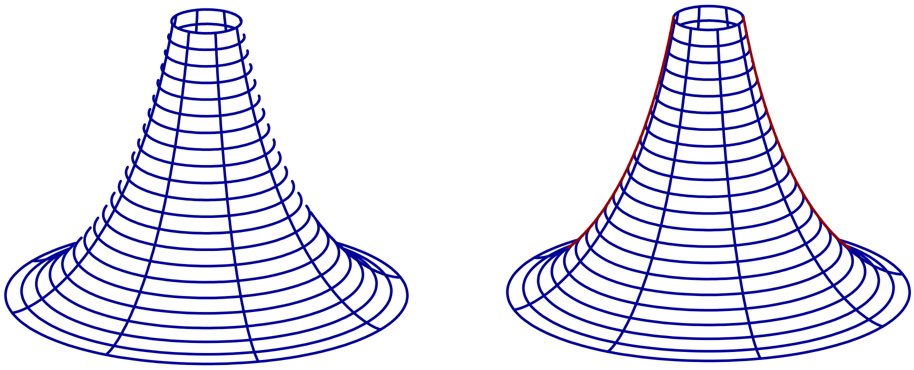

Figure 8: A pseudo–sphere without and with its contour line

(ii) Let E be the plane through P spanned by the vectors $\overrightarrow{CP}$ and $\vec{N}$, and γ be the curve of intersection of S and E. Then there is a neighbourhood of P in which γ runs on one side of the projection ray and has no points of intersection with it other than P (Figure 9).

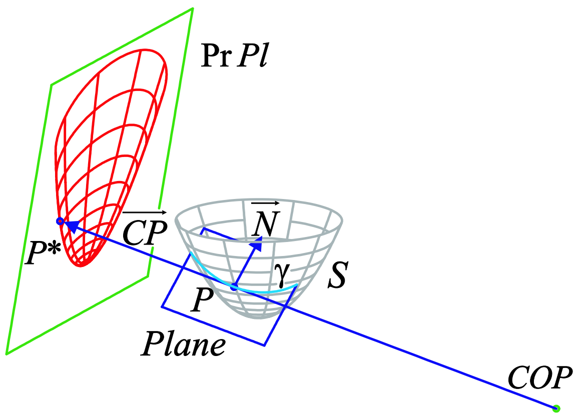

Figure 9: The definition of a contour point of a surface

We only use condition (4) to determine contour points, since checking condition (ii) is very time consuming and would only apply in a few cases which can be avoided much more conveniently by a slight change of the perspective. In Figure 10 we show a point that satisfies condition (i) but not (ii).

The *contour line of a surface* is the set of all its contour points defined by the condition in (4).

Since the contour line of a surface depends on the geometry of the surface, the methods for drawing contour lines have to be developed separately for each type of surface.

Here we outline the idea what is needed to solve contour problem for the Enneper's surface with the parametric representation (2). Since

$$\vec{x}_1(\rho,\phi) = \{\cos\phi - \rho^2\cos(3\phi), \sin\phi + \rho^2\sin(3\phi), 2\rho\cos(2\phi)\},$$
$$\vec{x}_2(\rho,\phi) = \{-\rho\sin\phi + \rho^3\sin(3\phi), \rho\cos\phi + \rho^3\cos(3\phi), -2\rho^2\sin(2\phi)\}$$

and

$$\vec{N}(\rho,\phi) = \frac{\vec{x}_1(\rho,\phi)\times\vec{x}_2(\rho,\phi)}{\|\vec{x}_1(\rho,\phi)\times\vec{x}_2(\rho,\phi)\|} = \frac{1}{1+\rho^2}\{-2\rho\cos\phi, 2\rho\sin\phi, 1-\rho^2\},$$

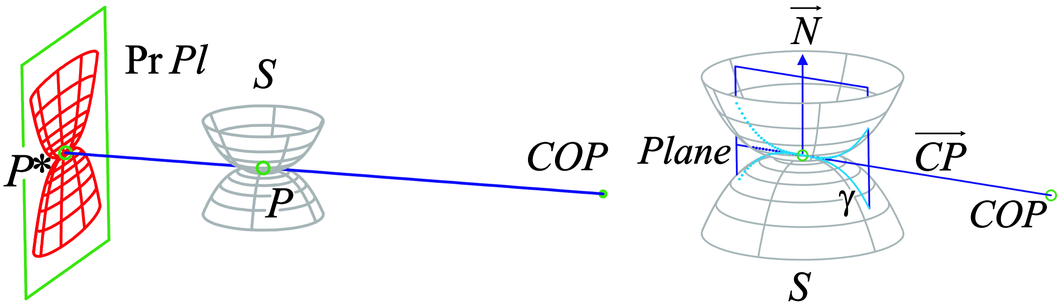

Figure 10: A point that satisfies condition (i) but not (ii)

the condition in (4) is equivalent to $(1+\rho^2)(\vec{N}(\rho,\phi)\bullet\overrightarrow{PC})=0$. We write $\vec{n}=(1+\rho^2)\vec{N}$ and obtain

$$\begin{aligned}\vec{n}\bullet\vec{p} &= \rho\{-2\rho\cos\phi, 2\rho\sin\phi, 1-\rho^2\}\bullet\left\{\cos\phi-\frac{\rho^2}{3}\cos(3\phi), \sin\phi+\frac{\rho^2}{3}\sin(3\phi), \rho\cos(2\phi)\right\}\\ &= \rho\left(-2\rho\cos^2\phi+\frac{2\rho^3}{3}\cos\phi\cos(3\phi)+2\rho\sin^2\phi+\frac{2\rho^3}{3}\sin\phi\sin(3\phi)\ +(1-\rho^2)\rho\cos(2\phi)\right)\\ &= \rho^2\left(-2\cos(2\phi)+\frac{2\rho^2}{3}\cos(2\phi)+(1-\rho^2)\cos(2\phi)\right)=-\rho^2\left(1+\frac{\rho^2}{3}\right)\cos(2\phi).\end{aligned}$$

Thus the contour line of Enneper's minimal surface is given by the zeros of the function

$$\Psi(\rho,\phi)=2\rho\left(c_2\sin\phi-c_1\cos\phi\right)+(1-\rho^2)c_3+\rho^2\left(1+\frac{\rho^2}{3}\right)\cos(2\phi). \tag{5}$$

We use the numerical method described in detail in [5] to find the zeros of the function Ψ in (5).
The contour line of an Enneper surface is shown in Figure 5.

We note that it is necessary to check the visibility of contour points (Figure 11).

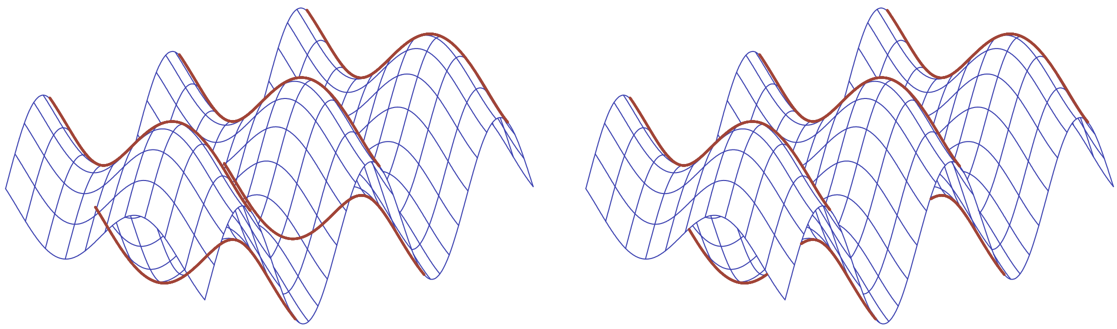

Figure 11: The contour line of a surface without and with visibility check

4. Some Applications

We apply line graphics to visualize the results of our research.
Geometry and differential geometry are natural choices for visualizing mathematical objects.
First we visualize an exponential cone. This is a surface given by a parametric representation

$$\vec{x}(u^i)=(u^1\cos u^2, u^1\sin u^2, (u^1)^\alpha e^{-\beta u^2}),\quad u^1>0, u^2\in(0,2\pi),$$

where α and β are real constants (Figure 12, left).

Next we consider potential surfaces. Potential surfaces are given by a parametric representation

$$\vec{x}(u_1, u_2) = h(u_1, u_2) \cdot \vec{y}(u_1, u_2), \quad (u_1, u_2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi),$$

where

$$\vec{y}(u_1, u_2) = \{\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1\} \quad \text{and} \quad h(u_1, u_2) > 0 \text{ on } R.$$

The representation of a mean curvature of a potential surface as a potential surface is shown in Figure 12, right.

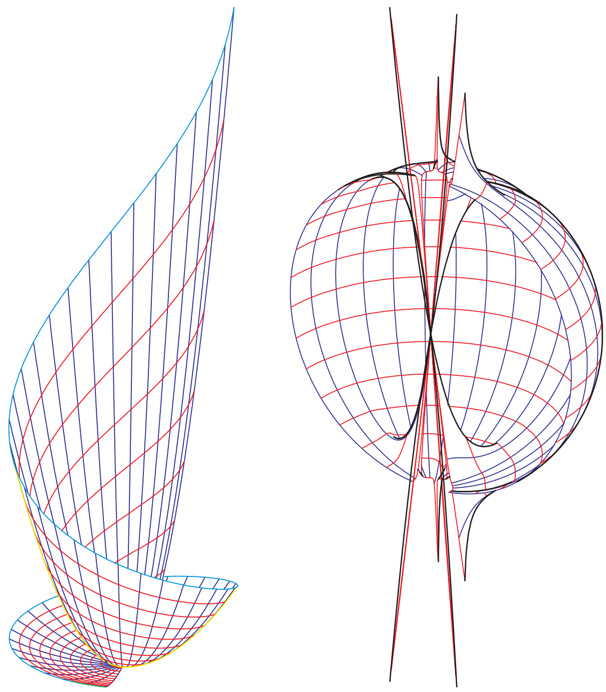

Figure 12: Left: An exponential cone. Right: Mean curvature of a potential surface.

In addition to this, we have visualized some objects and principles from some of the fields of mathematics that are not such obvious choices, such as topology [14], functional analysis [10, 11], and crystallography [9, 13]. Figure 13 shows the potential surface of the $v_\infty(\Lambda)$ norm defined in [13] and the corresponding Wulff's crystal.

Figure 14 shows some relative topologies given by metrics on Enneper's surface with the parametric representation (2). If the topology is given by the metric d, then the sphere of radius r and center at a point X_0 in the relative topology on Enneper's surface is given by the zeros of the function

$$\Psi(\rho, \phi) = d(X(\rho, \phi), X_0) - r,$$

where $X(\rho, \phi)$ is the point with the position vector $\vec{x}(\rho, \phi)$ in the parametric representation (2).

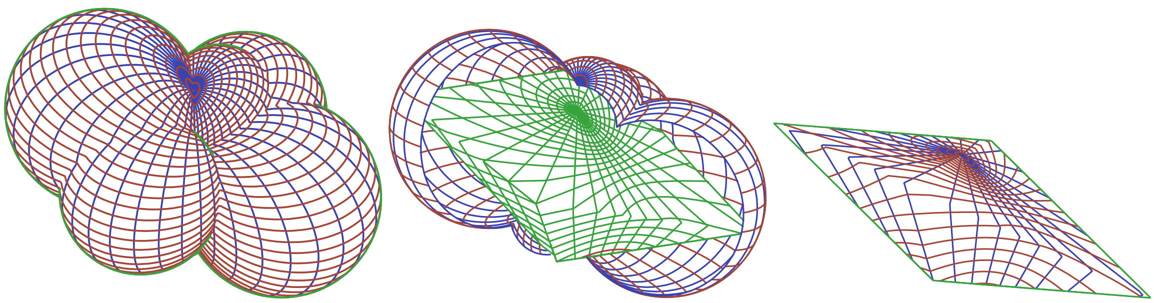

Figure 13: Potential surface and the corresponding Wulff's crystal

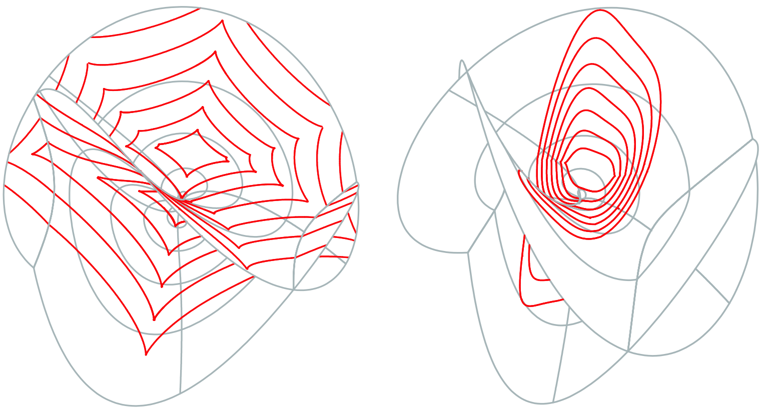

Figure 14: Neighbourhoods in relative topologies on Enneper's surface

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