Filomat 37:4 (2023), 1001–1016
https://doi.org/10.2298/FIL2304001D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Boundedness of Hardy–Cesàro operators on variable exponent Morrey–Herz spaces

Kieu Huu Dung[a], Do Lu Cong Minh[a], Tran Thi Nang[a]

[a]*Faculty of Fundamental Sciences, Van Lang University, Ho Chi Minh City, Vietnam*

Abstract. In this paper, we give the necessary and sufficient conditions for the boundedness of Hardy–Cesàro operators on some weighted function spaces such as the weighted central Morrey, weighted local central Morrey, weighted non-local central Morrey, weighted Herz and weighted Morrey–Herz type spaces with variable exponent.

1. Introduction

Let f be a non-negative measurable function on $\mathbb{R}^+$, and one-dimensional Hardy operator be given by

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt,\ x > 0.$$

G. H. Hardy [16] obtained a famous integral inequality as follows.

$$\|H\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} \leq \frac{q}{q-1},$$

where $q \in (1,\infty)$ and $\frac{q}{q-1}$ is the sharpest constant.

In 1984, C. Carton–Lebrun and M. Fosset [5] defined the weighted Hardy–Littlewood average operator U_ψ:

$$U_\psi(f) = \int_0^1 \psi(t)f(tx)dt,\ x \in \mathbb{R}^n,$$

where $\psi : [0,1] \to [0,\infty)$ is a measurable function and f is a measurable function on $\mathbb{R}^n$. It is clear to see that if we choose $n = 1$ and $\psi \equiv 1$, the Hardy–Littlewood average operator U_ψ reduces to the classical Hardy operator H. Next, J. Xiao [30] proved that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\mathcal{A}_{n,p,\psi} := \int_0^1 t^{-n/p}\psi(t)dt < \infty.$$

2020 *Mathematics Subject Classification*. Primary 46E30; Secondary 42B35, 26D10
Keywords. Hardy–Cesàro operator; variable exponent; Morrey space; Herz space; Morrey–Herz space
Received: 31 January 2022; Accepted: 19 April 2022
Communicated by Dragan S. Djordjević
This study was financially supported by Van Lang University, Vietnam.
Email addresses: dung.kh@vlu.edu.vn (Kieu Huu Dung), minh.dlc@vlu.edu.vn (Do Lu Cong Minh), nang.tt@vlu.edu.vn (Tran Thi Nang)

Moreover,

$$\|U_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \mathcal{A}_{n,p,\psi}.$$

On the other hand, J. Xiao also obtained the boundedness of this operator on the *BMO* spaces. In this view, we consider the Hardy–Cesàro operator as follows.

Definition 1.1. *Let $\psi : [0,1]^d \to [0,\infty)$, $s : [0,1]^d \to \mathbb{R}$ be measurable functions. The Hardy–Cesàro operator is defined by*

$$U_{\psi,s,d}(f)(x) = \int\limits_{[0,1]^d} \psi(t) f(s(t)x)dt,$$

for a measurable complex-valued function f on $\mathbb{R}^n$.

In case $d = 1$, the Hardy–Cesàro operator $U_{\psi,s,1}$ was researched by N. M. Chuong and H. D. Hung [10]. The authors gave the sufficient and necessary conditions for the boundedness of $U_{\psi,s,1}$ on the weighted Lebesgue and weighted *BMO* spaces. Note that, when we take $d = 1$ and $s(t) = t$, the Hardy–Cesàro operator $U_{\psi,s,d}$ becomes to the Hardy–Littlewood average operator U_ψ.

It is well-known that the Hardy–Cesàro operators and their commutators have attracted much more attention on real Euclidean spaces (see [8, 9, 14, 17, 25]).

The theory of function spaces with variable exponents is developed in the field of electronic fluid mechanics, elasticity, fluid dynamics, recovery of graphics, differential equations, harmonic analysis and partial differential equations (see [6], [12], [13], [18]). In particular, the maximal operators, the Calderón-Zygmund singular operators, the Kantorovich operators, the Hardy-type operators and their commutators have been extensively studied on the Lebesgue, Herz, Morrey, and Morrey-Herz spaces with variable exponent (see, e.g., [2], [3], [4], [7], [11], [15], [19], [21], [22], [23], [24], [26], [27], [28], [29] and others).

Motivated by above mentioned results, the goal of this paper is to establish the necessary and sufficient conditions for the boundedness of $U_{\psi,s,d}$ on the weighted central Morrey, weighted local central Morrey, weighted non-local central Morrey, weighted Herz and weighted Morrey-Herz spaces with variable exponent. In each case, the estimates for operator norms are also discussed.

Our paper is organized as follows. In Section 2, we give the necessary preliminaries on weighted Lebesgue spaces, central Morre spaces, Herz spaces and Morrey-Herz spaces with variable exponent. Our main theorems are given and proved in Section 3.

2. Preliminaries

Let us give some basic facts and notations which will be used throughout this paper. The letter C denotes a positive constant which is independent of the main parameters, but may be different from line to line. For any $a \in \mathbb{R}^n$ and $r > 0$, let $B(a,r)$ denote the ball centered at a with radius r. With a given measurable set Ω, let χ_Ω be a characteristic function, $\chi_k = \chi_{C_k}$, $C_k = B_k \setminus B_{k-1}$ and $B_k = \left\{x \in \mathbb{R}^n : |x| \le 2^k\right\}$, for all $k \in \mathbb{Z}$. Next, we write $a \lesssim b$ to mean that there is a positive constant C, independent of the main parameters, such that $a \le Cb$. The symbol $f \simeq g$ means that f is equivalent to g (i.e. $C^{-1}f \le g \le Cf$). In this paper, as usual, we will let $\omega(\cdot)$ represent a non-negative weighted function on $\mathbb{R}^n$.

Now, we give some notations and definitions of Lebesgue, Herz and Morrey-Herz spaces with constant parameters (see in [20]).

Definition 2.1. *Let $1 \le p < \infty$, we define the weighted Lebesgue space $L^p(\omega)$ of a measurable function f by*

$$\left\|f\right\|_{L^p(\omega)} = \Big(\int\limits_{\mathbb{R}^n} \left|f(x)\right|^p \omega(x)dx\Big)^{\frac{1}{p}} < \infty.$$

Definition 2.2. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, and $0 < p < \infty$. The weighted homogeneous Herz-type space $\dot{K}_q^{\alpha,p}(\omega)$ is defined by*

$$\dot{K}_q^{\alpha,p}(\omega) = \left\{ f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega)} < \infty \right\},$$

where $\|f\|_{\dot{K}_q^{\alpha,p}(\omega)} = \left(\sum\limits_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\omega)}^p \right)^{\frac{1}{p}}$.

Definition 2.3. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty, 0 < q < \infty, \lambda \geq 0$. The homogeneous weighted Morrey-Herz type space $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega)$ is defined by*

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\omega) = \left\{ f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega)} < \infty \right\},$$

where $\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega)} = \sup\limits_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum\limits_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\omega)}^p \right)^{\frac{1}{p}}$.

Remark 1. It is helpful to note that $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $0 < p < \infty$; $\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for all $0 < p < \infty$ and $\alpha \in \mathbb{R}$. Since $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, it follows that the Herz space is a special case of Morrey-Herz space.

Let us present the definition of the Lebesgue space with variable exponent. The reader may find in the works [6], [12] and [13].

Definition 2.4. *Let $\mathcal{P}_b(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) : \mathbb{R} \to [1, \infty)$ such that*

$$1 < p_- \leq p(x) \leq p_+ < \infty, \text{ for all } x \in \mathbb{R}^n,$$

where $p_- = ess\,inf_{x\in\mathbb{R}^n} p(x)$ and $p_+ = ess\,sup_{x\in\mathbb{R}^n} p(x)$. For $p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f defined on $\mathbb{R}^n$ such that there exists constant $\eta > 0$ satisfying

$$F_p(f/\eta) = \int\limits_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ then becomes a normed space equipped with a norm given by

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{ \eta > 0 : F_p\left(\frac{f}{\eta}\right) \leq 1 \right\}.$$

For $p \in \mathcal{P}_b(\mathbb{R}^n)$, it is useful to remark that we have the following inequalities which are usually used in the sequel.

$$\begin{aligned} &[i] \text{ If } F_p(f) \leq C, \text{ then } \|f\|_{L^{p(\cdot)}} \leq \max\left\{ C^{\frac{1}{p_-}}, C^{\frac{1}{p_+}} \right\}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n), \\ &[ii] \text{ If } F_p(f) \geq C, \text{ then } \|f\|_{L^{p(\cdot)}} \geq \min\left\{ C^{\frac{1}{p_-}}, C^{\frac{1}{p_+}} \right\}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n). \end{aligned} \tag{1}$$

The space $\mathcal{P}_\infty(\mathbb{R}^n)$ is defined by the set of all measurable functions $p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and there exists a constant p_∞ such that

$$p_\infty = \lim_{|x|\to\infty} p(x).$$

For $p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, the weighted variable exponent Lebesgue space $L_\omega^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f such that $f\omega$ belongs to the $L^{p(\cdot)}(\mathbb{R}^n)$ space, and the norm of f is given by $\|f\|_{L_\omega^{p(\cdot)}} = \|f\omega\|_{L^{p(\cdot)}}$. The set $L_{\omega,\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ consists of all measurable functions f on $\mathbb{R}^n \setminus \{0\}$ satisfying

$f.\chi_K \in L^{p(\cdot)}_{\omega}(\mathbb{R}^n)$ for any compact set $K \subset \mathbb{R}^n \setminus \{0\}$.
Let $\mathbf{C}_0^{\log}(\mathbb{R}^n)$ denote the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$|\alpha(x) - \alpha(0)| \leq \frac{C_0^{\alpha}}{\log\left(e + \frac{1}{|x|}\right)}, \text{ for all } x \in \mathbb{R}^n.$$

Denote by $\mathbf{C}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_\infty^{\alpha}}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

Next, we would like to give the definition of variable exponent weighted Herz spaces $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$, variable exponent weighted Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ (see [21], [27] for more details) and variable exponent weighted central Morrey spaces $\dot{M}^{\kappa,q(\cdot)}_{\omega}$.

Definition 2.5. *Let $0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Herz space $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$ is defined by*

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega} = \left\{ f \in L^{q(\cdot)}_{\omega,loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} < \infty \right\}$$

with $\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} = \left(\sum\limits_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}_{\omega}} \right)^{1/p}$.

Definition 2.6. *Assume that $0 \leq \lambda < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ is defined by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega} = \left\{ f \in L^{q(\cdot)}_{\omega,loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} < \infty \right\},$$

where $\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} = \sup\limits_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum\limits_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}_{\omega}} \right)^{1/p}$.

Note that, when $p(\cdot)$, $q(\cdot)$ and $\alpha(\cdot)$ are constant, it is obvious to see that

$$L^p_{\omega^{1/p}} = L^p(\omega), \ \dot{K}^{\alpha,p}_{q,\omega^{1/q}} = \dot{K}^{\alpha,p}_{q}(\omega) \text{ and } M\dot{K}^{\alpha,\lambda}_{p,q,\omega^{1/q}} = M\dot{K}^{\alpha,\lambda}_{p,q}(\omega).$$

Theorem 2.7 (Proposition 3.8 in [1]). *If $p \in (0, \infty)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}_0^{log}(\mathbb{R}^n) \cap \mathbf{C}_\infty^{log}(\mathbb{R}^n)$, then we have*

$$\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} \approx \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p} \|f\chi_k\|^p_{L^{q(\cdot)}_{\omega}} \right)^{1/p} + \left(\sum_{k=0}^{\infty} 2^{k\alpha_\infty p} \|f\chi_k\|^p_{L^{q(\cdot)}_{\omega}} \right)^{1/p}.$$

Theorem 2.8 (Proposition 2.5 in [21]). *If $\lambda \in [0, \infty)$, $p \in (0, \infty)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}_0^{log}(\mathbb{R}^n) \cap \mathbf{C}_\infty^{log}(\mathbb{R}^n)$, then we obtain*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \approx \max\left\{ \sup_{k_0 \leq 0, k_0 \in \mathbb{Z}} \mathcal{M}_{1,k_0}, \sup_{k_0 > 0, k_0 \in \mathbb{Z}} \left(\mathcal{M}_{2,k_0} + \mathcal{M}_{3,k_0} \right) \right\}.$$

Here

$$\mathcal{M}_{1,k_0} = 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}_\omega}\right)^{1/p},\ \mathcal{M}_{2,k_0} = 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}_\omega}\right)^{1/p},$$

$$\mathcal{M}_{3,k_0} = 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}\|f\chi_k\|^p_{L^{q(\cdot)}_\omega}\right)^{1/p}.$$

From the definition of weighted Morrey-Herz spaces with variable exponent and Proposition 2.5 in [21], we have the following result.

Lemma 2.9. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $p \in (0,\infty)$ and $\lambda \in [0,\infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\left\|f\chi_j\right\|_{L^{q(\cdot)}_\omega} \leq C.2^{j(\lambda-\alpha(0))}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}},\ \textit{for all } j \in \mathbb{Z}^-;$$
$$\left\|f\chi_j\right\|_{L^{q(\cdot)}_\omega} \leq C.2^{j(\lambda-\alpha_\infty)}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}},\ \textit{for all } j \in \mathbb{N}.$$

Proof. The proof of this lemma is found in [27]. □

Definition 2.10. *Assume that $\kappa > 0, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$. The variable exponent weighted central Morrey space $\dot{M}^{\kappa,q(\cdot)}_\omega$ is defined by*

$$\dot{M}^{q(\cdot),\kappa}_\omega = \left\{f \in L^{q(\cdot)}_{\omega,loc}(\mathbb{R}^n) : \|f\|_{\dot{M}^{q(\cdot),\kappa}_\omega} < \infty\right\},$$

where $\|f\|_{\dot{M}^{q(\cdot),\kappa}_\omega} = \sup\limits_{k\in\mathbb{Z}} \frac{1}{\omega(B_k)^\kappa}\|f\|_{L^{q(\cdot)}_\omega(B_k)}$.

Definition 2.11. *Let $\kappa > 0, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$. The variable exponent weighted local central Morrey space $\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}$ is defined by*

$$\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}} = \{f \in L^{q(\cdot)}_{\omega,\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} < \infty\},$$

where $\|f\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} = \sup\limits_{0<R<1} \frac{1}{\omega(B(0,R))^\kappa}\|f\|_{L^{q(\cdot)}_\omega(B(0,R))}$.

Definition 2.12. *Let $\kappa > 0, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$. The variable exponent weighted non-local central Morrey space $B^{q(\cdot),\kappa}_\omega$ is defined by*

$$B^{q(\cdot),\kappa}_\omega = \{f \in L^{q(\cdot)}_{\omega,\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{B^{q(\cdot),\kappa}_\omega} < \infty\},$$

where $\|f\|_{B^{q(\cdot),\kappa}_\omega} = \sup\limits_{R\geq 1} \frac{1}{\omega(B(0,R))^\kappa}\|f\|_{L^{q(\cdot)}_\omega(B(0,R))}$.

3. Main results and their proofs

Let us introduce some notations which will be used throughout this section. Assume that $p \in (1,\infty)$, $q \in \mathcal{P}_b(\mathbb{R}^n)$, $\alpha \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}^{\log}_0(\mathbb{R}^n) \cap \mathbf{C}^{\log}_\infty(\mathbb{R}^n)$ and ω be a weighted function. For simplicity of notation, we put

$$\theta_{\sup}(t) = \operatorname{ess\,supp}_{z\in\mathbb{R}^n}\frac{\omega(s^{-1}(t)z)}{\omega(z)} \text{ and } \theta_{\inf}(t) = \operatorname{ess\,inf}_{z\in\mathbb{R}^n}\frac{\omega(s^{-1}(t)z)}{\omega(z)},$$

$$\mathcal{K}_{s,\theta_{sup}}(t) = \theta_{sup}(t).\max\left\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\right\} \text{and } \mathcal{K}_{s,\theta_{inf}}(t) = \theta_{inf}(t).\min\left\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\right\}.$$

Now, we are ready to state our first main result in this paper.

Theorem 3.1. *Let $q(s^{-1}(t)\cdot) = q(\cdot)$ for almost everywhere $t \in \mathrm{supp}(\psi)$, $\lambda > 0$ and $\alpha(0) - \alpha_\infty \geq 0$.*
(i) If

$$C_{1,\sup} = \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{sup}}(t).\max\left\{|s(t)|^{\lambda-\alpha(0)}, |s(t)|^{\lambda-\alpha_\infty}\right\}dt < \infty,$$

then $U_{\psi,s,d}$ is bounded from $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ to itself.
(ii) Suppose that $U_{\psi,s,d}$ is bounded from $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ to itself, and either $\lambda = \alpha(0)$ or $q_+ = q_-$. We have

$$C_{1,\inf} = \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{inf}}(t).|s(t)|^{\lambda-\alpha(0)}dt < \infty.$$

Moreover,

$$C_{1,\inf} \leq \left\|U_{\psi,s,d}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega} \to M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.$$

Proof. Now, we will prove for the case (i). By the Minkowski inequality, we have

$$\left\|U_{\psi,s,d}(f)\chi_k\right\|_{L^{q(\cdot)}_\omega} \lesssim \int\limits_{[0,1]^d} \psi(t)\left\|f(s(t).)\chi_k\right\|_{L^{q(\cdot)}_\omega}dt. \tag{2}$$

For $\eta > 0$ and $t \in [0,1]^n$ such that $|s(t)| \neq 0$, we estimate

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \Big(\frac{\big|f(s(t)x)\big|\chi_k(x)\omega(x)}{\eta}\Big)^{q(x)}dx &= \int\limits_{s(t)C_k} \Big(\frac{\big|f(z)\big|\omega(s^{-1}(t)z)}{\eta}\Big)^{q(s^{-1}(t)z)}\left|s^{-1}(t)\right|^n dz \\
&= \left|s^{-1}(t)\right|^n \int\limits_{s(t)C_k} \Big(\frac{\big|f(z)\big|\omega(z)}{\eta}.\frac{\omega(s^{-1}(t)z)}{\omega(z)}\Big)^{q(s^{-1}(t)z)}dz. \\
&\leq \int\limits_{\mathbb{R}^n} \left(\frac{\mathcal{K}_{s,\theta_{\sup}}(t).\Big(|f(z)|\chi_{k+\ell-1}(z) + |f(z)|\chi_{k+\ell}(z)\Big)\omega(z)}{\eta}\right)^{q(z)} dz,
\end{aligned}$$

where $\ell = \ell(t) \in \mathbb{Z}$ such that $2^{\ell-1} < |s(t)| \leq 2^\ell$. Thus

$$\left\|f(s(t).)\chi_k\right\|_{L^{q(\cdot)}_\omega} \leq \mathcal{K}_{s,\theta_{sup}}(t).\Big(\left\|f\chi_{k+\ell-1}\right\|_{L^{q(\cdot)}_\omega} + \left\|f\chi_{k+\ell}\right\|_{L^{q(\cdot)}_\omega}\Big).$$

From this, by (2), we have

$$\left\|U_{\psi,s,d}(f)\chi_k\right\|_{L^{q(\cdot)}_\omega} \lesssim \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{sup}}(t).\Big(\left\|f\chi_{k+\ell-1}\right\|_{L^{q(\cdot)}_\omega} + \left\|f\chi_{k+\ell}\right\|_{L^{q(\cdot)}_\omega}\Big)dt. \tag{3}$$

Next, by applying Lemma 2.9 and $|s(t)| \simeq 2^\ell$, we get

$$\begin{aligned}
\left\|f\chi_{k+\ell-1}\right\|_{L^{q(\cdot)}_\omega} &\lesssim 2^{\max\{(k+\ell-1)(\lambda-\alpha(0)),(k+\ell-1)(\lambda-\alpha_\infty)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \\
&= \max\{2^{(\ell-1)(\lambda-\alpha(0))}, 2^{(\ell-1)(\lambda-\alpha_\infty)}\}.2^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \\
&\lesssim \max\left\{|s(t)|^{\lambda-\alpha(0)}, |s(t)|^{\lambda-\alpha_\infty}\right\}.2^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.
\end{aligned}$$

By estimating as above, we also have

$$\left\|f\chi_{k+\ell}\right\|_{L^{q(\cdot)}_{\omega}} \lesssim \max\left\{|s(t)|^{\lambda-\alpha(0)}, |s(t)|^{\lambda-\alpha_\infty}\right\}.2^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}.\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.$$

From these above, by (3), it is clear to see that

$$\left\|U_{\psi,s,d}(f)\chi_k\right\|_{L^{q(\cdot)}_{\omega}} \lesssim C_{1,\sup}.2^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}.\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}. \tag{4}$$

By Theorem 2.8, we infer

$$\left\|U_{\psi,s,d}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \lesssim \max\Big\{\sup_{k_0\leq 0, k_0\in\mathbb{Z}} T_1, \sup_{k_0>0,k_0\in\mathbb{Z}} (T_2+T_3)\Big\}, \tag{5}$$

where

$$T_1 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}\left\|U_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}},\ T_2 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\left\|U_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}},$$

$$T_3 = 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}\left\|U_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}}.$$

By using the inequality (4) with $k_0 \leq 0$ and $k_0 \in \mathbb{Z}$, we will estimate T_1 as follows.

$$\begin{aligned}
T_1 &\leq C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}.2^{\max\{k(\lambda-\alpha(0))p,\, k(\lambda-\alpha_\infty)p\}}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}\\
&= C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{\max\{k(\lambda-\alpha(0))p+k\alpha(0)p,\, k(\lambda-\alpha_\infty)p+k\alpha(0)p\}}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}\\
&= C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{kp.\min\{\lambda,\, \lambda-\alpha_\infty+\alpha(0)\}}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.
\end{aligned}$$

Consequently, by $\min\{\lambda, \lambda-\alpha_\infty+\alpha(0)\} > 0$ and $\alpha(0)-\alpha_\infty \geq 0$, $k_0 \leq 0$ and $k_0 \in \mathbb{Z}$, one has

$$\begin{aligned}
T_1 &\lesssim C_{1,\sup}.2^{k_0(\min\{\lambda,\lambda-\alpha_\infty+\alpha(0)\}-\lambda)}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} = C_{1,\sup}.2^{k_0.\min\{0,-\alpha_\infty+\alpha(0)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}\\
&= C_{1,\sup}.\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.
\end{aligned} \tag{6}$$

By evaluating as T_1, we also have

$$T_2 \lesssim C_{1,\sup}.2^{-k_0\lambda}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}. \tag{7}$$

Next step, we obtain

$$\begin{aligned}
T_3 &\lesssim C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}.2^{\max\{k(\lambda-\alpha(0))p,\, k(\lambda-\alpha_\infty)p\}}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}\\
&= C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{kp.\max\{\lambda-\alpha(0)+\alpha_\infty),\, \lambda\}}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} = C_{1,\sup}.2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{kp\lambda}\Big)^{\frac{1}{p}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}\\
&\lesssim C_{1,\sup}.2^{-k_0\lambda}\Big(2^{k_0\lambda}+1\Big)\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}.
\end{aligned}$$

Thus, by (5)-(7), it follows that

$$\|U_{\psi,s,d}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \lesssim C_{1,\sup}.\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}},$$

which gives the proof of case (i) of Theorem 3.1 is finished. Next, let us consider for case (ii). We choose

$$f_0(x) = |x|^{-\alpha(0)-\frac{n}{q(x)}+\lambda}.\omega(x)^{-1}.$$

It is obvious to see that

$$\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} > 0.$$

On the other hand, we calculate

$$F_q(f_0.\omega.\chi_k) = \int\limits_{C_k} |x|^{(\lambda-\alpha(0)).q(x)-n}dx = \int\limits_{2^{k-1}}^{2^k}\int\limits_{S^{n-1}} r^{(\lambda-\alpha(0)).q(r.x')-1}d\sigma(x')dr \lesssim 2^{\max\{k(\lambda-\alpha(0)).q_+,\ k(\lambda-\alpha(0)).q_-\}}.$$

Combining this with the inequality (1), we get

$$\begin{aligned}\|f_0\chi_k\|_{L^{q(\cdot)}_\omega} &\lesssim \max\Big\{2^{\max\{k(\lambda-\alpha(0)),\ k(\lambda-\alpha(0)).q_-/q_+\}}, 2^{\max\{k(\lambda-\alpha(0)).q_+/q_-,\ k(\lambda-\alpha(0))\}}\Big\}\\ &= 2^{\max\{k(\lambda-\alpha(0)).q_-/q_+, k(\lambda-\alpha(0)).q_+/q_-\}}.\end{aligned} \tag{8}$$

Besides, by Theorem 2.8, we estimate

$$\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \lesssim \max\Big\{\sup_{k_0\leq 0, k_0\in\mathbb{Z}} G_1,\ \sup_{k_0>0,k_0\in\mathbb{Z}} (G_2+G_3)\Big\}. \tag{9}$$

Here

$$G_1 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}\|f_0\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{\frac{1}{p}},\ G_2 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\|f_0\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{\frac{1}{p}},$$

$$G_3 = 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}\|f_0\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{\frac{1}{p}}.$$

By either $\alpha(0) = \lambda > 0$ or $q_+ = q_-$, we infer

$$\alpha(0) + \min\{(\lambda-\alpha(0)).q_-/q_+,\ (\lambda-\alpha(0)).q_+/q_-\} - \lambda = 0.$$

Hence, by (8), one has

$$\begin{aligned}G_1 &\lesssim 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}.2^{\max\{k(\lambda-\alpha(0)).q_-/q_+, k(\lambda-\alpha(0)).q_+/q_-\}p}\Big)^{\frac{1}{p}}\\ &\lesssim 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{kp\big(\alpha(0)+\min\{(\lambda-\alpha(0)).q_-/q_+, (\lambda-\alpha(0)).q_+/q_-\}\big)}\Big)^{\frac{1}{p}}\\ &\lesssim 2^{k_0(\alpha(0)+\min\{(\lambda-\alpha(0)).q_-/q_+, (\lambda-\alpha(0)).q_+/q_-\}-\lambda)} = 1.\end{aligned} \tag{10}$$

By estimating as (10) above, we also have

$$G_2 \lesssim 2^{-k_0\lambda}2^{-(\alpha(0)+\min\{(\lambda-\alpha(0)).q_-/q_+, (\lambda-\alpha(0)).q_+/q_-\})} \lesssim 2^{-k_0\lambda}. \tag{11}$$

By applying the inequality (8) again, we deduce

$$\begin{aligned}
G_3 &\lesssim 2^{-k_0\lambda}\Big(\sum_{k=1}^{k_0} 2^{kp(\alpha_\infty+\max\{(\lambda-\alpha(0)).q_-/q_+,\,(\lambda-\alpha(0)).q_+/q_-\})}\Big)^{\frac{1}{p}}\\
&\lesssim \begin{cases} 2^{-k_0\lambda}(k_0^{1/p}+1), \text{ if } \alpha_\infty+\max\{(\lambda-\alpha(0)).q_-/q_+,\,(\lambda-\alpha(0)).q_+/q_-\}=0,\\ 2^{-k_0\lambda}+2^{-k_0(\lambda-\alpha_\infty-\max\{(\lambda-\alpha(0)).q_-/q_+,\,(\lambda-\alpha(0)).q_+/q_-\})}, \text{ otherwise.}\end{cases}\\
&\lesssim 2^{-k\lambda}(k_0^{1/p}+1)+2^{-k_0(\lambda-\alpha_\infty-\max\{(\lambda-\alpha(0)).q_-/q_+,\,(\lambda-\alpha(0)).q_+/q_-\})}. \qquad (12)
\end{aligned}$$

By the assumptions $\lambda = \alpha(0) > 0$ or $q_+ = q_-$, we get

$$\zeta = \lambda - \alpha_\infty - \max\{(\lambda-\alpha(0)).q_-/q_+,\,(\lambda-\alpha(0)).q_+/q_-\} \geq 0.$$

From this, by (9)-(12), we derive

$$\big\|f_0\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \lesssim \max\Big\{\sup_{k_0\leq 0, k_0\in\mathbb{Z}} 1,\ \sup_{k_0>0,k_0\in\mathbb{Z}}\Big(2^{-k_0\lambda}(k_0^{1/p}+2)+2^{-k_0\zeta}\Big)\Big\} < \infty.$$

In addition, we get

$$U_{\psi,s,d}(f_0)(x) = \Big(\int\limits_{[0,1]^d}\psi(t).|s(t)|^{-\alpha(0)-\frac{n}{q(x)}+\lambda}.\frac{\omega(x)}{\omega(s(t)x)}dt\Big).|x|^{-\alpha(0)-\frac{n}{q(x)}+\lambda}\omega(x)^{-1} \geq C_{1,\inf}.f_0(x).$$

This leads to

$$C_{1,\inf} \leq \big\|U_{\psi,s,d}\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}\to M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} < \infty.$$

Therefore, the proof of this theorem is completed. □

Next, we obtain the boundedness for the Hardy–Cesàro operators on the weighted Herz spaces with variable exponent as follows.

Theorem 3.2. *Let $\alpha(0) = \alpha_\infty$ and the assumptions of Theorem 3.1 hold.*
(*i*) *If*

$$C_{2,\sup} = \int\limits_{[0,1]^d}\psi(t).\mathcal{K}_{s,\theta_{\sup}}(t).|s(t)|^{-\alpha(0)}dt < \infty,$$

then $U_{\psi,s,d}$ is a bounded operator from $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$ to itself.
(*ii*) *If $U_{\psi,s,d}$ is a bounded operator from $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$ to itself, and either $\alpha_\infty = 0$ or $q_+ = q_-$, then*

$$C_{2,\inf} = \int\limits_{[0,1]^d}\psi(t).\mathcal{K}_{s,\theta_{\inf}}(t).|s(t)|^{-\alpha(0)}dt < \infty.$$

Moreover,

$$C_{2,\inf} \lesssim \|U_{\psi,s,d}\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}\to\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}.$$

Proof. First, we will consider the proof for the case (*i*). By Theorem 2.7 and the relation $\alpha(0) = \alpha_\infty$, we infer

$$\big\|U_{\psi,s,d}\big(f\big)\big\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} \lesssim \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\big\|U_{\psi,s,d}\big(f\big)\chi_k\big\|^p_{L^{q(\cdot)}_\omega}\Big)^{\frac{1}{p}} := \mathcal{U}. \qquad (13)$$

By having the inequality (3) and using the Minkowski inequality, we deduce

$$\begin{aligned}\mathcal{U} &\lesssim \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{\sup}}(t)\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\Big(\big\|f\chi_{k+\ell-1}\big\|_{L^{q(\cdot)}_{\omega}} + \big\|f\chi_{k+\ell}\big\|_{L^{q(\cdot)}_{\omega}}\Big)^p\Big\}^{\frac{1}{p}} dt\\ &\lesssim \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{\sup}}(t)\left(\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\big\|f\chi_{k+\ell-1}\big\|^p_{L^{q(\cdot)}_{\omega}}\Big\}^{\frac{1}{p}} + \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\big\|f\chi_{k+\ell}\big\|^p_{L^{q(\cdot)}_{\omega}}\Big\}^{\frac{1}{p}}\right) dt. \qquad (14)\end{aligned}$$

On the other hand, by $|s(t)| \simeq 2^{\ell(t)}$, we derive

$$\Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\big\|f\chi_{k+\ell-1}\big\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}} = \Big(\sum_{r=-\infty}^{\infty} 2^{(r-\ell+1)\alpha(0)p}\big\|f\chi_{r}\big\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}} \lesssim 2^{-\ell\alpha(0)}\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} \lesssim |s(t)|^{-\alpha(0)}\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}.$$

By making as above, we also have

$$\Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p}\big\|f\chi_{k+\ell}\big\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{\frac{1}{p}} \lesssim |s(t)|^{-\alpha(0)}\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}.$$

From these, by (13) and (14), it is clear to see that

$$\big\|U_{\psi,s,d}\big(f\big)\big\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} \lesssim C_{2,\sup}.\big\|f\big\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}.$$

Therefore, the proof of this case is achieved.

Now, let us prove the case (ii). For any $\varepsilon > 0$, we choose the functions f_ε as follows

$$f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \le 1,\\ |x|^{-\alpha_\infty - \frac{n}{q(x)} - \varepsilon}\omega(x)^{-1}, & \text{otherwise.}\end{cases}$$

This leads to

$$\|f_\varepsilon\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} > 0, \text{for all } \varepsilon > 0.$$

For ε small enough, we get

$$F_q(f_\varepsilon.\omega.\chi_k) = \int\limits_{C_k} |x|^{(-\alpha_\infty-\varepsilon)q(x)-n}dx = \int\limits_{2^{k-1}}^{2^k}\int\limits_{S^{n-1}} r^{(-\alpha_\infty-\varepsilon)q(r.x')-1}d\sigma(x')dr.$$

This gives

$$2^{\min\{-k(\alpha_\infty+\varepsilon)q_-,\,-k(\alpha_\infty+\varepsilon)q_+\}} \lesssim F_q(f_\varepsilon.\omega.\chi_k) \lesssim 2^{\max\{-k(\alpha_\infty+\varepsilon)q_-,\,-k(\alpha_\infty+\varepsilon)q_+\}}.$$

Hence, by the inequality (1), we derive

$$2^{\min\{-k(\alpha_\infty+\varepsilon)q_-/q_+,\,-k(\alpha_\infty+\varepsilon)q_+/q_-\}} \lesssim \big\|f_\varepsilon\chi_k\big\|_{L^{q(\cdot)}_{\omega}} \lesssim 2^{\max\{-k(\alpha_\infty+\varepsilon)q_-/q_+,\,-k(\alpha_\infty+\varepsilon)q_+/q_-\}}.$$

Now, we put

$$\zeta_\varepsilon = \alpha_\infty - \min\{(\alpha_\infty+\varepsilon)q_-/q_+,\ (\alpha_\infty+\varepsilon)q_+/q_-\} \text{ and } \beta_\varepsilon = \alpha_\infty - \max\{(\alpha_\infty+\varepsilon)q_-/q_+,\ (\alpha_\infty+\varepsilon)q_+/q_-\}.$$

By applying the conditions $\alpha_\infty = 0$ or $q_+ = q_-$, we get $\zeta_\varepsilon, \beta_\varepsilon < 0$. Furthermore, it follows that

$$\lim_{\varepsilon\to 0^+}\zeta_\varepsilon = 0,\ \lim_{\varepsilon\to 0^+}\beta_\varepsilon = 0 \text{ and } \lim_{\varepsilon\to 0^+}\frac{\beta_\varepsilon}{\zeta_\varepsilon} = \begin{cases} 1, & \text{if } q_+ = q_-,\\ q_+^2/q_-^2, & \text{if } \alpha_\infty = 0.\end{cases} \qquad (15)$$

As a consequence, by Theorem 2.7, for each $r \in \mathbb{Z}^+$, we obtain

$$\|f_\varepsilon\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} \leq \left\{\sum_{k=1}^{\infty} 2^{k\alpha_\infty p}\left\|f_\varepsilon\chi_k\right\|^p_{L^{q(\cdot)}_\omega}\right\}^{1/p} \lesssim \left\{\sum_{k=1}^{\infty} 2^{kp\zeta_\varepsilon}\right\}^{1/p} \lesssim \frac{2^{\zeta_\varepsilon}}{\left(1-2^{\zeta_\varepsilon p}\right)^{1/p}},$$

$$\text{and} \left\{\sum_{k=r}^{\infty} 2^{k\alpha_\infty p}\left\|f_\varepsilon\chi_k\right\|^p_{L^{q(\cdot)}_\omega}\right\}^{1/p} \gtrsim \left\{\sum_{k=r}^{\infty} 2^{kp\beta_\varepsilon}\right\}^{1/p} \gtrsim \frac{2^{r\beta_\varepsilon}}{\left(1-2^{\beta_\varepsilon p}\right)^{1/p}}. \tag{16}$$

By setting

$$V^s_x = \{t \in [0,1]^d : |s(t).x| \geq 1\} \text{ and } U^s_\varepsilon = \{t \in [0,1]^d : |s(t)| \geq \varepsilon\}.$$

This gives

$$U^s_\varepsilon \subset V^s_x, \text{ for all } x \in B^c(0,\varepsilon^{-1}).$$

Instantly, for $0 < \varepsilon \leq 1$, we get

$$\begin{aligned}
U_{\psi,s,d}(f)(x) &\geq \int_{V^s_x} \psi(t).|s(t)x|^{-\alpha_\infty-\frac{n}{q(x)}-\varepsilon}\omega(s(t)x)^{-1}dt \\
&\geq \Big(\int\limits_{U^s_\varepsilon} \psi(t).|s(t)|^{-\alpha_\infty-\frac{n}{q(x)}-\varepsilon}\frac{\omega(x)}{\omega(s(t)x)}dt\Big)|x|^{-\alpha_\infty-\frac{n}{q(x)}-\varepsilon}\omega(x)^{-1}\chi_{B^c(0,\varepsilon^{-1})}(x) \\
&\geq \Big(\int\limits_{U^s_\varepsilon} \psi(t).\mathcal{K}_{s,\theta_{inf}}(t).|s(t)|^{-\alpha(0)-\varepsilon}dt\Big)f_\varepsilon(x).\chi_{B^c(0,\varepsilon^{-1})}(x).
\end{aligned}$$

Hence, by using Theorem 2.7 again and (16) with k_0 is the smallest integer number so that $2^{k_0-1} \geq \varepsilon^{-1} \geq 1$, we infer

$$\begin{aligned}
\left\|U_{\psi,s,d}(f)\right\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} &\gtrsim \Big(\int\limits_{U^s_\varepsilon} \psi(t).\mathcal{K}_{s,\theta_{inf}}(t).|s(t)|^{-\alpha(0)-\varepsilon}dt\Big)\Big(\sum_{k=k_0}^{\infty} 2^{k\alpha_\infty p}\left\|f_\varepsilon\chi_k\right\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p} \\
&\gtrsim \frac{(1-2^{\zeta_\varepsilon p})^{1/p}.2^{k_0\beta_\varepsilon}}{(1-2^{\beta_\varepsilon p})^{1/p}.2^{\zeta_\varepsilon}}\varepsilon^{-\varepsilon}\Big(\int\limits_{U^s_\varepsilon} \psi(t).\mathcal{K}_{s,\theta_{inf}}(t).|s(t)|^{-\alpha(0)}(|s(t)|^{-1}\varepsilon)^\varepsilon dt\Big)\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}. \\
&\gtrsim \frac{(1-2^{\zeta_\varepsilon p})^{1/p}.\varepsilon^{-\beta_\varepsilon}}{(1-2^{\beta_\varepsilon p})^{1/p}.2^{\zeta_\varepsilon}}\varepsilon^{-\varepsilon}\Big(\int\limits_{U^s_\varepsilon} \psi(t).\mathcal{K}_{s,\theta_{inf}}(t).|s(t)|^{-\alpha(0)}(|s(t)|^{-1}\varepsilon)^\varepsilon dt\Big)\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}}.
\end{aligned} \tag{17}$$

On the other hand, by (15) and $\lim_{\varepsilon\to 0^+} \varepsilon^{-\beta_\varepsilon} = 1$, we deduce

$$\lim_{\varepsilon\to 0^+} \frac{(1-2^{\zeta_\varepsilon p})^{1/p}.\varepsilon^{-\beta_\varepsilon}}{(1-2^{\beta_\varepsilon p})^{1/p}.2^{\zeta_\varepsilon}}\varepsilon^{-\varepsilon} = \begin{cases} 1, & \text{if } q_+ = q_-, \\ (q_+^2/q_-^2)^{1/p}, & \text{if } \alpha_\infty = 0. \end{cases}.$$

From this, by (17) and the dominated convergence theorem of Lebesgue, it is clear to see that

$$C_{2,\inf} \lesssim \left\|U_{\psi,s,d}\right\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}\to\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} < \infty.$$

Thus, the proof for this case is finished. □

Now, we state the boundedness of Hardy–Cesàro operators on weighted central Morrey spaces.

Theorem 3.3. *Let $q(s^{-1}(t)\cdot) = q(\cdot)$ for almost everywhere $t \in \mathrm{supp}(\psi)$, $\kappa > 0$ and $\omega(x) = |x|^\gamma$ with $\gamma > -n$.*
(i) *If*

$$C_{3,\max} = \int_{[0,1]^d} \psi(t).|s(t)|^{\kappa(\gamma+n)-\gamma} \max\Big\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\Big\}dt < \infty,$$

then the operator $U_{\psi,s,d}$ is bounded from $\dot{M}^{\kappa,q(\cdot)}_\omega$ to itself.
(ii) *If $U_{\psi,s,d}$ is bounded from $\dot{M}^{\kappa,q(\cdot)}_\omega$ to itself and $q_+ = q_-$, we have*

$$C_3 = \int_{[0,1]^d} \psi(t).|s(t)|^{\kappa(\gamma+n)-\gamma-\frac{n}{q}} dt < \infty.$$

Moreover,

$$\|U_{\psi,s,d}\|_{\dot{M}^{\kappa,q}_\omega \to \dot{M}^{\kappa,q}_\omega} \simeq C_3.$$

Proof. (i) By the same arguments as (3) above, one has

$$\big\|U_{\psi,s,d}(f)\big\|_{L^{q(\cdot)}_\omega(B_k)} \lesssim \int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{sup}}(t).\big\|f\big\|_{L^{q(\cdot)}_\omega(B_{k+\ell})} dt,$$

where $\ell = \ell(t) \in \mathbb{Z}$ such that $2^{\ell-1} < |s(t)| \le 2^\ell$. Accordingly, we infer

$$\begin{aligned}\|U_{\psi,s,d}(f)\|_{\dot{M}^{\kappa,q(\cdot)}_\omega} &= \sup_{k\in\mathbb{Z}} \frac{1}{\omega(B_k)^\kappa} \|U_{\psi,s,d}(f)\|_{L^{q(\cdot)}_\omega(B_k)} \\ &\lesssim \sup_{k\in\mathbb{Z}} \Big(\int\limits_{[0,1]^d} \psi(t).\mathcal{K}_{s,\theta_{sup}}(t).\frac{\omega(B_{k+\ell})^\kappa}{\omega(B_k)^\kappa}.dt\Big) \|f\|_{\dot{M}^{\kappa,q(\cdot)}_\omega}. \end{aligned} \tag{18}$$

Here

$$\mathcal{K}_{s,\theta_{sup}}(t) = \mathrm{ess\,supp}_{z\in\mathbb{R}^n} \frac{\omega(s^{-1}(t)z)}{\omega(z)}.\max\Big\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\Big\} = |s(t)|^{-\gamma}\max\Big\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\Big\}.$$

And by $|s(t)| \simeq 2^\ell$, we have

$$\frac{\omega(B_{k+\ell})}{\omega(B_k)} \simeq \frac{2^{(k+\ell)(\gamma+n)}}{2^{k(\gamma+n)}} \simeq |s(t)|^{\gamma+n}.$$

From these, by (18), we obtain

$$\|U_{\psi,s,d}(f)\|_{\dot{M}^{\kappa,q(\cdot)}_\omega} \lesssim C_{3,\max}.\|f\|_{\dot{M}^{\kappa,q(\cdot)}_\omega}.$$

This gives the proof of the case (i) is ended.

(ii) Suppose that $U_{\psi,s,d}$ is a bounded operator from $\dot{M}^{\kappa,q(\cdot)}_\omega$ to itself and $q_+ = q_-$. Let us choose

$$h_0(x) = |x|^{\kappa(n+\gamma)-\frac{n}{q}}.\omega^{-1}(x).$$

It is not hard to see that

$$\|h_0\|_{L^q_\omega(B_k)} = \Big(\int\limits_{B_k} |x|^{\kappa(n+\gamma).q-n} dx\Big)^{1/q} \simeq 2^{k.\kappa(n+\gamma)}.$$

Thus

$$\|h_0\|_{\dot{M}^{\kappa,q}_\omega} = \sup_{k\in\mathbb{Z}} \frac{1}{\omega(B_k)^\kappa} \|h_0\|_{L^q_\omega(B_k)} \simeq 1.$$

Besides, we deduce

$$U_{\psi,s,d}(h_0)(x) = \Big(\int_{[0,1]^d} \psi(t).|s(t)|^{\kappa(n+\gamma)-\frac{n}{q}-\gamma} dt\Big)|x|^{\kappa(n+\gamma)-\frac{n}{q}}.\omega(x)^{-1} = C_3.h_0(x).$$

Instanly, we obtain

$$C_3 \leq \|U_{\psi,s,d}\|_{\dot{M}^{\kappa,q}_{\omega} \to \dot{M}^{\kappa,q}_{\omega}} < \infty.$$

Hence, we achieve the proof of this case. □

Finally, we present the boundedness for the Hardy-Cesàro operators on weighted local central Morrey spaces and weighted non-local central Morrey spaces with variable exponent as follows.

Theorem 3.4. *Let $q(s^{-1}(t)\cdot) = q(\cdot)$ for almost everywhere $t \in \mathrm{supp}(\psi)$, $\kappa > 0$ and $\omega(x) = |x|^{\gamma}$ with $\gamma > -n$.*
(i) *If $C_{3,\max}$ in Theorem 3.3 is finite, then the operator $U_{\psi,s,d}$ is bounded from $\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}$ to itself.*
(ii) *If $U_{\psi,s,d}$ is bounded from $\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}$ to itself, we then have*

$$\tilde{C}_{3,\min} = \int_{[0,1]^d} \psi(t)|s(t)|^{-\gamma+\kappa(n+\gamma)q_+/q_-} \min\left\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\right\} dt < \infty.$$

Furthermore,

$$\tilde{C}_{3,\min} \leq \|U_{\psi,s,d}\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}} \to \mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}}.$$

Proof. By estimating as in the proof of Theorem 3.3(i), we immediately have

$$\|U_{\psi,s,d}\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}} \to \mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} \lesssim C_{3,\max}.$$

Hence, we accomplish the proof of case (i). Next, let us prove the case (ii). For any $m \in \mathbb{N}^*$, we set

$$g_m(x) = |x|^{\mu_m \kappa(n+\gamma) - \frac{n}{q(x)}}.\omega^{-1}(x),$$

with $\mu_m = \left(1 + \frac{1}{m}\right)\frac{q_+}{q_-}$. Thus, it is easy to see that

$$\|g_m\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} > 0, \text{ for any } m \in \mathbb{N}^*.$$

On the other hand, for any $R \in (0,1)$ and $m \in \mathbb{N}^*$, we have

$$\begin{aligned} F_q(g_m.\omega.B(0,R)) &= \int_{B(0,R)} |x|^{\mu_m \kappa(n+\gamma)q(x)-n} dx = \int_0^R \int_{S^{n-1}} r^{\mu_m \kappa(n+\gamma)q(rx')-1} d\sigma(x') dr \\ &\lesssim \int_0^R r^{\min\left\{\mu_m \kappa(n+\gamma)q_+ - 1, \mu_m \kappa(n+\gamma)q_- - 1\right\}} dr = \frac{R^{\mu_m \kappa(n+\gamma)q_-}}{\mu_m \kappa(n+\gamma)q_-}. \end{aligned}$$

Consequently, by using the inequality (1), we get

$$\begin{aligned} \|g_m\|_{L^{q(\cdot)}_{\omega}(B(0,R))} &\lesssim \max\left\{\frac{R^{\mu_m \kappa(n+\gamma)}}{[\mu_m \kappa(n+\gamma)q_-]^{1/q_-}}, \frac{R^{\mu_m \kappa(n+\gamma)q_-/q_+}}{[\mu_m \kappa(n+\gamma)q_-]^{1/q_+}}\right\} \\ &\lesssim \eta_m R^{\mu_m \kappa(n+\gamma)q_-/q_+}, \text{ for all } R \in (0,1) \text{ and } m \in \mathbb{N}^*. \end{aligned}$$

Here $\eta_m = \max\left\{[\mu_m \kappa(n+\gamma)q_-]^{-1/q_-}, [\mu_m \kappa(n+\gamma)q_-]^{-1/q_+}\right\}$.
Thus, for any $m \in \mathbb{N}^*$, we obtain

$$\|g_m\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} = \sup_{0<R<1} \frac{1}{\omega(B(0,R))^{\kappa}} \|g_m\|_{L^{q(\cdot)}_{\omega(B(0,R))}} = \eta_m \sup_{0<R<1} R^{\kappa(n+\gamma)[\mu_m \kappa(n+\gamma)q_-/q_+ - 1]} = \eta_m < \infty.$$

Besides, one has

$$U_{\psi,s,d}(g_m)(x) = \Big(\int_{[0,1]^d} \psi(t).|s(t)|^{\mu_m \kappa(n+\gamma) - \frac{n}{q(x)} - \gamma} dt\Big) g_m(x) \geq C_{3,m,\min}.g_m(x).$$

Here

$$C_{3,m,\min} = \int_{[0,1]^d} \psi(t).|s(t)|^{\mu_m\kappa(n+\gamma)-\gamma} \min\left\{|s(t)|^{\frac{-n}{q_-}}, |s(t)|^{\frac{-n}{q_+}}\right\} dt.$$

From this, by $U_{\psi,s,d}$ is a bounded operator from $\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}$ to itself, we infer

$$C_{3,m,\min} \le \|U_{\psi,s,d}\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}} \to \mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} < \infty.$$

By applying $\lim\limits_{m\to\infty} \mu_m = \frac{q_+}{q_-}$ and the dominated convergence theorem of Lebesque, we get

$$\tilde{C}_{3,\min} \le \|U_{\psi,s,d}\|_{\mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}} \to \mathcal{B}^{q(\cdot),\kappa}_{\omega,\mathrm{loc}}} < \infty.$$

Hence, the proof for this case is solved. □

Theorem 3.5. *Let $q(s^{-1}(t)\cdot) = q(\cdot)$ for almost everywhere $t \in \mathrm{supp}(\psi)$, $\kappa > 0$ and $\omega(x) = |x|^\gamma$ with $\gamma > -n$.*
(i) If $C_{3,\max}$ in Theorem 3.3 is finite, then the operator $U_{\psi,s,d}$ is bounded from $B^{q(\cdot),\kappa}_\omega$ to itself.
(ii) If $U_{\psi,s,d}$ is bounded from $B^{q(\cdot),\kappa}_\omega$ to itself, we then have

$$C^*_{3,\min} = \int_{[0,1]^d} \psi(t)|s(t)|^{-\gamma+\kappa(n+\gamma)q_-/q_+} \min\left\{|s(t)|^{\frac{-n}{q_+}}, |s(t)|^{\frac{-n}{q_-}}\right\} dt < \infty.$$

Moreover,

$$C^*_{3,\min} \le \|U_{\psi,s,d}\|_{B^{q(\cdot),\kappa}_\omega \to B^{q(\cdot),\kappa}_\omega}.$$

Proof. From Theorem 3.3(i), let us obtain the proof of case (i). Now, we will prove case (ii). By choosing the functions g_m as in Theorem 3.4 with $\mu_m = \left(1 - \frac{1}{m}\right)\frac{q_-}{q_+}$, we also have

$$\|g_m\|_{B^{q(\cdot),\kappa}_\omega} > 0, \text{ for any } m \in \mathbb{N}^*.$$

On the other hand, for any $R \ge 1$ and $m \in \mathbb{N}^*$, we infer

$$\begin{aligned}
F_q(g_m.\omega.B(0,R)) &= \int_0^R \int_{S^{n-1}} r^{\mu_m\kappa(n+\gamma)q(rx')-1} d\sigma(x')dr \\
&= \int_0^1 \int_{S^{n-1}} r^{\mu_m\kappa(n+\gamma)q(rx')-1} d\sigma(x')dr + \int_1^R \int_{S^{n-1}} r^{\mu_m\kappa(n+\gamma)q(rx')-1} d\sigma(x')dr \\
&\lesssim \int_0^1 r^{\mu_m\kappa(n+\gamma)q_- -1} dr + \int_1^R r^{\mu_m\kappa(n+\gamma)q_+ -1} dr \\
&= \frac{R^{\mu_m\kappa(n+\gamma)q_+}}{\mu_m\kappa(n+\gamma)q_+} + \nu_m,
\end{aligned}$$

where $\nu_m = \frac{1}{\mu_m\kappa(n+\gamma)q_-} - \frac{1}{\mu_m\kappa(n+\gamma)q_+} \ge 0$. Thus, by applying the inequality (1) again, we deduce

$$\begin{aligned}
\|g_m\|_{L^{q(\cdot)}_\omega(B(0,R))} &\lesssim \max\left\{\left(\frac{R^{\mu_m\kappa(n+\gamma)q_+}}{\mu_m\kappa(n+\gamma)q_+} + \nu_m\right)^{\frac{1}{q_+}}, \left(\frac{R^{\mu_m\kappa(n+\gamma)q_+}}{\mu_m\kappa(n+\gamma)q_+} + \nu_m\right)^{\frac{1}{q_-}}\right\} \\
&\lesssim \max\left\{\frac{R^{\mu_m\kappa(n+\gamma)}}{[\mu_m\kappa(n+\gamma)q_+]^{1/q_+}} + \nu_m{}^{1/q_+}; \frac{R^{\mu_m\kappa(n+\gamma)q_+/q_-}}{[\mu_m\kappa(n+\gamma)q_+]^{1/q_-}} + \nu_m{}^{1/q_-}\right\} \\
&\lesssim \tilde{\eta}_m R^{\mu_m\kappa(n+\gamma)q_+/q_-} + \tilde{\nu}_m, \text{ for any } R \ge 1 \text{ and } m \in \mathbb{N}^*.
\end{aligned}$$

with $\tilde{\eta}_m = \max\left\{[\mu_m\kappa(n+\gamma)q_+]^{-1/q_+}; [\mu_m\kappa(n+\gamma)q_+]^{-1/q_-}\right\}$ and $\tilde{\nu}_m = \max\left\{\nu_m^{1/q_+}, \nu_m^{1/q_-}\right\}$.
Hence, from the definition of non-local central Morrey spaces, one has

$$\begin{aligned}\|g_m\|_{B_\omega^{q(\cdot),\kappa}} &= \sup_{R\geq 1}\frac{1}{\omega(B(0,R))^\kappa}\|g_m\|_{L_\omega^{q(\cdot)}(B(0,R))}\\ &\lesssim \sup_{R\geq 1} R^{-\kappa(n+\gamma)}\left(\tilde{\eta}_m\cdot R^{\mu_m\kappa(n+\gamma)q_+/q_-}+\tilde{\nu}_m\right)\\ &= \sup_{R\geq 1}\left\{\tilde{\eta}_m\cdot R^{\kappa(n+\gamma)(\mu_m q_+/q_- -1)}+\tilde{\nu}_m\cdot R^{-\kappa(n+\gamma)}\right\}\\ &\lesssim \tilde{\eta}_m+\tilde{\nu}_m<\infty, \text{ for any } m\in\mathbb{N}^*.\end{aligned}$$

By the same arguments as in the final section of the proof of Theorem 3.4(ii) and $\lim\limits_{m\to\infty}\mu_m = \frac{q_-}{q_+}$, we also obtain

$$C_{3,\min}^* \leq \|U_{\psi,s,d}\|_{B_\omega^{q(\cdot),\kappa}\to B_\omega^{q(\cdot),\kappa}} < \infty.$$

Therefore, we accomplish the proof of this case. □

Acknowledgments

The authors are grateful to the anonymous reviewer for the valuable suggestions and comments which lead to the improvement of the paper.

References

[1] A. Almeida, D. Drihem, Maximal, Potential and singular type operators on Herz spaces with variable exponents, Journal of Mathematical Analysis and Applications 394 (2012) 781–795.
[2] R. Ayazoglu (Mashiyev), S. Akbulut and E. Akkoyunlu, Uniform boundedness of Kantorovich operators in variable exponent Lebesgue spaces, Filomat 33(18) (2019) 5755–5765.
[3] R. A. Bandaliev, The boundedness of multidimensional Hardy operators in weighted variable Lebesgue spaces, Lithuanian Mathematical Journal 50 (2010) 249–259.
[4] C. Capone, D. Cruz-Uribe and A. Fiorenza, The fractional maximal operator and fractional integrals on variable L_p spaces, Revista Matematica Iberoamericana 23 (2007) 743–770.
[5] C. Carton-Lebrun and M. Fosset, Moyennes et quotients de Taylor dans BMO, Bull Soc Roy Sci Liége 53 (1984) 85–87.
[6] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Springer-Basel, 2013.
[7] N. M. Chuong, D. V. Duong and K. H. Dung, Multilinear Hausdorff operator on variable exponent Morrey–Herz type spaces, Integral Transforms and Special Functions 31 (2020) 62–86.
[8] N. M. Chuong, D. V. Duong and H. D. Hung, Bounds for the weighted Hardy–Cesàro operator and its commutator on weighted Morrey–Herz type spaces, Journal of Analysis and its Applications 35 (2016) 489–504.
[9] N. M. Chuong, N. T. Hong and H. D. Hung, Multilinear Hardy–Cesàro operator and commutator on the product of Morrey–Herz spaces, Analysis Mathematica 43 (2017) 547–565.
[10] N. M. Chuong and H. D. Hung, Bounds of weighted Hardy–Cesàro operators on weighted Lebesgue and BMO spaces, Integral Transforms and Special Functions 25 (2014) 697–710.
[11] D. V. Duong, K. H. Dung and N. M. Chuong, Weighted estimates for commutators of multilinear Hausdorff operators on variable exponent Morrey–Herz type spaces, Czechoslovak Mathematical Journal 70 (2020) 833–865.
[12] L. Diening and M. Ružička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics, Journal fur die Reine und Angewandte Mathematik 563 (2003) 197–220.
[13] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, Springer-Verlag, 2011.
[14] J. E. Galé, P. J. Miana and L. Sánchez-Lajusticia, RKH spaces of Brownian type defined by Cesàro-Hardy operators, Analysis and Mathematical Physics 11(119) (2021).
[15] V. S. Guliyev, J. Hasanov and S. Samko, Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces, Mathematica Scandinavica 107 (2010) 285–304.
[16] G. H. Hardy, Note on a theorem of Hilbert, Mathematische Zeitschrift 6 (1920) 314–317.
[17] H. D. Hung and L. D. Ky, New weighted multilinear operators and commutators of Hardy–Cesàro type, Acta Math Sci Ser B Engl Ed 35 (2015) 1411–1425.
[18] W. Hoh, Pseudodifferential operators with negative definite symbols of varable order, Revista Matematica Iberoamericana 18 (2000) 219–241.
[19] V. Kokilashvili and S. Samko, The maximal operator in weighted variable exponent spaces on metric spaces, Georgian Mathematical Journal 15 (2008) 683–712.
[20] S. Z. Lu, D. C. Yang and G. E. Hu, Herz type spaces and their applications, Beijing Sci Press, 2008.

[21] Y. Lu and Y. P. Zhu, Boundedness of multilinear Calderón-Zygmund singular operators on Morrey–Herz spaces with variable exponents, Acta Math Sin (Engl Ser) 30 (2014) 1180–1194.

[22] F. I. Mamedov and A. Harman, On a Hardy type general weighted inequality in spaces $L^{p(\cdot)}$, Integral Equations and Operator Theory 66 (2010) 565–592.

[23] R. Mashiyev, B. Çekiç, F. I. Mamedov and S. Ogras, Hardy's inequality in power-type weighted $L^{p(\cdot)}(0,\infty)$, Journal of Mathematical Analysis and Applications 334 (2007) 289–298.

[24] Y. Mizuta, Duality of Herz–Morrey spaces of variable exponent, Filomat 30(7) (2016) 1891–1898.

[25] T. V. Pedersen, Generalized Hardy–Cesàro operators between weighted spaces, Glasgow Mathematical Journal 61 (2019) 13–24.

[26] Y. Sawano, X. Tian and J. Xu, Uniform boundedness of Szász-Mirakjan-Kantorovich operators in Morrey spaces with variable exponents, Filomat 34(7) (2020) 2109–2121.

[27] J. L. Wu and W. J. Zhao, Boundedness for fractional Hardy-type operator on variable exponent Herz–Morrey spaces, Kyoto Journal of Mathematics 56 (2016) 831–845.

[28] L. W. Wang and L. S. Shu, Higher order commutators of fractional integrals on Morrey type spaces with variable exponents, Mathematische Nachrichten 291 (2018) 1437–1449.

[29] L. W. Wang and L. S. Shu, Boundedness of some sublinear operators on weighted variable Herz–Morrey spaces, Journal of Mathematical Inequalities 12 (2018) 31–42.

[30] J. Xiao, L^p and BMO bounds of weighted Hardy–Littlewood averages, Journal of Mathematical Analysis and Applications 262 (2001) 660–666.