Filomat 37:3 (2023), 905–913
https://doi.org/10.2298/FIL2303905O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Certain curves along Riemannian submersions

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Abstract. In this paper, when a given curve on the total manifold of a Riemannian submersion is transferred to the base manifold, the character of the corresponding curve is examined. First, the case of a Frenet curve on the total manifold being a Frenet curve on the base manifold along a Riemannian submersion is investigated. Then, the condition that a circle on the total manifold (respectively a helix) is a circle (respectively, a helix) or a geodesic on the base manifold along a Riemannian submersion is obtained. We also investigate the curvatures of the original curve on the total manifold and the corresponding curve on the base manifold in terms of Riemannian submersions.

1. Introduction

The Frenet frame and its derivative equations defined along a curve in Euclidean 3-space are the most important tools for the curve theory. Since the concept of the derivative defined in Euclidean space does not apply on manifolds, it is not possible to carry this concept directly. The special curves on the surfaces were first moved to the submanifolds by Nomizu and Yano in [1]. The authors defined the notion of the circle. They showed that when a circle on the submanifolds is carried along the immersion to the target manifold, such submanifolds are umbilical and their mean curvature vector field is parallel. The notion of an ordinary helix was first defined in a Riemannian manifold by Ikawa in [2]. Ikawa showed that if a helix on the submanifold is carried in the manifold through immersion, this submanifold is umbilical and the mean curvature vector field satisfies a certain identity.

Circles and related concepts have been studied by many authors [3], [4], [5], [6], [7], [8]. For example, in [5], the author showed that if a circle on a submanifold is carried to the curve with the constant first curvature on the manifold by immersion, then the submanifold is isotropic. Similar results were also obtained for complex manifolds and Sasakian manifolds, see for instances [6], [9]. These results have clarified geometrical meaning of many concepts previously defined as algebraic notions.

As an analogue of isometric immersions, the notion of Riemannian submersions was introduced [10] and [11]. However, there are a few results for characterizations of curves along Riemannian submersions (see [12]). It seems that there are many research problems by considering different curves along Riemannian

2020 *Mathematics Subject Classification*. Primary 53B20; Secondary 58D17.
Keywords. Circle; Helix; Geodesic; Riemannian submersion; O'Neill's tensors; Second fundamental form; Isotropic submersion.
Received: 26 July 2021; Accepted: 21 November 2021
Communicated by Ljubica Velimirović
Research supported by by The Scientific and Technological Council of Turkey (TUBITAK)with number 119F025.)
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submersions. The main purpose of this article is to investigate certain properties of both curves and Riemannian submersions in terms of Riemannian submersions for a Frenet curve, a circle and a helix curve given on the total manifold to be a Frenet curve, a circle and a helix under Riemannian submersion.

The paper is organized as follows. In section 2, we recall some important general notions and formulas needed throughout the paper. In section 3, we first consider general Frenet curve on the total manifold. In particular, if the Frenet curve on the total manifold of a Riemannian submersion is also a Frenet curve on the base manifold, the relations between the curvatures of these two curves are investigated. When a circle on the total manifold of a Riemannian submersion is transferred to the base manifold, the conditions for the corresponding curve to be a geodesic curve and a circle curve on the base manifold are investigated. Same problem is also studied for a helix curve. In addition, if a given circle on the total manifold is a helix on the base manifold, the properties of the two curves are compared with the terms of the Riemannian submersion.

2. Preliminaries

This section recalls some fundamental formulas and notions for later use. Let M and N be two Riemannian manifolds with Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$. We suppose that $F : M \to N$ is a map. There exists a unique pullback connection $\overset{2}{\nabla}{}^{F}$ of $\overset{2}{\nabla}$ along F such that for $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$

$$\overset{2}{\nabla}{}^{F}_{X}(Y \circ F) = \overset{2}{\nabla}_{F_*(X)}Y.$$

The second fundamental form of F [13] is the map $\nabla F_* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TN)$ defined by

$$(\nabla F_*)(X, Y) = \overset{2}{\nabla}{}^{F}_{X} F_*(Y) - F_*\left(\overset{1}{\nabla}_X Y\right). \tag{1}$$

We now recall basic information for Riemannian submersion from [10] and [12]. Let (M, g_M) and (N, g_N) be m and $n-$dimensional Riemannian manifolds $(m > n)$, respectively. A Riemannian submersion $F : M \to N$ is a surjective map of M onto N satisfying the following axioms:

(i) F has maximal rank,

(ii) F_* preserves the lengths of horizontal vectors.

Suppose that F is a Riemannian submersion of (M^m, g_M) onto (N^n, g_N). For each $q \in N$, the submanifolds $F^{-1}(q)$ are called fibers. A vector field on M is called vertical and horizontal if it is always tangent and orthogonal to fibers, respectively. A vector field X on M is called basic if X is horizontal and $F-$related to a vector field X_* on N. The geometry of Riemannian submersions is characterized by O'Neill's tensor fields T and A defined for vector fields E and F on M by

$$T_E F = h\overset{1}{\nabla}_{vE} vF + v\overset{1}{\nabla}_{vE} hF \tag{2}$$

and

$$A_E F = h\overset{1}{\nabla}_{hE} vF + v\overset{1}{\nabla}_{hE} hF. \tag{3}$$

For any $E \in \Gamma(TM)$, T_E and A_E reverse the horizontal and vertical distributions. T and A satisfy

$$T_V W = T_W V$$

and

$$A_X Y = -A_Y X = \frac{1}{2} v[X, Y] \tag{4}$$

for $V, W \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma\left((\ker F_*)^{\perp}\right)$. The following formulas are consequences of (2) and (3):

$$\begin{aligned} \overset{1}{\nabla}_V W &= T_V W + \hat{\nabla}_V W, & (5)\\ \overset{1}{\nabla}_V X &= h\overset{1}{\nabla}_V X + T_V X, & (6)\\ \overset{1}{\nabla}_X V &= A_X V + v\overset{1}{\nabla}_X V & (7) \end{aligned}$$

and

$$\overset{1}{\nabla}_X Y = h\overset{1}{\nabla}_X Y + A_X Y, \tag{8}$$

where $\hat{\nabla}_V W = v\nabla_V W$. If X is basic, then $h\nabla_V X = A_X V$. Note that the restriction of A to $\chi^h(M) \times \chi^h(M)$ measures the integrability of the horizontal distribution, and the restriction of T to $\chi^h(M) \times \chi^h(M)$ acts as the second fundamental form of any fiber. In particular, the vanishing of T means that any fiber of F is a totally geodesic submanifold of M. The converse statement is also true [12].

3. Certain curves along Riemannian submersions

In this section, we first compare curvatures of both curves when the Riemannian submersion maps a Frenet curve on the total manifold onto the target manifold. We first recall the notion of a Frenet curve. Let M be a Riemannian manifold and α is a curve parametrized by an arclength parametrization. If there is an orthonormal frame $\{V_1 = \alpha', V_2, V_3, ...V_d\}$ along α and positive functions $\kappa_0 = \kappa_d = 0, \kappa_1 = \kappa, \kappa_2 = \tau, ..., \kappa_{d-1}$ with the following differential equation

$$\nabla_{\alpha'} V_j(s) = -\kappa_{j-1}(s)\, V_{j-1}(s) + \kappa_j(s)\, V_{j+1}(s), \quad V_0 = V_{d+1} = 0,$$

then the curve is known Frenet curve with degree d, where the orthonormal frame field $\{V_1 = \alpha', V_2, ...V_d\}$ and the functions $\kappa_1, \kappa_2, ..., \kappa_{d-1}$ are known Frenet frame fields and curvatures, respectively (see, [7]).

Theorem 3.1. *Let $F : M_1 \to M_2$ be a Riemannian submersion, $\alpha : I \to M_1$ a Frenet curve and $\gamma = F \circ \alpha$ the corresponding Frenet curve on M_2. Then we have the following cases;*

(i) The curve $\gamma = F \circ \alpha$ has the same curvature with α if the unit normal vector field N of α has only horizontal component and $(2A_H V + T_V V)(t) = 0$, where H and V are the horizontal and vertical parts of $\dot{\alpha}$.

(ii) If α is a horizontal Frenet curve in M_1, then the unit normal vector field of α is horizontal and the corresponding Frenet curve $\gamma = F \circ \alpha$ along a Riemannian submersion has the same curvature with α.

Proof. Let $\alpha : I \subset R \to M_1$, $t \to \alpha(t)$, be a Frenet curve in M_1 with curvature κ and $\gamma = F \circ \alpha$ be the corresponding Frenet curve with curvature $\tilde{\kappa}$ along a Riemannian submersion $F : M_1 \to M_2$. Then we have at point $\gamma(t) = (F \circ \alpha)(t)$

$$\tilde{\kappa}^2(t) = g_{M_2}\left(\overset{2}{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t), \overset{2}{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t)\right). \tag{9}$$

From (1), $\tilde{\kappa}^2(t)$ is equal to

$$g_{M_2}(F_*\left(\overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t)\right) + (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t)), F_*\left(\overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t)\right) + (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t))).$$

By using Frenet equations of α at $\alpha(t)$, we obtain

$$\begin{aligned} \tilde{\kappa}^2(t) = \; & g_{M_2}(F_*(\kappa(t)N(t)), F_*(\kappa(t)N(t))) + 2g_{M_2}(F_*(\kappa(t)N(t)), (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t))) \\ & + g_{M_2}((\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t)), (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t))). \end{aligned}$$

Considering the horizontal component $N_h(t)$ and the vertical component $N_v(t)$ of the vector field $N(t)$ in above equation, we derive

$$\begin{aligned}\tilde{\kappa}^2(t) &= \kappa^2(t)g_{M_2}\left(F_*\left(N_h(t)+N_v(t)\right), F_*\left(N_h(t)+N_v(t)\right)\right)\\ &\quad + 2g_{M_2}\left(F_*\left(N_h(t)+N_v(t)\right), (\nabla F_*)\left(H(t)+V(t), H(t)+V(t)\right)\right)\\ &\quad + g_{M_2}\left((\nabla F_*)\left(H(t)+V(t), H(t)+V(t)\right), (\nabla F_*)\left(H(t)+V(t), H(t)+V(t)\right)\right)\end{aligned}$$

where $H(t)$ and $V(t)$ are the horizontal part and vertical part of the vector field $\dot{\alpha}(t)$, respectively. By using (5) - (7), we obtain

$$\begin{aligned}\tilde{\kappa}^2(t) = \;& \kappa^2(t)g_{M_1}\left(N_h(t), N_h(t)\right) - 2\kappa(t)g_{M_1}\left(N_h(t), 2A_{H(t)}V(t) + T_{V(t)}V(t)\right)\\ & + g_{M_1}\left(2A_{H(t)}V(t) + T_{V(t)}V(t), 2A_{H(t)}V(t) + T_{V(t)}V(t)\right),\end{aligned}$$

where the all horizontal vector fields are considered as basic vector fields on M_1. Since $N_v(t)$, $2v\nabla_{H(t)}V(t)$ and $\hat{\nabla}_{V(t)}V(t)$ vertical vector fields, we have

$$\tilde{\kappa}(t) = \left(\kappa^2(t)\left\|N_h(t)\right\|^2 - 2\kappa(t)g_{M_1}\left(N_h(t), (2A_HV + T_VV)(t)\right) + \|(2A_HV + T_VV)(t)\|^2\right)^{\frac{1}{2}}. \tag{10}$$

Now we can prove the statements of the theorem. *i*) We suppose that α has unit normal vector field with vanishing vertical part. If we use $(2A_HV + T_VV)(t) = 0$ in (10), we obtain $\tilde{\kappa}(t) = \kappa(t)$. *ii*) Let $\dot{\alpha}(t) = H(t)$ be a horizontal vector field. From (8) and (4), we have

$$\overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t) = \overset{1}{\nabla}_{H(t)}H(t) = h\overset{1}{\nabla}_{H(t)}H(t) = \kappa(t)N(t),$$

that is, the vertical component of $N(t)$ vanishes. Thus, from (10), we can see that α and γ have the same curvature.

□

We now recall the following result from [12] "*Let $F : M_1 \to M_2$ be a Riemannian submersion and α a geodesic in the total space. The corresponding curve $\gamma = F \circ \alpha$ is a geodesic in M_2 if and only if the following equation is satisfied*

$$(2A_HV + T_VV)(t) = 0."$$

Remark 3.2. *For a vertical Frenet curve in the total manifold, there is no a corresponding regular curve in the base manifold along a Riemannian submersion. So, we do not take into consideration this situation.*

Consider a Frenet curve associated with the horizontal part of an arbitrary geodesic in the total manifold. The following theorem shows that the image of this curve under a Riemannian submersion is also a Frenet curve.

Theorem 3.3. *Let $F : (M_1, g_{M_1}) \to (M_2, g_{M_2})$ be a Riemannian submersion and α a geodesic in M_1. Considering a unit speed Frenet curve formed by horizontal part of $\dot{\alpha}$, the corresponding curve $\gamma = F \circ \alpha$ is also a unit speed Frenet curve in M_2.*

Proof. Let $F : M_1 \to M_2$ be a Riemannian submersion and α a geodesic in M_1. From Proposition 1.8 in [12], we have

$$\bar{\nabla}_{H(t)}H(t) = -\left(2A_HV + T_VV\right)(t), \tag{11}$$

where $\bar{\nabla}$ is the Schouten connection satisfying the following relation

$$\nabla_EF = \bar{\nabla}_EF + T_EF + A_EF, \tag{12}$$

$H(t)$ and $V(t)$ are horizantal and vertical part of $\dot{\alpha}$. Taking into consideration (8) and (12), we can write

$$\bar{\nabla}_{H(t)}H(t) = \overset{1}{\nabla}_{H(t)}H(t) = h\overset{1}{\nabla}_{H(t)}H(t). \tag{13}$$

The Frenet frame field of unit speed Frenet curve formed by horizontal part of velocity vector of α at $\alpha(t)$ is denoted by $\{V_1(t) = H(t) = T_H(t), V_2(t) = N_H(t), V_3(t) = B_H(t), ...V_d(t)\}$ with formulas

$$\overset{1}{\nabla}_{H(t)}V(t)_j = -\kappa(t)_{(H)j-1}V(t)_{j-1} + \kappa_{(H)j}(t)V_{j+1}(t), \tag{14}$$

where $T_H = H, N_H$ and B_H are the unit tangent, unit normal and binormal vector fields of α at the point $\alpha(t)$, respectively, $V_0(t) = V(t)_{d+1} = 0$, $\kappa_{(H)0} = \kappa_{(H)d} = 0$ and $\kappa_{(H)1} = \kappa_{(H)}, \kappa_{(H)2} = \tau_{(H)}, ..., \kappa_{(H)d-1}$ are curvatures of α. So, $\gamma = F \circ \alpha$ is a regular curve because of

$$g_{M_2}\left(\tilde{T}(t), \tilde{T}(t)\right) = g_{M_2}(F_*(H(t)), F_*(H(t))) = g_{M_1}(H(t), H(t)) = 1$$

and the unit tangent vector field is found as

$$\tilde{T}(t) = \dot{\gamma}(t) = F_*(\dot{\alpha}(t)) = F_*(H(t) + V(t)) = F_*(H(t)).$$

Theorem 3.1 shows that if α is a horizontal geodesic, then γ is also a geodesic. So, we suppose that α has non-zero horizontal and vertical parts. The curvature $\tilde{\kappa}$ of γ is obtained from (10) as follows

$$\tilde{\kappa}(t) = \|(2A_HV + T_VV)(t)\| = \left\|\bar{\nabla}_{H(t)}H(t)\right\| \neq 0. \tag{15}$$

Now, we find a perpendicular vector field to $\tilde{T}$. Since $\dot{\gamma} \perp \overset{2}{\nabla}_{\dot{\gamma}}\dot{\gamma}$, we consider $\overset{2}{\nabla}_{\dot{\gamma}}\dot{\gamma}$ as an orthogonal vector field. Because of α is a geodesic in M_1, (1) reduces to

$$\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}F_*(\dot{\alpha}(t)) = (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t)).$$

If we use horizontal and the vertical components of the $\dot{\alpha}(t)$, we have:

$$(\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t)) = -F_*\left(2\overset{1}{\nabla}_{H(t)}V(t) + \overset{1}{\nabla}_{V(t)}V(t)\right).$$

By using (5) - (7), we get

$$(\nabla F_*)\left(\dot{\alpha(t)}, \dot{\alpha(t)}\right) = -F_*\left((2A_HV + v\overset{1}{\nabla}_HV + T_VV + \hat{\nabla}_VV)(t)\right).$$

Since $F_*\left(\left(v\overset{1}{\nabla}_HV + \hat{\nabla}_VV\right)(t)\right) = 0$, we obtain

$$\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}F_*(\dot{\alpha}(t)) = (\nabla F_*)(\dot{\alpha}(t), \dot{\alpha}(t)) = -F_*((2A_HV + T_VV)(t)). \tag{16}$$

Thus, from (9), we get

$$\begin{aligned}\tilde{\kappa}^2(t) &= g_{M_2}(F_*((2A_HV + T_VV)(t)), F_*((2A_HV + T_VV)(t))) \\ &= g_{M_1}((2A_HV + T_VV)(t), (2A_HV + T_VV)(t)).\end{aligned}$$

The unit vector field $\tilde{N}$ is given by

$$\tilde{N}(t) = \frac{\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}F_*(\dot{\alpha}(t))}{\left\|\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}F_*(\dot{\alpha}(t))\right\|} = \frac{\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}F_*(\dot{\alpha}(t))}{\tilde{\kappa}(t)},$$

this gives

$$\overset{2}{\nabla}{}^{F}_{\dot{\alpha}(t)} F_{*}(\dot{\alpha}(t)) = \tilde{\kappa}(t)\tilde{N}(t). \tag{17}$$

Using (17), (11), (13) and (15) in (16), we have

$$\begin{aligned} \tilde{\kappa}(t)\tilde{N}(t) &= -F_{*}\left((2A_H V + T_V V)(t)\right) \\ \left\| h\overset{1}{\nabla}_H H \right\| \tilde{N}(t) &= \kappa_{(H)}(t)\tilde{N}(t) = F_{*}\left(h\overset{1}{\nabla}_H H\right). \end{aligned}$$

So, we obtain

$$\tilde{N} = F_{*}\left(\frac{h\overset{1}{\nabla}_{H(t)} H(t)}{\kappa_{(H)}(t)}\right) = F_{*}(N_H(t)).$$

To create an orthonormal frame perpendicular to $\tilde{V}_1(t) = \tilde{T}(t)$ and $\tilde{V}_2(t) = \tilde{N}(t)$ vector fields, we can choose vector fields, without loss of generality, as follows

$$0 = g_{M_1}\left(H(t), V_j(t)\right) = g_{M_1}\left(F_{*}(H(t)), F_{*}\left(V_j(t)\right)\right) = g_{M_2}\left(\tilde{V}_1(t), \tilde{V}_j(t)\right).$$

Thus, we get an orthonormal frame $\{\tilde{V}_1(t) = F_{*}(H(t)), \tilde{V}_2(t) = F_{*}(N(t)), \tilde{V}_3(t) = F_{*}(B(t)), \dots \tilde{V}_d(t) = F_{*}(V_d(t))\}$ along γ with degree d since $\tilde{V}_0 = F_{*}(V_0) = 0$, $\tilde{V}_{d+1} = F_{*}(V_{d+1}) = 0$. By using second fundamental formula, (14) and (8), curvatures of γ are found as follow

$$\begin{aligned} \tilde{\kappa}_j(t) &= g_{M_2}\left(\overset{2}{\nabla}_{F_{*}(\dot{\alpha}(t))}\tilde{V}_j(t), \tilde{V}_{j+1}(t)\right) = g_{M_2}\left(\overset{2}{\nabla}_{F_{*}(\dot{\alpha}(t))}F_{*}\left(V_j(t)\right), F_{*}\left(V_{j+1}(t)\right)\right) \\ &= g_{M_2}\left(F_{*}\left(\overset{1}{\nabla}_{H(t)}V_j(t)\right), F_{*}\left(V_{j+1}(t)\right)\right) = g_{M_1}\left(h\overset{1}{\nabla}_{H(t)}V_j(t), V_{j+1}(t)\right) \\ &= \kappa_{(H)j}(t). \end{aligned}$$

On the other hand, if $\gamma = F\circ\alpha$ is a Frenet curve, it has an orthonormal frame $\{\tilde{V}_1(t) = \tilde{T}(t), \tilde{V}_2 = \tilde{N}(t), \tilde{V}_3(t) = \tilde{B}(t), \dots \tilde{V}_d(t)\}$ along γ with formulas

$$\overset{2}{\nabla}_{\dot{\gamma}(t)}\tilde{V}_j(t) = -\tilde{\kappa}_{j-1}(t)\tilde{V}_{j-1}(t) + \tilde{\kappa}_j(t)\tilde{V}_{j+1}(t), \quad \tilde{V}_0(t) = \tilde{V}_{d+1}(t) = 0.$$

□

We will now investigate the necessary conditions for a circle on the total manifold to be a helix, a circle or a geodesic on the base manifold under the Riemannian submersion.
Now we recall that a smooth curve α on a Riemannian manifold M parametrized by its arclength is called a circle if it satisfies

$$\nabla^2_{\dot{\alpha}}\dot{\alpha} + \kappa^2\dot{\alpha} = 0 \tag{18}$$

with some nonnegative constant κ, where $\nabla_{\dot{\alpha}}$ denotes the covariant differentiation along α with respect to the Riemannian connection ∇ on M. This condition is equivalent to the condition that there exist a nonnegative constant κ and a field of unit vectors Y along this curve which satisfies the following differential equations:

$$\nabla_{\dot{\alpha}}\dot{\alpha} = \kappa Y \quad \text{and} \quad \nabla_{\dot{\alpha}}Y = -\kappa\dot{\alpha}. \tag{19}$$

Here κ is called curvature of α(see, [1]).
On the other hand, a regular curve α parametrized by its arclength is called an ordinary helix if there exist unit vector fields X and Y along α and constants $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ such that

$$\nabla_{\dot{\alpha}}\dot{\alpha} = \kappa_1 Y, \quad \nabla_{\dot{\alpha}} Y = -\kappa_1\dot{\alpha} + \kappa_2 X \quad \text{and} \quad \nabla_{\dot{\alpha}} X = -\kappa_2 Y,$$

where κ_1 and κ_2 is called curvature and torsion of the helix, respectively. An ordinary helix can be characterized by the following differential equation

$$\nabla^3_{\dot{\alpha}}\dot{\alpha} + K^2\nabla_{\dot{\alpha}}\dot{\alpha} = 0, \tag{20}$$

where $K^2 = \kappa^2 + \tau^2$ is constant (see, [2]). We now define vector field $Z = \nabla^2_{\dot{\alpha}}\dot{\alpha} + K^2\dot{\alpha}$. It is easy to see that (20) can be written as $\nabla_{\dot{\alpha}} Z = 0$. Then we can give the following theorem.

Theorem 3.4. *Let $F : M_1 \to M_2$ be a Riemannian submersion and $\alpha : I \to M_1$ a circle with curvature κ. If the curve $\gamma = F \circ \alpha$ along F is a helix on M_2, then the following two assertions determine the third assertion:*

(i) $\tilde{K}^2 = \kappa^2$,
(ii) $2A_{_H}V + T_{_V}V$ *is a parallel vector field,*
(iii) γ *is reduced to a circle with curvature* $\tilde{K}$ *in* M_2,

where $\tilde{K} = \tilde{\kappa}^2 + \tilde{\tau}^2$.

Proof. We suppose that $F : M_1 \to M_2$ is a Riemannian submersion and $\alpha : I \to M_1$ is a circle with curvature κ. From (18) and (19), we have from

$$\overset{1}{\nabla}_{\dot{\alpha}}\dot{\alpha} = \kappa Y, \ \overset{1}{\nabla}_{\dot{\alpha}} Y = -\kappa\dot{\alpha}, \text{ or } \overset{1}{\nabla}{}^2_{\dot{\alpha}}\dot{\alpha} + g_{_{M_1}}(\overset{1}{\nabla}_{\dot{\alpha}}\dot{\alpha}, \overset{1}{\nabla}_{\dot{\alpha}}\dot{\alpha})\dot{\alpha} = 0. \tag{21}$$

If $\gamma = F \circ \alpha$ along F is a helix with curvature $\tilde{K}$, then we have along γ

$$\overset{2}{\nabla}_{\dot{\gamma}}\tilde{Z} = 0,$$

where

$$\tilde{Z} = \overset{2}{\nabla}{}^2_{\dot{\gamma}}\dot{\gamma} + \tilde{K}^2\dot{\gamma}. \tag{22}$$

Now we need to find $\overset{1}{\nabla}{}^2_{\dot{\alpha}(t)}\dot{\alpha}(t)$, so we use the second fundamental form of F as follows

$$F_*\left(\overset{1}{\nabla}_{\dot{\alpha}(t)}(\overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t))\right) = \overset{2}{\nabla}{}^F_{\dot{\alpha}(t)} F_*\left(\overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t)\right) - (\nabla F_*)\left(\dot{\alpha}(t), \overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t)\right). \tag{23}$$

Then we get from (1) and (23),

$$\begin{aligned} F_*(\overset{1}{\nabla}{}^2_{\dot{\alpha}(t)}\dot{\alpha}(t)) = (\ &\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)})^2 F_*(\alpha(t)) - \overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}\left((\nabla F_*)\left(\dot{\alpha}(t), \dot{\alpha}(t)\right)\right) \\ &- (\nabla F_*)\left(\dot{\alpha}(t), \overset{1}{\nabla}_{\dot{\alpha}(t)}\dot{\alpha}(t)\right). \end{aligned} \tag{24}$$

Considering (16) and using (1) and (8), we have

$$\overset{2}{\nabla}{}^F_{\dot{\alpha}(t)}((\nabla F_*)\left(\dot{\alpha}(t), \dot{\alpha}(t)\right)) = -F_*(h\overset{1}{\nabla}_{H(t)}(2A_{_H}V + T_{_V}V)(t)). \tag{25}$$

Taking into consideration (25), (21) and (22) in (24), we obtain

$$\begin{aligned} (\tilde{K}^2 - \kappa^2)(t)F_*(\dot{\alpha}(t)) = \ & F_*((h\overset{1}{\nabla}_H(2A_{_H}V + T_{_V}V))(t)) \\ & -\kappa(t)(\nabla F_*)(\dot{\alpha}(t), Y) + \tilde{Z}. \end{aligned}$$

Changing Y into $-Y$, we have

$$(\tilde{K}^2-\kappa^2)(t)F_*(\dot{\alpha}(t)) = F_*((h\overset{1}{\nabla}_H(2A_{_H}V+T_{_V}V))(t))+\tilde{Z}.$$

The last equation gives the desired results. □

From above theorem, we have the following corollary.

Corollary 3.5. *Let $F: M_1 \to M_2$ be a Riemannian submersion. Suppose that α is a circle with κ on M_1, then*
a) $\gamma = F\circ\alpha$ is a geodesic if and only if the following equation holds

$$(-\kappa^2 H)(t) = h\overset{1}{\nabla}_H(2A_H V+T_V V)(t).$$

Moreover, if $2A_H V+T_V V$ is a parallel vector field, then the Riemannian submersion F is a totally geodesic.
b) $\gamma = F\circ\alpha$ is a circle if the following equation holds

$$(\tilde{\kappa}^2-\kappa^2)(t)H(t) = (h\overset{1}{\nabla}_H(2A_{_H}V+T_{_V}V),$$

where $\tilde{\kappa}$ is the curvature of γ.

Now, we will present the necessary conditions for a helix on the total manifold to be a helix, a circle or a geodesic on the base manifold under the Riemannian submersion. With the help of this result, we find a characterization for Riemannian submersions.

Theorem 3.6. *Let $F: M_1 \to M_2$ be a Riemannian submersion and let $\alpha: I \to M_1$ be a helix with constant curvature κ and constant torsion τ. If the curve $\gamma = F\circ\alpha$ along F is a helix on M_2 with constant curvature $\tilde{\kappa}$ and constant torsion $\tilde{\tau}$ then the following two assertions determine the third assertion:*

(i) $\tilde{K}^2 = K^2$,
(ii) $2A_{_H}V+T_{_V}V$ *is a parallel vector field,*
(iii) *F transfers the parallel vector field Z to the parallel vector field $\tilde{Z}$*

where $K^2 = \kappa^2+\tau^2$ and $\tilde{K}^2 = \tilde{\kappa}^2+\tilde{\tau}^2$.

Proof. Assume that α is a helix on M_1. Then we have $\overset{1}{\nabla}_{\dot{\alpha}}Z = 0$, where

$$Z = (\overset{1}{\nabla}{}^2_{\dot{\alpha}}\dot{\alpha}+K^2\dot{\alpha}). \tag{26}$$

If the corresponding curve $\gamma = F\circ\alpha$ is a helix on M_2, so we have from $\overset{2}{\nabla}_{\dot{\gamma}}\tilde{Z} = 0$ with (22). By using (26), (22) and (25) in (24), we get

$$\begin{aligned}(\tilde{K}^2-K^2)(t)F_*(H(t)) = \;& F_*(h\overset{1}{\nabla}_H(2A_{_H}V+T_{_V}V)(t))\\ & -\kappa\,(\nabla F_*)\,(\dot{\alpha}(t),N)+\tilde{Z}(t)-F_*(Z(t)).\end{aligned} \tag{27}$$

Changing N into $-N$, we have

$$(\tilde{K}^2-K^2)(t)F_*(H(t)) = F_*(h\overset{1}{\nabla}_H(2A_{_H}V+T_{_V}V)(t))+\tilde{Z}(t)-F_*(Z(t)).$$

So, the last equation completes the proof. □

Taking into consideration above theorem, we can give the following corollary.

Corollary 3.7. *Let $F : M_1 \to M_2$ be a Riemannian submersion. Suppose that α is a helix on M_1, then*

a) $\gamma = F \circ \alpha$ is a geodesic if and only if the following equation holds

$$-K^2(t)H(t) = (h\overset{1}{\nabla}_H(2A_{_H}V + T_{_V}V) - Z_h)(t).$$

b) $\gamma = F \circ \alpha$ is a circle if the following equation holds

$$(\tilde{\kappa}^2 - K^2)(t)H(t) = (h\overset{1}{\nabla}_H(2A_{_H}V + T_{_V}V) - Z_h)(t)$$

where $Z_h(t)$ is the horizontal part of $Z(t) = (\overset{1}{\nabla}{}^2_{\dot{\alpha}}\dot{\alpha} + K^2\dot{\alpha})(t)$, $H(t)$ and $V(t)$ are the horizontal and vertical parts of $\dot{\alpha}(t)$, respectively.

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