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On metrics projectively and holomorphically projectively equivalent to metrics of parabolic Riemannian and Kähler manifolds

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Abstract. We prove a number of nonexistence theorems for metrics projectively and holomorphically projectively equivalent to parabolic metrics and metrics of finite volume of complete Riemannian and Kähler manifolds, respectively.

1. Introduction

Two metrics on a manifold are said to be pointwise projectively equivalent if the geodesics of one are, after suitable reparametrization, the geodesics of the other. Pointwise projectively equivalent metrics on manifolds are a classical object of research in differential geometry. Back in 1865, Beltrami was the first to formulate the problem of finding all pairs of projectively equivalent metrics. For example, his classical theorem states that a metric projectively equivalent to a metric of constant curvature is itself a metric of constant curvature. Later, projectively equivalent metrics were considered by Dini, Levi-Civita, Weyl, Cartan, Thomas, Eisenhart, Shirokov, Sinyukov, Solodovnikov, Petrov, Lichnerowicz, Aminova, Mikeš, Venzi, Formella, Sobchuk, Voss, Taber, Pogorelov, Matveev and other geometers. The theory of pointwise projectively equivalent metrics has a very long and interesting history, which is described in more detail in the voluminous monograph [20] of the authors of this article. Moreover, many beautiful local tensor properties of projectively equivalent metrics can be found in this monograph and the following paper of last years [3–6, 16, 17, 21, 28, 29, 31].

However, the global behavior of projectively equivalent metrics is not understood completely. To make up for this, we will consider metrics that are pointwise projectively equivalent to the metrics of complete non-compact Riemannian manifolds.

We also consider Kähler metrics that are pointwise holomorphically projectively equivalent to the metrics of complete Kähler manifolds (see details in [13], [19] and [24]). In both cases, we mainly focus on parabolic Riemannian and Kähler manifolds and, in particular, manifolds of finite volume. By applying

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Liouville-type theorems for subharmonic and superharmonic functions on complete non-compact Riemannian manifolds, we prove some vanishing theorems for nontrivial projective transformations of metrics of complete Riemannian manifolds and nontrival holomorphic projective transformations of metrics of complete Kähler manifolds. In particular, we generalize theorems from [8] and [12] proved for compact Riemannian manifolds. All results of this article belong to the *generalized Bochner technique* that falls under the general heading of "curvature and topology of a complete Riemannian manifold" (see, for example, [10], [15], [25]). On the other hand, the *classical Bochner technique* works on compact Riemannian manifolds (see [35] and [23, p. 333–363], for instance). More details about the Bochner technique and its generalization can be found in the monograph [33].

This paper is organized as follows. In the next section of the paper, we consider subharmonic and superharmonic functions on complete noncompact parabolic Riemannian manifolds and, in particular, manifolds of finite volume. In the other two sections, we give applications of these results to the theories of projectively equivalent Riemannian metrics on a connected non-compact smooth manifold and holomorphically projective equivalent Kähler metrics on a connected manifold with an almost complex structure. More precisely, we shall prove a number of nonexistence theorems for metrics projectively and holomorphically projectively equivalent to parabolic metrics and metrics of finite volume of complete Riemannian and Kähler manifolds, respectively.

In the presented paper, we continue our research begun in [2] and [27].

2. Vanishing theorems for subharmonic and superharmonic functions on a complete Riemannian manifold having finite volume

We further assume that (M, g) is a connected bounderless complete Riemannian manifold with the Levi-Civita connection ∇. The Laplace-Beltrami operator with respect to the Riemannian metric g is defined by the identity $\Delta := \operatorname{div} \circ \nabla$. In this case, the function $u\colon M \to \mathbb{R}$ is said to be *harmonic* if it satisfies the equation $\Delta u = 0$. In turn, when the equality is replaced by the inequality $\Delta u \geq 0$ we will say that u is *subharmonic*. Finally reversing the inequality $\Delta u \leq 0$ we get a *superharmonic* function.

Remark 2.1 Easy application of the *Hoph maximum principle* shows that (see [35, p. 30]): every harmonic (subharmonic, superharmonic) function is constant on a compact manifold (M, g). Therefore, throughout the article, we will consider these functions defined on complete non-compact manifolds.

At the same time, we know a similar statement from [1] about superharmonic functions on a complete noncompact Riemannian manifold with finite volume.

Theorem 2.1. *A complete Riemannian manifold of finite volume does not carry non-constant positive superharmonic functions.*

We recall here that a complete Riemannian manifold is said to be *parabolic* if it admits no non-constant positive superharmonic function (see [11] and [10, p. 313], for instance). Then a complete Riemannian manifold of finite volume is an example of a parabolic manifold.

Some of the most natural spaces of smooth functions on a complete manifold (M, g) are those consisting of L^p-functions on (M, g), denoted this space by $L^p(M, g)$, where integration is defined with respect to the Riemannian measure $d\operatorname{vol}_g = \sqrt{\det\left(g_{ij}\right)}\, dx^1 \wedge \ldots \wedge dx^n$ where $x^1, \ldots, x^n$ is a local coordinate system on (M, g). Moreover, the L^p-norm of a function or a tensor is denoted by $\int_M \|\cdot,\cdot\|^p d\, vol_g$ where to simplify the notation, we will write $\|\cdot,\cdot\| = \sqrt{g\,(\cdot,\cdot)}$.

Yau proved the following famous theorem: let u be a nonnegative smooth subharmonic function on a complete Riemannian manifold (M, g), then $\int_M u^p d\operatorname{vol}_g = +\infty$ for any $p > 1$, unless u is a constant function (see Theorem 3 in [37]). In particular, let $u \in L^p(M, g)$ be a positive subharmonic function, then by Yau's theorem it can be a positive constant $C > 0$ for some $p > 1$. In this case, the inequality $\int_M u^p d\operatorname{vol}_g < +\infty$ becomes $C^p \int_M d\operatorname{vol}_g < +\infty$ and, therefore, $\operatorname{Vol}(M, g) < +\infty$. Then we can formulate the following statement which is a modification of Yau's theorem.

Theorem 2.2. *If a complete Riemannian manifold (M, g) admits a nonnegative subharmonic function u such that $u \in L^p(M, g)$ for some $p > 1$, then u must be identically constant. Moreover, in this case (M, g) is a manifold of finite volume, unless the constant is zero.*

Remark 2.2 Obviously, a statement similar to Theorem 2.2 is also true for harmonic L^p-functions on a complete manifold (M, g).

Let u be a subharmonic function defined on a complete manifold of finite volume (M, g). Then the following theorem holds (see [36, p. 318]).

Theorem 2.3. *Let (M, g) be a complete manifold of finite volume. If a subharmonic function u satisfies the integral inequality $\int_M \| du \|^2 \, d\,\mathrm{vol}_g < \infty$ then it is a constant function.*

Convex functions are an example of subharmonic functions. In addition, we recall here that $u \in C^2 M$ is called a *convex function* if its Hessian $\mathrm{Hess}_g u := \nabla du$ is positive semi-definite. In this case, $\Delta u = trace_g\ (\nabla du) \geq 0$ and, therefore, u is a subharmonic function. It's obvious that the existence of subharmonic functions is a much weaker condition than the existence of convex functions.

The following theorem of Bishop and O'Neil is well known: if (M, g) is a connected complete Riemannian manifold having finite volume, then all convex functions on (M, g) are constant (see [7]). On the other hand, Yau gave in [36] a proof of another wonderful result: if (M, g) is a complete Riemannian manifold on which there exists a non-constant convex function then the volume of the manifold is infinite. In this case, taking into account Theorem 2.2, we can conclude that the following statement holds.

Corollary 2.4. *Let (M, g) be a complete Riemannian manifold. Then there no non-constant nonnegative convex L^p-functions for any $p > 1$.*

3. Applications to the theory of pointwise projectively equivalent metrics

Let (M, g) be a connected Riemannian manifold (without boundary) of dimension $n \geq 3$. We say that another metric $\bar{g}$ on M is pointwise projectively equivalent to g, if each geodesic in (M, g) is a geodesic in $(\bar{M}, \bar{g})$, with possibly different parameterizations. As it was known, the two Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$, respectively, have the same geodesics, if and only if these connections are related by Levi-Civita equation (see [14], [9, p. 133] and [20, p. 329])

$$\bar{\nabla} = \nabla + id_{TM} \otimes d\psi + d\psi \otimes id_{TM} \tag{1}$$

for the scalar function

$$\psi = \frac{1}{2\,(n+1)}\ \ln\left(\frac{\det\,\bar{g}}{\det\,g}\right). \tag{2}$$

A geodesic equivalence is called *trivial* or, in the other words, *affine equivalence* if $\bar{\nabla} = \nabla$. In this case, the Riemann curvatures of $\bar{g}$ and g are equal.

We obtain from (1) that the Ricci tensors Ric and $\overline{Ric}$ of g and $\bar{g}$, respectively, are related by equation (see [14], [9, p. 135] and [20, p. 299])

$$\overline{Ric} = Ric + (n-1)\ (\nabla\, d\,\psi - d\,\psi \otimes d\,\psi)\,. \tag{3}$$

The identity (3) can be rewritten in the form

$$\mathrm{Hess}_g \psi = \frac{1}{n-1}\left(\overline{Ric} - Ric\right) + d\psi \otimes d\psi. \tag{4}$$

If we suppose now that ψ is a non-constant function and $\overline{Ric} \geq Ric$, then from (4) we conclude that the Hessian of ψ is positive semi-definite and, therefore, ψ is a convex function. Taking into account the theorem of Bishop and O'Neil, we can formulate the following statement.

Theorem 3.1. *Let (M, g) be a connected complete n-dimensional $(n \geq 3)$ Riemannian manifold having finite volume. Assume that $\bar{g}$ is another metric on (M, g) which is pointwise projectively equivalent to g. If $\overline{Ric} \geq Ric$ for the Ricci curvatures of $\bar{g}$ and g, respectively, then the geodesic equivalence is trivial.*

Let the geodesic equivalence defined by equations (1) be nontrivial, then the function ψ defined by formula (2) is a non-constant convex function. Taking into account Yau's theorem, we can formulate the following statement.

Theorem 3.2. *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 3$. Let $\bar{g}$ be another Riemannian metric on (M, g) which is pointwise projectively equivalent to g. If the geodesic equivalence is not trivial and $\overline{Ric} \geq Ric$ for the Ricci curvatures of $\bar{g}$ and g, respectively, then (M, g) has an infinite volume.*

From (4) we obtain the following equation

$$\Delta \psi = \frac{1}{n-1} \left(\mathrm{trace}_g \overline{Ric} - s \right) + \left\| d\psi \right\|^2 \tag{5}$$

where $\Delta \psi = \mathrm{trace}_g (\nabla d\psi)$ and $s = \mathrm{trace}_g Ric$ is the scalar curvature of (M, g). If we suppose that $\mathrm{trace}_g \overline{Ric} \geq s$ (and, in particular, $\overline{Ric} \geq Ric$), then from (5) we obtain $\Delta \psi \geq 0$. In this case, ψ is a subharmonic function. If it belongs to $L^p(M, g)$ for some $1 < p < +\infty$, then by Theorem 2.2 the manifold (M, g) has a finite volume. At the same time, we proved above that (M, g) has an infinite volume if $\overline{Ric} \geq Ric$. Therefore, we can formulate the corollary of our Theorem 2.1.

Corollary 3.3. *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 3$. Let $\bar{g}$ be another metric on (M, g) such that $\overline{Ric} \geq Ric$ for the Ricci curvatures of the metrics $\bar{g}$ and g, respectively. If the scalar function ψ defined by formula (2) satisfies the condition $\int_M \psi^p d\mathrm{vol}_g < +\infty$ for some $1 < p < +\infty$, then $\bar{g}$ and g can not be pointwise projectively equivalent.*

Remark 3.1 In contrast to our results, it was shown in [8] that if two Riemannian metrics $\bar{g}$ and g are pointwise projectively equivalent and their Ricci curvatures satisfy $\overline{Ric} \leq Ric$, then the geodesic equivalence is trivial provided that g is complete. In this case, $\bar{\nabla} = \nabla$ and, therefore, the Riemann curvatures of g and $\bar{g}$ are equal. It is easy to conclude that the projective equivalence is an affine equivalence.

Let now (M, g) be a complete manifold of finite volume. Then from Theorem 2.2 we conclude that if the subharmonic function ψ defined by formula (2) satisfies the integral inequality $\int_M \left\| d\psi \right\|^2 d\,\mathrm{vol}_g < \infty$, then it is a constant function. Therefore, the following theorem holds.

Theorem 3.4. *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 3$ having finite volume. Suppose that there is another metric $\bar{g}$ that is pointwise projectively equivalent to the metric g. If $\mathrm{trace}_g \overline{Ric} \geq s$ for the Ricci tensor $\overline{Ric}$ and the scalar curvature s of $\bar{g}$ and g, respectively, and $\left\| d\psi \right\| \in L^2(M, g)$ for the scalar function (2), then the metrics $\bar{g}$ and g are affine equivalent. Moreover, if (M, g) is irreducible, then $\bar{g} = \mathrm{const} \cdot g$.*

Remark 3.2 Recall that a Riemannian manifold (M, g) is *irreducible* if it cannot be represented as a non-trivial Riemannian product. It is irreducible if and only if its holonomy group $\mathrm{Hol}\,(g_x)$ acts irreducibly on $T_x M$ for any $x \in M$. In this case, if $\varphi \in C^\infty S^2 M$ is parallel with respect to the Levi-Civita connection of g, then $\varphi = c \cdot g$ for some constant c.

The equality (2) can be rewritten in the form

$$\psi = \frac{1}{n+1} \ln \left(\frac{d\mathrm{vol}_{\bar{g}}}{d\mathrm{vol}_{\bar{g}}} \right) \tag{6}$$

where $d\,\mathrm{vol}_{\bar{g}} = \sqrt{\det \left(\bar{g}_{ij} \right)}\; dx^1 \wedge \ldots \wedge dx^n$ is the Riemannian measure of $\bar{g}$. From (6) we obtain the equality

$$d\,\mathrm{vol}_{\bar{g}} = e^{(n+1)\,\psi} d\,\mathrm{vol}_g. \tag{7}$$

which is true at every point $x \in M$. In this case, if we denote by $f = e^{\psi} > 0$, then (7) can be rewritten in the form

$$d\,\mathrm{vol}_{\bar{g}} = f^{(n+1)} d\,\mathrm{vol}_g. \tag{8}$$

Therefore, if we assume that the complete Riemannian manifold $(\bar{M}, \bar{g})$ has a finite volume, then from (8) we obtain the inequality

$$\int_M f^{(n+1)} d\,\mathrm{vol}_g < \infty.$$

Moreover, in this case (5) can be rewritten in the form

$$\Delta f = \frac{1}{n-1}\left(\overline{Ric} - Ric\right) f + \frac{2}{f}\,\left\| df \right\|^2. \tag{9}$$

If we suppose here that $\mathrm{trace}_g\overline{Ric} \geq s$, then from (9) we obtain $\Delta f \geq 0$. Then f is a positive subharmonic function such that $f \in L^{(n+1)}(M, g)$. Taking into account Theorem 2.1, we can conclude that the following theorem holds.

Theorem 3.5. *Let $\bar{g}$ and g be two complete Riemannian metrics on a connected smooth manifold M such that they are pointwise projectively equivalent and their Ricci curvature $\overline{Ric}$ and scalar curvature s satisfy the inequality* $\mathrm{trace}_g\overline{Ric} \geq s$. *If the Riemannian manifold $(\bar{M}, \bar{g})$ has a finite volume, then the geodesic equivalence is trivial. Moreover, if (M, g) is irreducible, then $\bar{g} = \mathrm{const} \cdot g$.*

On the other hand, if we assume now that $\psi = -\ln f$ then

$$f = \left(\frac{\det\, g}{\det\, \bar{g}}\right)^{\frac{1}{2(n+1)}}. \tag{10}$$

In this case, (4) can be rewritten in the form

$$\nabla df = \frac{1}{n-1}\left(Ric - \overline{Ric}\right) f. \tag{11}$$

Therefore, the inequalities $\nabla df \geq 0$ and $Ric \geq \overline{Ric}$ are equivalent. Taking into account Remark 3.1, we can formulate the following statement.

Corollary 3.6. *Let (M, g) be a connected complete n-dimensional $(n \geq 3)$ Riemannian manifold. Suppose that $\bar{g}$ is another metric on (M, g) such that function (10) is a non-constant convex function, then the metrics g and $\bar{g}$ are not pointwise projectively equivalent.*

In turn, from (11) we obtain

$$\Delta f = \frac{1}{n-1}\left(\mathrm{trace}_g\overline{Ric} - s\right) f. \tag{12}$$

If $\mathrm{trace}_g\overline{Ric} \leq s$ then from (12) we obtain the inequality $\Delta f \leq 0$. Then f is a positive superharmonic function. In this case, we can formulate the following theorem in accordance with Theorem 2.3.

Theorem 3.7. *Let (M, g) be a connected complete n-dimensional $(n \geq 3)$ parabolic Riemannian manifold (in particular, a complete manifold of finite volume). Suppose there is another metric $\bar{g}$ on (M, g) which is pointwise projectively equivalent to the metric g. If $\mathrm{trace}_g\overline{Ric} \leq s$ for the Ricci tensor $\overline{Ric}$ and the scalar curvature s of $\bar{g}$ and g, respectively, then the metrics $\bar{g}$ and g are affine equivalent. Moreover, if (M, g) is irreducible, then $\bar{g} = \mathrm{const} \cdot g$.*

Remark 3.2 Our Theorems 2.3, 3.1 and 3.2 generalize similar results from [12] and [8], which proved in the case of compact Riemannian manifolds.

4. Applications to the theory of pointwise holomophically projectively equivalent metrics

Let (M, J) be an almost complex manifold, where M is a connected smooth $2n$-dimensional manifold (without boundary) and J is a smooth section of the tensor bundle $T^*M \otimes TM$ such that $J^2 = \varepsilon\, \mathrm{id}_{TM}$ for $\varepsilon = -1$ in the classical (elliptic) and $\varepsilon = +1$ in the hyperbolic case. The Riemannian metric g on (M, J) is *Kähler* if $g(J, J) = \varepsilon\, g$ and $\nabla J = 0$ for the Levi-Civita connection ∇ of the metric g. The triplet (M, g, J) is called a *Kähler manifold* (see [20, p. 160] and [11] for instance).

Remark 4.1 It is well known (see [34]) that a projective diffeomorphism of Kähler manifolds under preserving the structure is an affine mapping. Therefore, for Kähler manifolds we consider not projective, but more general holomorphically projective transformations.

A curve $\gamma : I \subset \mathbb{R} \to M$ is said to be *analytically planar* or *holomorphically almost geodesic* (see [30] and [20, p. 485]) if any its tangent vector X_x after translation along this curve from a point $x \in \gamma$ to a point $y \in \gamma$ belongs to the linear span of the vectors X_y and $(JX)_y$.

Two Kähler metrics g and $\bar{g}$ are *holomorphically projectively equivalent* if the identity mapping $\mathrm{id} : (M, g, J) \to (M, \bar{g}, J)$ is holomorphically projective (see [30] and [20, p. 654]) which is equivalent to the fact that any curve γ on M is analytically planar of the metric $\bar{g}$ if and only if it is analytically planar of the metric g have general analytically planar curves. By definition, two Kähler metrics g and $\bar{g}$ are holomorphically projectively equivalent if their Levi-Civita connections ∇ and $\bar{\nabla}$ satisfy (see [13], [19], [20, p. 483] and etc.)

$$\bar{\nabla} = \nabla + id_{TM} \otimes \bar{\psi} + \bar{\psi} \otimes id_{TM} + d\psi \otimes J + J \otimes d\psi \tag{13}$$

where $\bar{\psi}(X) = d\psi\,(JX)$ for an arbitrary vector field X and the scalar function

$$\psi = \frac{1}{2\,(n+2)}\ \ln\left(\frac{\det\, \bar{g}}{\det\, g}\right).$$

From (13) we obtain the differential equations (see [19] and [20, p. 485])

$$\overline{Ric} = Ric + (n+2)\ (\nabla\, d\,\psi - d\,\psi \otimes d\,\psi - \varepsilon\ ((d\psi)\,J) \otimes ((d\psi)\,J)) \tag{14}$$

where $d\psi = (n+2)^{-1}\, d\, \ln\left(d\mathrm{vol}_g \big/ d\mathrm{vol}_{\bar{g}}\right)$. Equations (14) can be rewritten in the form

$$\mathrm{Hess}_g \psi = \frac{1}{n+2}\left(\overline{Ric} - Ric\right) + d\psi \otimes d\psi + \varepsilon\ ((d\psi)\,J) \otimes ((d\psi)\,J)\,. \tag{15}$$

In this case, we can formulate statement which is analog of our Theorem 3.1.

Theorem 4.1. *Let (M, g, J) be a connected $2n$-dimensional ($n \geq 2$) complete Kähler manifold of hyperbolic type of finite volume and $\bar{g}$ be another Kähler metric on (M, J) such that it is pointwise holomorphically projectively equivalent to g. If $\overline{Ric} \geq Ric$ for the Ricci curvatures of $\bar{g}$ and g, respectively, then these metrics are affine equivalent.*

Taking the convolutions of the left- and right-hand sides of (15) with components of the tensor g^{-1} we obtain the equation

$$\Delta\,\psi = \frac{1}{n+2}\left(\mathrm{trace}_g \overline{Ric} - s\right) + 2\,\left\| d\psi \right\|^2. \tag{16}$$

Then, we can formulate a statement which is analog of our Theorem 3.4.

Theorem 4.2. *Let (M, g, J) be a connected $2n$-dimensional ($n \geq 2$) complete Kähler manifold of finite volume and $\bar{g}$be another Kähler metric on (M, g, J) such that it is pointwise holomorphically projectively equivalent to g and $\| d\psi \| \in L^2(M, g)$ for $d\psi = (n+2)^{-1} d\, \ln\,(d\mathrm{vol}_g / d\mathrm{vol}_{\bar{g}})$. If $\mathrm{trace}_g \overline{Ric} \geq s$ for the Ricci tensor $\overline{Ric}$ of $\bar{g}$ and the scalar curvature s of g, then the metrics $\bar{g}$ and g are affine equivalent.*

In turn, from (16) we obtain

$$\Delta f = \frac{1}{2(n+2)} \left(s - \mathrm{trace}_g \overline{Ric} \right) f$$

for $\psi = -2 \ln f$. Therefore, if $s \leq \mathrm{trace}_g \overline{Ric}$ then f is a positive superharmonic function. In this case, we can formulate a statement which is analog of our Theorem 3.7.

Theorem 4.3. *Let (M, g, J) be a connected $2n$-dimensional $(n \geq 2)$ complete parabolic Kähler manifold (in particular, a complete Kähler manifold of finite volume) and $\bar{g}$ be another Kähler metric on (M, g, J) such that it is pointwise holomorphically projectively equivalent to g. If $\mathrm{trace}_g \overline{Ric} \geq s$ for the Ricci tensor $\overline{Ric}$ and the scalar curvature s of $\bar{g}$ and g, respectively, then the metrics $\bar{g}$ and g are affine equivalent.*

On the other hand, if (M, g, J) has quasi-positive (resp. quasi-negative) Ricci curvature *Ric* and integrable scalar curvature s, then (M, g, J) is a parabolic manifold (see [22]). Therefore, we can formulate the following statement.

Theorem 4.4. *Let (M, g, J) be a connected $2n$-dimensional $(n \geq 2)$ complete Kähler manifold of quasi-positive (resp. quasi-negative) Ricci curvature Ric and integrable scalar curvature s. Let $\bar{g}$ be another Kähler metric on (M, g, J) such that it is pointwise holomorphically projectively equivalent to g and its Ricci curvature $\overline{Ric}$ satisfies the inequality $\mathrm{trace}_g \overline{Ric} \geq s$, then the metrics $\bar{g}$ and g are affine equivalent.*

Remark 4.2 We recall that (M, g, J) has quasi-positive Ricci curvature if the Ricci curvature nonnegative and positive at one point of (M, g, J). In turn, (M, g, J) has quasi-negative Ricci curvature if the Ricci curvature nonpositive and negative at one point of (M, g, J).

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