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α-Baskakov-Durrmeyer type operators and their approximation properties

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Abstract. In the present research article, we construct a new family of summation-integral type hybrid operators in terms of shape parameter $\alpha \in [0,1]$. Further, basic estimates, rate of convergence and the order of approximation with the aid of Korovkin theorem and modulus of smoothness are investigated. Moreover, numer4ical simulation and graphical approximations are studied. For these sequences of positive linear operators, we study the local approximation results using Peetre's K-functional, Lipschitz class and modulus of smoothness of second order. Next, we obtain the approximation results in weighted space. Lastly, A-statistical-approximation results are presented.

1. Introduction

The theory of linear positive operators deals with question that arise in the approximate representation of an arbitrary function by the simplest analytical expedients possible. Operator theory is a growing and fascinating field of research of approximation theory for the last two decades with the advent of computer. Several Mathematicians, *e.g.*, Acar et al. ([1], [2]), Moihuddine et al. [3], Ana et al. [4], İçöz et al. ([5]), [6]), Kajla et al. ([7], [8]) constructed new sequences of linear positive operators to investigate the rapidity of convergence and order of approximation in diffrent functional spaces in terms of several generating functions. In the recent past, for $g \in [0,1]$, $m \in \mathbb{N}$ and $\alpha \in [-1,1]$, Chen et al. [10], and Nadeem et al. ([24], [25], [**?**]), constructed a sequnce of new linear positive operators as:

$$T_{m,\alpha}(g;y) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p_{m,i}^{\alpha}(y) \qquad (y \in [0,1]), \tag{1}$$

where $p_{1,0}^{(\alpha)} = 1-y, \quad p_{1,1}^{(\alpha)} = y$ and

$$\begin{aligned} p_{m,i}^{\alpha}(y) \;\; = \;\; & \left[(1-\alpha)y\binom{m-2}{i} + (1-\alpha)(1-y)\binom{m-2}{i-2} + \alpha y(1-y)\binom{m}{i}\right] \\ & y^{i-1}(1-y)^{m-i-1} \quad (m \geq 2). \end{aligned} \tag{2}$$

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The operators defined in (1) are named as $\alpha-$Berstein operators of order m.

Remark 1.1. *One can not that for $\alpha = 1$, the relation (1) is reduced to classical Bernstein operators [11].*

These operators are restricted for the space of continuous functions only. To approximate the wider class than the class of continuous function, *i.e.*, space of Lebesgue integrable functions, Mohiuddine et al. [12] constructed Kantorovich-type of α-Bernstein operators and Stancu-type α-Bernstein-Kantorovich operators. Cai et al. [13] introduced a generalization of classical Bernstein operators based on shape parameter $\alpha \in [0, 1]$. These operators are termed as $\alpha-$Bernstein operators of degree m and defined as:

$$T_{m,\alpha}(g; u) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p_{m,i}^{\alpha}(u), \qquad (u \in [0, 1]), \tag{3}$$

where $p_{m,i}^{(\alpha)}(u)$ is defined by (2).

Remark 1.2. *Note that, $p_{m,i}^{\alpha}$ in the relation (3) is called $\alpha-$Berstein polynomials of order m and the binomial coefficients*

$$\binom{p}{q} = \begin{cases} \frac{p!}{q!(p-q)!}, & 0 \leq q \leq p \\ 0 & otherwise. \end{cases}$$

Later on, Aral and Erbay [15] introduced the parametric form of Baskakov-Durrmeyer operators as:

$$L_{m,\alpha}(g; z) \;=\; \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(z) g\Big(\frac{s}{m}\Big), \tag{4}$$

where $f \in C_B[0, \infty)$, $m \geq 1$, $z \in [0, \infty)$ and for $\alpha \in [0, 1]$

$$\begin{aligned} P_{m,s}^{\alpha}(z) &= \frac{z^{s-1}}{(1+z)^{m+s-1}} \Bigg\{ \frac{\alpha z}{1+z} \binom{m+s-1}{s} - (1-\alpha)(1+z) \binom{m+s-3}{s-2} \\ &+ (1-\alpha) z \binom{m+s-1}{s} \Bigg\} \end{aligned} \tag{5}$$

with $\binom{m-3}{-2} = \binom{m-2}{-1} = 0$. The sequences (4) are resstricted for the space of continuous functions only. Motivated by the above developement, we construct a sequence of hybrid operators to approximate in a wider class, i.e., space of Lebesgue integrable functions as follows:

$$A_{m,\alpha}^{*}(g; z) \;=\; \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(z) \frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)} \int_{0}^{\infty} t^{s+\lambda} e^{-mt} g(t) dt, \tag{6}$$

where $P_{m,s}^{(\alpha)}(z)$ is given by (5) and the gamma function as:

$$\Gamma n = \int_{0}^{\infty} z^{m-1} e^{-z} dz, \quad \Gamma z = (z-1)\Gamma(z-1) = (z-1)!.$$

In the subsequent sections, we investigate basic Lemmas, rate of convergence, order of approximation locally and globally in terms of modulus of smoothness, Peetre's K-functional, second modulus of smoothness, Lipschitz space maximal function and weighted modulus of smoothness. Lastly, A-Statistical approximation result is studied.

Remark 1.3. *One can note that, for $\alpha = \lambda = 0$, the operators constructed by us in (6) reduced to the classical Baskakov-Durrmeyer operators [14].*

2. Basic Estimates and approximation

Lemma 2.1. *[15] For $m \in \mathbb{N}$, the $\alpha-$Baskakov operator has the following identities:*

$$\begin{aligned} L_{m,\alpha}(1;z) &= 1, \\ L_{m,\alpha}(t;z) &= z + \frac{2}{m}(\alpha-1)z, \\ L_{m,\alpha}(t^2;z) &= z^2 + \frac{4\alpha-3}{m}z^2 + \frac{z}{m^2}(m+4\alpha-4). \end{aligned}$$

Lemma 2.2. *Let the operators $A^*_{m,\alpha}(.;.)$ defined by (6) and $e_i(z) = z^i, i \in \{0,1,2\}$ bethe test function, we have the following identities:*

$$\begin{aligned} A^*_{m,\alpha}(e_0;z) &= 1, \\ A^*_{m,\alpha}(e_1;z) &= z + \frac{2}{m}(\alpha-1)z + \frac{\lambda+1}{m}, \\ A^*_{m,\alpha}(e_2;z) &= z^2\Big(1+\frac{4\alpha-3}{m}\Big) + z\Big(\frac{2\lambda+3}{m} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\Big) \\ &+ \frac{\lambda^2+3\lambda+2}{m^2}, \end{aligned}$$

where $m \in \mathbb{M}$ and $\alpha \in [-1,1]$.

Proof. In view of Lemma (2.2) and for the operator given by (6), we have, for $e_0 = 1$

$$\begin{aligned} A^*_{m,\alpha}(e_0;z) &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\int_0^{\infty} t^{s+\lambda}e^{-mt}dt \\ &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\frac{\Gamma(s+\lambda+1)}{m^{s+\lambda+1}} \\ &= 1. \end{aligned}$$

For $e_1 = t$, we have

$$\begin{aligned} A^*_{m,\alpha}(e_1;z) &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\int_0^{\infty} t^{s+\lambda}e^{-mt}tdt \\ &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\int_0^{\infty} t^{s+\lambda+1}e^{-mt}dt \\ &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\frac{\Gamma(s+\lambda+2)}{m^{s+\lambda+2}} \\ &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{1}{m}\frac{(s+\lambda+1)}{\Gamma(s+\lambda+1)}\Gamma(s+\lambda+1) \\ &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{s}{m} + \frac{\lambda+1}{m} \\ &= z + \frac{2}{m}(\alpha-1)z + \frac{\lambda+1}{m}. \end{aligned}$$

For $e_2 = t^2$

$$\begin{aligned}
A^*_{m,\alpha}(e_2;z) &= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\int_0^{\infty} t^{s+\lambda}e^{-mt}t^2dt \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\int_0^{\infty} t^{s+\lambda+2}e^{-mt}dt \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\frac{\Gamma(s+\lambda+3)}{m^{s+\lambda+3}} \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{m^{s+\lambda+1}}{\Gamma(s+\lambda+1)}\frac{(s+\lambda+2)(s+\lambda+1)(\Gamma s+\lambda+1)}{m^{s+\lambda+3}} \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{(s+\lambda+2)(s+\lambda+3)}{m^2} \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{s^2+(2\lambda+3)s+(\lambda^2+3\lambda+2)}{m^2} \\
&= \sum_{s=0}^{\infty} P^{\alpha}_{m,s}(z)\frac{s^2}{m^2}+\frac{(2\lambda+3)s}{m^2}+\frac{\lambda^2+3\lambda+2}{m^2}
\end{aligned}$$

$$\begin{aligned}
A^*_{m,\alpha}(e_2;z) &= z^2+\frac{(4\alpha-3)z^2}{m}+\frac{z(m+4\alpha-4)}{m^2}+\frac{2\lambda+3}{m}\left(z+\frac{2(\alpha-1)z}{m}\right) \\
&+ \frac{\lambda^2+3\lambda+2}{m^2} \\
&= z^2\Big(1+\frac{4\alpha-3}{m}\Big)+z\Big(\frac{2\lambda+3}{m}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\Big) \\
&+ \frac{\lambda^2+3\lambda+2}{m^2}.
\end{aligned}$$

□

Lemma 2.3. *Let the central moments $\eta_j(z) = (t-z)^j$, for $j \in \{0,1,2\}$. Then, for the operator $A^*_{m,\alpha}(.;.)$ given by (6), we have the following equalities :*

$$\begin{aligned}
A^*_{m,\alpha}(\eta_0;z) &= 1, \\
A^*_{m,\alpha}(\eta_1;z) &= \frac{2(\lambda-1)z}{m}+\frac{\lambda+1}{m}, \\
A^*_{m,\alpha}(\eta_2;z) &= 0\Big(\frac{1}{m}\Big)\Big(z^2+z+1\Big).
\end{aligned}$$

Proof. In the light of linearity property and Lemma (2.2) with the use of the operators $A^*_{m,\alpha}(.;.)$, we prove the desired Lemma (2.3). □

Definition 2.4. *Let $f \in C[0,\infty)$. Then, modulus of continuity for a uniformly continuous function f is defined as*

$$\omega(f;\delta) = \sup_{|t_1-t_2|\le\delta} |f(t_1)-f(t_2)|, \qquad t_1,t_2 \in [0,\infty).$$

For a uniformly continuous function f in $C[0,\infty)$ and $\delta > 0$, we get

$$|f(t_1)-f(t_2)| \le \Big(1+\frac{(t_1-t_2)^2}{\delta^2}\Big)\omega(f;\delta). \tag{7}$$

Theorem 2.5. *Suppose that* $Q_\varphi = \left\{\varphi : z \geq 0, \frac{\varphi(z)}{1+z^2} \text{ stands for convergent when } z \to \infty\right\}$. *Then, for any* $\varphi \in C[0,\infty) \bigcap Q_\varphi$, *the operators* $A^*_{m,\alpha}(.;.)$ *given by* (6) *converges to function* φ *uniformly.*

Proof. Taking into account the property (vi) of Theorem 4.1.4 [16], it is enough to show that

$$A^*_{m,\alpha}(e_j;z) \to e_j(z), \text{ if } j = 0,1,2.$$

In view of Lemma 2.2, it is obvious that $A^*_{m,\alpha}(e_j;z) \to e_j(z)$ for $j = 0,1,2$ when $m \to \infty$. Which gives the prove of Theorem 2.5. □

Theorem 2.6. *(See [17]) Let* $L : C([a,b]) \to B([a,b])$ *be a linear and positive operator and let* φ_x *be the function defined by*

$$\varphi_x(t) = |t-x|, \ (x,t) \in [a,b] \times [a,b].$$

If $f \in C_B([a,b])$ *for any* $x \in [a,b]$ *and any* $\delta > 0$, *the operator* L *verifies:*

$$|(Lf)(x) - f(x)| \leq |f(x)||(Le_0)(x) - 1| + \{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\}\omega_f(\delta).$$

Theorem 2.7. *For any* $g \in C_B[0,\infty)$, *the sequence of operators* $A^*_{m,\alpha}(.;.)$ *defined by* (6) *verify the inequality*

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq 2\omega(g;\delta),$$

where $\delta = \sqrt{A^*_{m,\alpha}(\eta_2;z)}$ *and* $C_B[0,\infty)$ *stands for space of all continuous and bounded functions on the interval* $[0,\infty)$.

Proof. In the light of Lemma 2.2, Lemma 2.3 and Theorem 2.6, it is easy to obtain

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq \{1 + \delta^{-1}\sqrt{A^*_{m,\alpha}(\eta_2;z)}\}\omega^*(g;\delta).$$

On taking $\delta = \sqrt{A^*_{m,\alpha}(\eta_2;z)}$, we arrive at the required result. □

3. Local and Global Approximation Results

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions equipped with the norm $||f|| = \sup_{0\leq x<\infty} |f(x)|$. For any $f \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as

$$K_2(g,\delta) = inf\{||f-h|| + \delta||h''|| : h \in C_B^2[0,\infty)\}, \tag{8}$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. From DeVore and Lorentz [[18], p.177, Theorem 2.4], there exists an absolute constant $C > 0$ in such a way

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}).$$

In order to get the proof of Theorem 3.2, we define the auxiliary operator $\widehat{A}_{m,\alpha}(g;z)$ as:

$$\widehat{A}_{(m,\alpha)}(g;z) = A^*_{m,\alpha}(g;z) + g(z) - g\left(z + \frac{2}{m}(\alpha-1)z + \frac{\lambda+1}{m}\right). \tag{9}$$

Lemma 3.1. *For* $g \in C_B^2[0,\infty)$, $z \geq 0$ *and* $i, \lambda \geq 0$, $\quad \alpha \in [0,1]$, *one get*

$$|\widehat{A}_{m,\alpha}(g;z) - g(z)| \quad \leq \quad \xi_m^z||g''||,$$

where

$$\xi_m^z = \widehat{A}_{m,\alpha}((t-z)^2;z;z) + \left(\frac{2}{m}(\alpha-1)z + \frac{\lambda+1}{m}\right).$$

Proof. From the auxiliary operator (9), we get that

$$\widehat{A}_{m,\alpha}(e_0;z) = 1,\ \widehat{A}_{m,\alpha}(\eta(z);z) = 0 \text{ and } |\widehat{A}_{m,\alpha}(g;z)| \quad \leq \quad 3\|g\|. \tag{10}$$

From Taylor series Expansion, for any $g \in C_B^2[0,\infty)$, we have

$$g(t) = g(z) + (t-z)g'(z) + \int_z^t (t-\rho)g''(\rho)d\rho. \tag{11}$$

On operating $\widehat{A}_{m,\alpha}(f;z)$ in (11), we obtain

$$\widehat{A}_{m,\alpha}(g;z) - g(z) \quad = \quad g'(z)\widehat{A}_{m,\alpha}(t-z;z) + \widehat{A}_{m,\alpha}\left(\int_z^t (t-\rho)g''(\rho)d\rho;z\right).$$

From the equalities (9) and (10), we have

$$\begin{aligned}\widehat{A}_{m,\alpha}(g;z) - g(z) &= \widehat{A}_{m,\alpha}\left(\int_z^t (t-\rho)g''(\rho)d\rho;z\right)\\ &= A^*_{m,\alpha}\left(\int_z^t (t-\rho)g''(\rho)d\rho;z\right)\\ &- \int_z^{z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}} \left(z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}-\rho\right)g''(\rho)d\rho.\end{aligned}$$

$$\begin{aligned}|\widehat{A}_{m,\alpha}(g;z) - g(z)| &\leq \left|A^*_{m,\alpha}\left(\int_z^t (t-\rho)g''(\rho)d\rho;z\right)\right|\\ &+ \left|\int_z^{z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}} \left(z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}-\rho\right)g''(\rho)d\rho\right|.\end{aligned} \tag{12}$$

Since

$$\left|\int_z^t (t-\rho)g''(\rho)d\rho\right| \leq (t-z)^2 \parallel g'' \parallel. \tag{13}$$

Then,

$$\begin{aligned}\left|\int_z^{z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}} \left(z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m} \right.\right. &- \left.\left.\rho\right)g''(\rho)d\rho\right| \\ &\leq \left(\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}\right)^2 \parallel g'' \parallel.\end{aligned} \tag{14}$$

Combining the equalities (12), (13) and (14), we see that

$$\begin{aligned}|\widehat{A}_{m,\alpha}(g;z)-g(z)| &\leq \left\{\widehat{A}_{m,\alpha}((t-z)^2;z;z)+\left(\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}\right)^2\right\}\|g''\| \\ &= \xi_m^z\|g''\|,\end{aligned}$$

which completes the proof. □

Theorem 3.2. *For any $g \in C_B^2[0,\infty)$, there exists a positive number C satisfying the inequality*

$$|\ A_{m,\alpha}(g;z)-g(z)\ |\leq C\omega_2^*\Big(g;\ \sqrt{\xi_m^z}\Big)+\omega(g;\widehat{A}_{m,\alpha}(\eta_1;z)),$$

where ξ_m^z is given by Lemma 3.1.

Proof. Let $h \in C_B^2[0,\infty)$ and $g \in C_B[0,\infty)$. Then, using the definition of $\widehat{A}_{m,\alpha}(.;.)$, we get

$$\begin{aligned}|A^*_{m,\alpha}(g;z)-g(z)| &\leq |\widehat{A}_{m,\alpha}(g-h;z)|+|(g-h)(z)|+|\widehat{D}_m^h(z)-h(z)| \\ &+ \left|g\Big(z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}\Big)-g(z)\right|.\end{aligned}$$

In view of Lemma 3.1 and the relations (10), one has

$$\begin{aligned}|A^*_{m,\alpha}(g;z)-g(z)| &\leq 4\|g-h\|+|\widehat{A}^*_{m,\alpha}(h;z)-h(z)| \\ &+ \left|g\Big(z+\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}\Big)-g(z)\right| \\ &\leq 4\|g-h\|+\xi_m(z)\|h''\|+\omega\Big(g;A^*_{m,\alpha}(\eta_1;z)\Big).\end{aligned}$$

Using the definition K-functional, we obtain

$$|A^*_{m,\alpha}(g;z)-g(z)|\leq C\omega_2^*\Big(g;\ \sqrt{\xi_m(z)}\Big)+\omega^*(g;A^*_{m,\alpha}(\eta_1;z)).$$

This gives the proof of Theorem 3.2. □

For any fixed two real positive numbers s_1 and s_2, the Lipschitz-class of functions defined in [19] by:

$$Lip_M^{s_1,s_2}(\beta):=\left\{g\in C_B[0,\infty):|g(t)-g(z)|\leq C\frac{|t-z|^\beta}{(t+s_1z+s_2z^2)^{\frac{\beta}{2}}}\ :z,t\in(0,\infty)\right\},$$

with the positive constant C and $0<\beta\leq 1$.

Theorem 3.3. *Let the function $g \in Lip_M^{s_1,s_2}(\beta)$, then, it follows that*

$$|A^*_{m,\alpha}(g;z)-g(z)|\leq C\left(\frac{A^*_{m,\alpha}(\eta_2;z)}{s_1z+s_2z^2}\right)^{\frac{\beta}{2}}, \tag{15}$$

where $z > 0$.

Proof. For $\beta = 1$, we have

$$\begin{aligned}|A^*_{m,\alpha}(g;z)-g(z)| &\leq A^*_{m,\alpha}(|g(t)-g(z)|;z) \\ &\leq CA^*_{m,\alpha}\left(\frac{|t-z|}{(t+s_1z+s_2z^2)^{\frac{1}{2}}};z\right).\end{aligned}$$

Since $\frac{1}{t+s_1z+s_2z^2} < \frac{1}{s_1z+s_2z^2}$, for every $z \in (0,\infty)$, we get that

$$\begin{aligned} |A^*_{m,\alpha}(g;z) - g(z)| &\leq \frac{M}{(s_1z+s_2z^2)^{\frac{1}{2}}}(A^*_{m,\alpha}((t-z)^2;z))^{\frac{1}{2}} \\ &\leq C\left(\frac{A^*_{m,\alpha}(\eta_2;z)}{s_1z+s_2z^2}\right)^{\frac{1}{2}}. \end{aligned}$$

Thus the Theorem 3.3 holds good when $\beta = 1$. Next, on choosing $0 < \beta < 1$) and applying Hölder's inequality for $p_1 = \frac{2}{\beta}$ and $p_2 = \frac{2}{2-\beta}$, we have

$$\begin{aligned} |A^*_{m,\alpha}(g;z) - g(z)| &\leq \left(A^*_{m,\alpha}(|g(t)-g(z)|;z)^{\frac{2}{\beta}}\right)^{\frac{\beta}{2}} \\ &\leq M\left(A^*_{m,\alpha}\left(\frac{|t-z|^2}{(t+s_1z+s_2z^2)};z\right)\right)^{\frac{\beta}{2}}. \end{aligned}$$

Since $\frac{1}{t+s_1z+s_2z^2} < \frac{1}{s_1z+s_2z^2}$ for every $z \in (0,\infty)$, we get

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq C\left(\frac{A^*_{m,\alpha}((|t-z|^2;z)}{s_1z+s_2z^2}\right)^{\frac{\beta}{2}}.$$

Thus, we get the prove of Theorem 3.3. □

To obtain the local type approximation results in r^{th} order the Lipschitz-maximal function given by Lenze [20] such that:

$$\widetilde{\omega}_r(g;z) = \sup_{t\neq z, t\in(0,\infty)} \frac{|g(t)-g(z)|}{|t-z|^r},\ z \in [0,\infty) \text{ and } 0 < r \leq 1. \tag{16}$$

Theorem 3.4. *Let $z \in [0,\infty)$ and $g \in C_B[0,\infty)$. Then, for any $r \in (0,1]$, we have*

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq \widetilde{\omega}_r(g;z)\left(A^*_{m,\alpha}((|t-z|^2;z)\right)^{\frac{r}{2}}.$$

Proof. Since we know that

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq A^*_{m,\alpha}(|g(t)-g(z)|;z).$$

Therefore, from equality (16),we yield

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq \widetilde{\omega}_r(g;z)A^*_{m,\alpha}(|t-z|^r;z).$$

Using the Ḧolder's inequality with $p_1 = \frac{2}{r}$ and $p_2 = \frac{2}{2-r}$, we have

$$|A^*_{m,\alpha}(g;z) - g(z)| \leq \widetilde{\omega}_r(g;z)\left(A^*_{m,\alpha}(|t-z|^2;z)\right)^{\frac{r}{2}},$$

which gives the desired result. □

From [21], we recall some notation to prove the global approximation results.
For the weight function $1+x^2$ and $0 \leq x < \infty$, we have
$B_{1+x^2}[0,\infty) = \{f(x) : |f(x)| \leq M_f(1+x^2), M_f$ is constant depending on $f\}$.
$C_{1+x^2}[0,\infty) \subset B_{1+x^2}[0,\infty)$ space of continuous functions endowed with the norm $\|f\|_{1+x^2} = \sup\limits_{x\in[0,\infty)} \frac{|f|}{1+x^2}$ and

$$C^k_{1+x^2}[0,\infty) = \{f \in C_{1+x^2} : \lim_{x\to\infty} \frac{f(x)}{1+x^2} = k,\ \text{where } k \text{ is a constant}\}.$$

Moreover, the modulus of smoothness for any function ϕ on closed interval $[0,\mu]$, $\mu>0$ and be defined by:

$$\omega^*_\mu(\phi,\delta)=\sup_{|t-z|\leq\delta}\ \sup_{z,t\in[0,\mu]}|\phi(t)-\phi(z)|. \tag{17}$$

Theorem 3.5. *Let $\omega^*_{\mu+1}(\phi;\delta)$ be the modulus of smoothness defined on $[0,\mu+1]\subset[0,\infty)$. Then for all $\phi\in C_{1+z^2}[0,\infty)$*

$$||A^*_{m,\alpha}(\phi;z)-\phi(z)||_{C[0,\mu]}\leq 6M_\phi(1+\mu^2)\delta_m(\mu)+2\omega^*_{\mu+1}(\phi;\sqrt{\delta_m(\mu)}),$$

where $\delta_m(\mu)=A^*_{m,\alpha}(\eta_2;\mu)$.

Proof. For all $z\in[0,\mu]$ and $t\in[0,\infty)$, one has

$$|\phi(t)-\phi(z)|\leq 6M_\phi(1+\mu^2)(t-z)^2+\left(1+\frac{|t-z|}{\delta}\right)\omega^*_{\mu+1}(\phi;\delta).$$

On applying the operators $A^*_{m,\alpha}$, we see that

$$|A^*_{m,\alpha}(\phi;z)-\phi(z)|\leq 6M_\phi(1+\mu^2)A^*_{m,\alpha}((t-z)^2;z)+\left(1+\frac{A^*_{m,\alpha}(|t-z|;z)}{\delta}\right)\omega^*_{\mu+1}(\phi;\delta).$$

Thus, for $z\in[0,\mu]$, if we apply the Lemma (2.2), we get

$$|A^*_{m,\alpha}(\phi;z)-\phi(z)|\leq 6M_\phi(1+\mu^2)\delta_m(\mu)+\left(1+\frac{\sqrt{\delta_m(\mu)}}{\delta}\right)\omega^*_{\mu+1}(\phi;\delta).$$

Taking $\delta=\delta_m(\mu)$, we reaches the proof of desired result. □

Theorem 3.6. *Suppose the operators $A^*_{m,\alpha}(.;.)$ acting from $C^k_{1+z^2}[0,\infty)$ to $B_{1+z^2}[0,\infty)$ satisfying the conditions*

$$\lim_{m\to\infty}||A^*_{m,\alpha}(e_i)-z^i||_{1+z^2}=0,\ i=0,1,2,$$

then, for each $C^k_{1+z^2}[0,\infty)$

$$\lim_{m\to\infty}||A^*_{m,\alpha}(\phi)-\phi||_{1+z^2}=0.$$

Proof. For the results of Theorem 3.6, we have to show that

$$\lim_{m\to\infty}||A^*_{m,\alpha}(e_i)-z^i||_{1+z^2}=0,\ i=0,1,2.$$

By using the Lemma 2.2, it is enough to show

$$||A^*_{m,\alpha}(e_0)-z^0||_{1+z^2}=\sup_{z\in[0,\infty)}\frac{|A^*_{m,\alpha}(e_0;z)-1|}{1+z^2}=0 \text{ for } i=0.$$

For $i=1$,

$$\begin{aligned}||A^*_{m,\alpha}(e_1)-z^1||_{1+z^2} &= \sup_{z\in[0,\infty)}\frac{\frac{2}{m}(\alpha-1)z+\frac{\lambda+1}{m}}{1+z^2}\\ &= \frac{2}{m}(\alpha-1)\sup_{z\in[0,\infty)}\frac{z}{1+z^2}+\frac{\lambda+1}{m}\sup_{z\in[0,\infty)}\frac{1}{1+z^2}.\end{aligned}$$

Which gives us $\|A^*_{m,\alpha}(e_1) - z^1\|_{1+z^2} \to 0$ as $m \to \infty$. If we take $i = 2$,

$$\begin{aligned}
\|A^*_{m,\alpha}(e_2) - z^2\|_{1+z^2} &= \sup_{z\in[0,\infty)} \frac{\left| z^2\left(\frac{4\alpha-3}{m}\right) + z\left(\frac{2\lambda+3}{m} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\right)\right|}{1+z^2} + \sup_{z\in[0,\infty)} \frac{\left|\frac{\lambda^2+3\lambda+2}{m^2}\right|}{1+z^2} \\
&= \frac{4\alpha-3}{m} \sup_{z\in[0,\infty)} \frac{z^2}{1+z^2} \\
&+ \left(\frac{2\lambda+3}{m} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\right) \sup_{z\in[0,\infty)} \frac{z}{1+z^2} \\
&+ \frac{(\lambda+1)(\lambda+2)}{m^2} \sup_{z\in[0,\infty)} \frac{1}{1+z^2}.
\end{aligned}$$

Which shows that $\|A^*_{m,\alpha}(e_2) - z^2\|_{1+z^2} \to 0$ as $m \to \infty$. □

Here, we want to prove the theorem to obtain the approximation of locally integrable functions belongs to $C^k_{1+z^2}[0,\infty)$. For such types of result is investigated by Gadjiev [21].

Theorem 3.7. *Suppose $\phi \in C^k_{1+z^2}[0,\infty)$. Then, for any $\theta > 0$, it satisfies that*

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{|A^*_{m,\alpha}(\phi;z)| - \phi(z)}{(1+z^2)^{1+\theta}} = 0.$$

Proof. Let z_0 be the fixed positive real number then, one has

$$\begin{aligned}
\sup_{z\in[0,\infty)} \frac{|A^*_{m,\alpha}(\phi;z)| - \phi(z)}{(1+z^2)^{1+\theta}} &\leq \sup_{z\leq z_0} \frac{|A^*_{m,\alpha}(\phi;z)| - \phi(z)}{(1+z^2)^{1+\theta}} + \sup_{z\geq z_0} \frac{|A^*_{m,\alpha}(\phi;z)| - \phi(z)}{(1+z^2)^{1+\theta}} \\
&\leq \||A^*_{m,\alpha}(\phi;z)| - \phi(z)\|_{C[0,z_0]} \\
&+ \|\phi\|_{1+z^2} \sup_{z\geq z_0} \frac{|A^*_{m,\alpha}(1+t^2;z)|}{(1+z^2)^{1+\theta}} + \sup_{z\geq z_0} \frac{|\phi(z)|}{(1+z^2)^{1+\theta}} \\
&= I_1 + I_2 + I_3 \ (say).
\end{aligned} \tag{18}$$

Since $|\phi(z)| \leq \|\phi\|_{1+z^2}(1+z^2)$, therefore, we take

$$\begin{aligned}
I_3 &= \sup_{z\geq z_0} \frac{|\phi(z)|}{(1+z^2)^{1+\theta}} \\
&\leq \sup_{z\geq z_0} \frac{\|\phi\|_{1+z^2}(1+z^2)}{(1+z^2)^{1+\theta}} \leq \frac{\|\phi\|_{1+z^2}}{(1+z^2)^{\theta}}.
\end{aligned}$$

For an arbitrary real number $\epsilon > 0$, in view of Theorem 2.5 there exists $m_1 \in \mathbb{N}$ satisfying

$$\begin{aligned}
I_2 &< \frac{1}{(1+z^2)^{\theta}} \|\phi\|_{1+z^2} \left(1 + z^2 + \frac{\epsilon}{3\|\phi\|_{1+z^2}}\right) \text{ for all } m_1 \geq m, \\
&< \frac{\|\phi\|_{1+z^2}}{(1+z^2)^{\theta}} + \frac{\epsilon}{3}, \text{ for all } m_1 \geq m.
\end{aligned}$$

This implies that

$$I_2 + I_3 < 2\frac{\|\phi\|_{1+z^2}}{(1+z^2)^{\theta}} + \frac{\epsilon}{3}.$$

For any large number of z_0, we have $\frac{\|\phi\|_{1+z2}}{(1+z^2)^{\theta}} < \frac{\epsilon}{6}$.

$$I_2 + I_3 < \frac{2\epsilon}{3} \text{ for all } m_1 \geq m. \tag{19}$$

In the light of Theorem 3.6, and for any $m_2 > m$, one has

$$I_1 = \|A^*_{m,\alpha}(\phi; z) - \phi\|_{C[0,z_0]} < \frac{\epsilon}{3} \text{ for all } m_2 \geq m. \tag{20}$$

Let $m_3 = max(m_1, m_2)$ and by combining the equality (18), (19) and (20), then we easily get

$$\sup_{z\in[0,\infty)} \frac{|A^*_{m,\alpha}(\phi; z)| - \phi(z)}{(1 + z^2)^{1+\theta}} < \epsilon.$$

Thus, the proof Theorem 3.7 is completed. □

4. A-Statistical approximation

The approximation theorems for a statistical convergence in operators theory introduced in [23]. We recall from [22] and suppose the non-negative infinite suitability matrix defined by $A = (u_{ms})$. Let the sequence $z = (z_s)$ and the A-transform of z be $Az : ((Az)_m)$ are such that

$$(Az)_m = \sum_{s=1}^{\infty} u_{ms} z_s,$$

provided the infinite series converges for each m. The suitability matrix A be regular for $\lim(Az)_m = M$ as $\lim z = M$. The sequence $z = (z_m)$ be a A-statistical convergent to M, i.e., $st_A - \lim z = M$, if for each positive real ϵ, we have $\lim_m \sum_{s:|z_s - M|\geq\epsilon} u_{ms} = 0$.

Theorem 4.1. *Let $A = (u_{ms})$ be the non-negative regular suitability matrix. Then, for $z \geq 0$ and $\phi \in C^k_{1+z^{2+\lambda}}[0, \infty)$ it satisfying that*

$$st_A - \lim_m \|A^*_{m,\alpha}(\phi; z) - \phi\|_{1+z^{2+\lambda}} = 0, \textit{ for all and } \lambda > 0.$$

Proof. For $\lambda = 0$ it is easy to get that

$$st_A - \lim_m \|A^*_{m,\alpha}(e_i; z) - e_i\|_{1+z^2} = 0, \text{ for } i \in \{0, 1, 2\}. \tag{21}$$

From Lemma 2.2, we see that

$$\begin{aligned} \|A^*_{m,\alpha}(e_1; z) - z\|_{1+z^2} &= \sup_{z\in[0,\infty)} \frac{z}{1+z^2}\left|\frac{2}{m}(\alpha - 1)\right| + \sup_{z\in[0,\infty)} \frac{1}{1+z^2}\left|\frac{\lambda+1}{m}\right| \\ &= \frac{2}{m}(\alpha - 1) \sup_{z\in[0,\infty)} \frac{z}{1+z^2} + \frac{\lambda+1}{m} \sup_{z\in[0,\infty)} \frac{1}{1+z^2}. \end{aligned}$$

Now, for a given positive ϵ, if we let the following sets

$$\begin{aligned} M_1 : &= \left\{ n : \|A^*_{m,\alpha}(e_1; z) - z\| \geq \epsilon \right\}, \\ M_2 : &= \left\{ n : \frac{2}{m}(\alpha - 1) \geq \frac{\epsilon}{2} \right\}, \\ M_3 : &= \left\{ n : \frac{\lambda+1}{m} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

This implies that $M_1 \subseteq M_1 \bigcup M_2$ which shows that $\sum_{s\in M_1} a_{ms} \leq \sum_{s\in M_2} au_{ms} + \sum_{s\in M_3} u_{ms}$. Hence, we have

$$st_A - \lim_m \|A^*_{m,\alpha}(e_1;z) - z\|_{1+z^2} = 0. \tag{22}$$

For $i = 2$ in the view of Lemma 2.2, we get

$$\begin{aligned}
\|A^*_{m,\alpha}(e_2;z) - z^2\|_{1+z^2} &= \sup_{z\in[0,\infty)} \frac{1}{1+z^2}\Bigg| z^2\Big(\frac{4\alpha-3}{m}\Big) \\
&+ z\Big(\frac{2\lambda+3}{m} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\Big) + \frac{(\lambda+1)(\lambda+2)}{m^2}\Bigg| \\
&= \frac{4\alpha-3}{m}\sup_{z\in[0,\infty)}\frac{z}{1+z^2} + \frac{2\lambda+3}{m}\sup_{z\in[0,\infty)}\frac{z^2}{1+z^2} \\
&+ \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2}\sup_{z\in[0,\infty)}\frac{z}{1+z^2} \\
&+ \frac{(\lambda+1)(\lambda+2)}{m^2}\sup_{z\in[0,\infty)}\frac{1}{1+z^2}.
\end{aligned}$$

Also for the positive given $\epsilon > 0$, we suppose the sets

$$\begin{aligned}
T_1 : &= \left\{n : \left\|A^*_{m,\alpha}(e_2;z) - z^2\right\| \geq \epsilon\right\}, \\
T_2 : &= \left\{n : \frac{4\alpha-3}{m} \geq \frac{\epsilon}{4}\right\}, \\
T_3 : &= \left\{n : \frac{2\lambda+3}{m} \geq \frac{\epsilon}{4}\right\}, \\
T_4 : &= \left\{n : \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{m^2} \geq \frac{\epsilon}{4}\right\}, \\
T_5 : &= \left\{n : \frac{(\lambda+1)(\lambda+2)}{m^2} \geq \frac{\epsilon}{4}\right\}.
\end{aligned}$$

This implies that $T_1 \subseteq T_2 \bigcup T_3 \bigcup T_4 \bigcup T_5$. By which, we get

$$\sum_{s\in T_1} u_{ms} \leq \sum_{s\in T_2} u_{ms} + \sum_{s\in T_3} u_{ms} + \sum_{s\in T_4} u_{ms} + \sum_{s\in T_5} u_{ms}.$$

As $m \to \infty$, we have

$$st_A - \lim_m \|A^*_{m,\alpha}(e_2;z) - z^2\|_{1+z^2} = 0. \tag{23}$$

This gives the desired proof of Theorem 4.1. □

Example 4.2. *Consider the function $g(t) = t^3 - t^2 + 8t - 10$ and for the set of following parameters $\alpha = 0.9$ and $\lambda = 1.5$. The convergence of the α-Baskakov-Gamma operators $A^*_{m,\alpha}(g;z)$ corresponding to the mentioned function $g(t)$ is improving as increasing the values of $n = 10, 20, 30$. The figure 1 shows the convergence of the operators and the error approximation $E^*_{m,\alpha}(g;z)$ is appeared in the figure 2 of the operator corresponding the function $g(t)$ for same values of $n = 10, 20, 30$.*

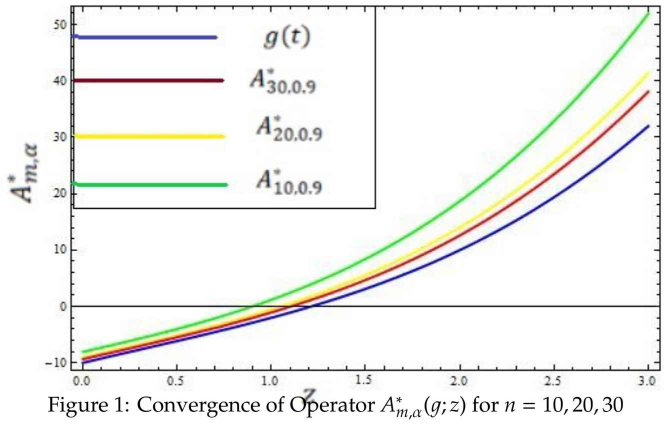

Figure 1: Convergence of Operator $A^*_{m,\alpha}(g;z)$ for $n = 10, 20, 30$

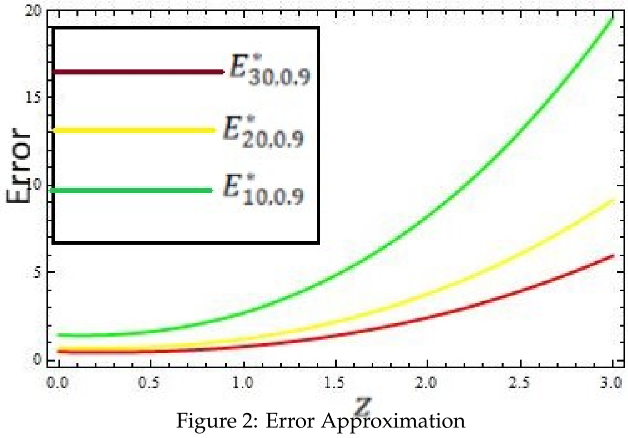

Figure 2: Error Approximation

Here, the error approximation table 1 is given below, which is supported our analytical and numerical results.

t	$E^*_{10,\alpha}(g;z)$	$E^*_{20,\alpha}(g;z)$	$E^*_{30,\alpha}(g;z)$
0.3	1.9664880000	0.9533007500	0.6302030822
0.6	2.2679490000	1.0630366250	0.6941154444
0.9	2.8974059999	1.3322615000	1.0645921831
1.2	3.8960070000	1.7809823749	1.1511241111
1.5	5.3049000000	2.4292062500	1.5706444444
1.8	7.1652329999	3.2969401250	2.1351247780
2.1	9.5181539999	4.4041910000	2.8577771111
2.4	12.404811000	5.7709658750	3.7518134444
2.7	15.866352000	7.4172717500	4.8304457777
3	19.943925000	9.3631156249	6.1068861111

Table 1: Error Approximation Table

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