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Stability of relative essential spectra involving relative demicompactness concept in Banach subalgebra

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Abstract. This paper develops the notion of relative demicompact elements of an algebra with respect to a Banach subalgebra as a generalization of relative demicompact linear operators acting on Banach spaces. Drawing on this novel notion, we build a new class of Fredholm perturbation regarding a given Banach subalgebra B which contains its inessential ideal k_B and the set of left Fredholm perturbations suggested in [6]. The developed class of Fredholm perturbation exhibits that is a two-sided closed ideal of B that is key in the characterization of the weyl spectrum of elements affiliated with B.

1. Introduction

The concept of relative demicompactness was introduced and delineated by B. Krichen [17] in 2014 as a generalization of the demicompactness introduced by W. V. Petryshyn [19]. By using this concept, B. Krichen developed some insightful results on Fredholm perturbations and spectral theories. He defines in [17] the relative demicompactness as follows: Let X be a Banach space, and $\phi : \mathcal{D}(\phi) \subset X \longrightarrow X$, $\varphi_0 : \mathcal{D}(\varphi_0) \subset X \longrightarrow X$ be densely defined linear operators with $\mathcal{D}(\phi) \subset \mathcal{D}(\varphi_0)$. ϕ is said to be φ_0-demicompact (or relative demicompact with respect to φ_0), if every bounded sequence $(x_n)_n$ in $\mathcal{D}(\phi)$, such that $\varphi_0 x_n - \phi x_n$ converges in X, has a convergent subsequence. We denote this class of φ_0-demicompact linear operators acting on a Banach space X by $\mathcal{DC}_{\varphi_0}(X)$. When $\varphi_0 = I_X$ (I_X denotes the identity operator of X), ϕ is simply said to be demicompact and $\mathcal{DC}(X)$ stands for the class of demicompact linear operators acting on X. In Fredholm theory, W. V. Petryshyn and W. Y. Akashi (see [1, 20]) employed the classes of demicompact and 1-set contraction linear operators (more generally condensing operators) to reach some interesting results on Fredholm perturbations. For more details on classes of perturbations involving the demicompactness (or relative demicompactness) concept, the reader may refer to the papers [7, 8, 14]. Provided that the class of Fredholm operators are stable by compact perturbations, the authors of [1, 20] used condensing operators instead of compacts operators to study perturbation of Fredholm operators. By using the concept of an inverse modulo compact operator for a given Fredholm operator, these authors established more perturbation results (see [1, Theorem 2.2]). Further, Considerable efforts have been pointed out by [7, 8, 17] who refined some stability results involving demicompact (or relatively demicompact) operators. In particular, these authors deployed the obtained results to study various essential spectra. Of note, throughout the paper, A stand for a complex unital algebra and $(A, +, .)$ its associated ring, where the two

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operations $(+)$ and $(\cdot)$ stand for the addition and multiplication respectively. We consider $(B, \|\cdot\|_B)$ as a Banach subalgebra of A that encompasses the unit. K_B stands for the closed inessential ideal of B that is, for every $a \in K_B$, the spectrum of a is a finite set or a sequence converging to 0. F_B is considered as a two-sided ideal of B satisfying $F_B \leq K_B$ and the quotient algebra F_B/K_B is a radical algebra. Throughout, we denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X, where X is a Banach space, and $\mathcal{K}(X)$ stands for the two-sided ideal of compact operators of $\mathcal{L}(X)$.

The next sets will be deployed in the sequel.

The set of elements in A such that their inverses are in B is denoted and defined as follows:

$$Inv_B(A) = \{a \in A : a^{-1} \in B\}.$$

For $a \in A$ and $s_0 \in A \setminus \{0\}$, $res_{B,\, s_0}(a)$, $\sigma_{B,\, s_0}(a)$, and $r_{\sigma_{B,\, s_0}}(a)$ denote, respectively, the resolvent, the spectrum and the spectral radius sets of the element a relative to B with respect to s_0 which are defined, respectively, by:

$$res_{B,\, s_0}(a) = \{\lambda \in \mathbb{C} : (\lambda s_0 - a) \in Inv_B(A)\},$$

$$\sigma_{B,\, s_0}(a) = \mathbb{C} \backslash res_{B,\, s_0}(a),$$

and

$$r_{\sigma_{B,\, s_0}}(a) = \sup\{|\lambda| : \lambda \in \sigma_{B,\, s_0}(a)\}.$$

If $s_0 = 1$, we recover the usual sets $res_B(a)$, $\sigma_B(a)$ and $r_{\sigma_B}(a)$ defined in [2].

Definition 1.1. An element $a \in A$ is said to be s_0-affiliated with B (or relatively affiliated with B with respect to s_0) if there exists $\lambda \in \mathbb{C}$ such that $(\lambda s_0 - a) \in Inv_B(A)$.

If $s_0 = 1$, a is simply said to be affiliated with B (i.e., $(\lambda - a) \in Inv_B(A)$).

Definition 1.2. (i) An element a in A is said to be a Fredholm element relative to K_B, if there exist $b, b' \in B$ and $j, j' \in K_B$ such that $ab = 1 - j$ and $b'a = 1 - j'$. We denote this set by Φ_B.

An element $a \in A$ is said to be in Φ_B^0, if there exists $f \in F_B$ such that $a + f \in Inv_B(A)$. Obviously, we have $Inv_B(A) \subset \Phi_B^0 \subset \Phi_B$.

(*ii*) An element a in A is said to be a left (resp. a right) Fredholm element relative to K_B, if there exist $b \in B$ and $j \in K_B$ such that $ba = 1 - j$ (resp. $ab = 1 - j$). The set of left (resp. right) Fredholm elements relative to the subalgebra B is denoted by Φ_B^l (resp. Φ_B^r). Observe that $\Phi_B = \Phi_B^r \cap \Phi_B^l$.

A. Ben Ali and N. Moalla defined (see [6, Definition 2.1]) the following sets of Fredholm perturbation:

(*i*) An element $p \in B$ is said to be a Fredholm perturbation, if $p + a \in \Phi_B$ for all $a \in \Phi_B$. This set is denoted by $Pr(\Phi_B)$.

(*ii*) An element $p \in B$ is said to be a left Fredholm perturbation (resp. right Fredholm perturbation) if $p + a \in \Phi_B^l$ for all $a \in \Phi_B^l$ (resp. $p + a \in \Phi_B^r$ for all $a \in \Phi_B^r$). We denote those sets by $Pr(\Phi_B^l)$ and $Pr(\Phi_B^r)$ respectively.

(*iii*) An element $p \in B$ is said to be a Weyl perturbation, if $p + a \in \Phi_B^0$ for all $a \in \Phi_B^0$. This set is denoted by $Pr(\Phi_B^0)$.

Many results relate to the ideal structure resonate with the outstanding closed ideals originating from operator theory and a number of applied studies. Of note, however, $\mathcal{DC}(X)$ couldn't be an ideal. Yet, several classes that are not ideals were developed so that to warrant stability results in Fredholm and perturbation theory. In particular, A. Ben Ali and N. Moalla suggested in [6] an extension of Fredholm and perturbation theory in a Banach subalgebra B of a given algebra A. These authors have demonstrated that the sets of Fredholm perturbation $Pr(\Phi_B^l)$, $Pr(\Phi_B^r)$, $Pr(\Phi_B)$ and $Pr(\Phi_B^0)$ are two-sided closed ideals of B. In addition, they deployed the aforesaid ideals to portray the stability of some essential spectra of an element $a \in A$. So, in this paper, we stretch out these pre-established constructions to a more general setting which is the relative demicompactness concept on a Banach subalgebra of a given algebra. Our suggestions depict a discerning generalization of the classical Fredholm theory. In Section 3, we will suggest a new larger class of all relative demicompact perturbations regarding to a Banach subalgebra B which is denoted by $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$. Among the main contributions of this paper is to show that $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ is a two-sided closed ideal of B containing K_B, the Fredholm spectrum $Pr(\Phi_B)$ and the left Fredholm spectrum $Pr(\Phi^l_B)$ of an element $a \in A$ as investigated in in [6] while ensuring a refinement of their stability results. Also, a number of our results is prompted by this set $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ that is key in the characterization of the weyl spectrum of elements affiliated with B. Moreover, we resettle some findings and remarks recapitulating the current knowledge in the literature (see, for examples [2, 4, 6, 7] and the references therein). As an analogous example with the Fredholm perturbation theory, it was shown (see [6, Remark 2.2]) that if X is a Banach space and if we take $A = B = \mathcal{L}(X)$, then $Pr(\Phi_B) = \mathcal{F}^b(X)$, $Pr(\Phi^l_B) = \mathcal{F}^b_+(X)$ and $Pr(\Phi^r_B) = \mathcal{F}^b_-(X)$, where $\mathcal{F}^b(X)$, $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b_-(X)$ are two-sided closed ideals of $\mathcal{L}(X)$ (see for instance [18]). The Weyl spectrum $W\sigma_B(a)$ and the left Fredholm spectrum $F\sigma_B(a)$ of such element $a \in A$ are defined in section 4, respectively, by:

$$W\sigma_B(a) = \bigcap_{k\in K_B} \sigma_B(a + k),$$

$$F\sigma_B(a) = \{\lambda \in \mathbb{C} : (\lambda - a) \notin \Phi_B\}.$$

Several studies examined the essential spectra of an element $a \in A$ (e.g., [2, 4, 6, 14]). It was shown in [6, Theorem 3.4], that if for some $\nu \in res_B(a) \cap res_B(a')$, $(\nu - a)^{-1} - (\nu - a')^{-1} \in Pr(\Phi^l_B)$, then $F\sigma^l_B(a) = F\sigma^l_B(a')$. In our paper, this finding is generalized to the larger class $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ showing, by Theorem 4.4 in Section 4, consistency if we replace $Pr(\Phi^l_B)$ by $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$. So, we use Theorem 4.4 to refine (see Corollary 4.5) the stability result of the left Fredholm spectrum $F\sigma^l_B(\cdot)$ as presented in [6, Corollary 3.1]. Perturbation results involving the Gustafson essential spectrum in Fredholm and perturbation theory in a Banach space are presented in [7]. Notably, Theorem 4.2 and Theorem 4.7 are considered as an extension of Theorem 7 and Theorem 9 in [7], respectively, to the Fredholm and perturbation theories in Banach subalgebra. These outcomes provided some perturbation findings and some connections between the left Fredholm spectrum of the sum of two elements in B and the left Fredholm spectrum of each of these elements. They enable a refinement of the stability result of the left Fredholm spectrum $F\sigma^l_B(\cdot)$. Finally, in [6, Theorem 3.5], it was shown that if an element $t \in A$ is affiliated with B, then the Fredholm spectrum of t as denoted and defined in [2] by $W\sigma_B(t) = \bigcap\limits_{k\in K_B} \sigma_B(t + k)$ Can be written in the following form:

$$W\sigma_B(t) = \bigcap_{a\in\Omega_B} \sigma_B(t + a),$$

where $\Omega_B = \{a = tk + j; k, j \in K_B\}$. We give, by Theorem 4.7, a fine description of the Weyl spectrum $W\sigma_B(\cdot)$ showing that [6, Theorem 3.5] remains true if we replace the set Ω_B by a larger set $S_B = \Big\{a = tk + j; k, j \in Pr\Big(\mathcal{DC}(B)\Big)\Big\}$ which is a two-sided ideal of B containing Ω_B.

We organize the paper as follows: Section 2 will present some key definitions and results that will be deployed in the paper. Then, in Section 3, we will show that the set $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ is a two-sided closed ideal of B containing $Pr\Big(\Phi^l_B\Big)$ which is relevant to set some properties relating to essential spectra. Finally, in Section 4, we present some perturbation findings of the left Fredholm spectrum and the Weyl spectrum involving the set $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

2. Preliminary results

Here, we present our main concepts, notations, and preliminary findings that are deployed in the paper.

Definition 2.1. [2, Definition 3] Let $t \in A$. An element $a \in A$ is called t-bounded if $a = tb + c$ or $a = bt + c$, where $b, c \in B$.

Definition 2.2. Let $t \in A$.

(i) if there exist $a_l \in B$ (resp. $a_r \in B$) and $j \in K_B$ such that $a_l t = 1 - j$ (resp. $ta_r = 1 - j$), then a_l (resp. a_r) is called a left (resp. a right) Fredholm inverse modulo K_B.

(ii) An element which is both a left and a right Fredholm inverse of t modulo K_B is said a two-sided Fredholm inverse of t modulo K_B.

Remark 2.1. (i) An element t belongs to Φ^l_B, Φ^r_B and Φ_B if it includes a left, a right and two-sided Fredholm inverse of t modulo K_B, respectively.
(ii) The set Φ^0_B can be seen as the set of all elements $b \in \Phi_B$ which have index function identically equal to zero (see for instance [5]).

Now, some properties presenting a generalization of some established findings in the Banach algebra for bounded linear operators will be recalled.

Proposition 2.3. [2, Proposition 9] Let $t \in A$. The following statements are equivalent:
(1) $t \in \Phi_B$.
(2) There exist $b, c \in B$ and $f, g \in F_B$ such that $tb = 1 - f$ and $ct = 1 - g$.
(3) There exist $b \in B$ and $p, q \in F_B$ such that $tb = 1 - p$ and $bt = 1 - q$.

Lemma 2.4. [6, Lemma 2.1] i) Let $t \in \Phi^l_B$ (resp. $t \in \Phi^r_B$), then there exists an $\eta > 0$ such that for all $t^{'} \in B$, with $\|t^{'}\|_B < \eta$ we have $t + t^{'} \in \Phi^l_B$ (resp. $t + t^{'} \in \Phi^r_B$).
ii) Assume that $t \in \Phi_B$, then there is an $\eta > 0$ such that for all $t^{'} \in B$ satisfying $\|t^{'}\|_B < \eta$ one has $t + t^{'} \in \Phi_B$.
(iii) Assume that $t \in \Phi^0_B$, then there is an $\eta > 0$ such that for all $t^{'} \in B$ satisfying $\|t^{'}\|_B < \eta$ one has $t + t^{'} \in \Phi^0_B$.

Lemma 2.5. [6, Lemma 2.2] (i) If $t \in \Phi_B$ and $t^{'} \in \Phi^l_B$ (resp. $t^{'} \in \Phi^r_B$), then $tt^{'} \in \Phi^l_B$ (resp. $t^{'}t \in \Phi^r_B$).
(ii) If $t \in \Phi_B \cap B$ and $t^{'} \in \Phi^l_B$ (resp. Φ^r_B), then $t^{'}t \in \Phi^l_B$ (resp. $tt^{'} \in \Phi^r_B$).
(iii) If $t, t^{'} \in \Phi_B$, then $tt^{'} \in \Phi_B$.
(iv) If $t, t^{'} \in \Phi^0_B$, then $tt^{'} \in \Phi^0_B$.

Proposition 2.6. (1) If $t \in \Phi^l_B$ and $t^{'} \in \Phi^l_B$, then $tt^{'} \in \Phi^l_B$.
(2) If $t \in \Phi^r_B$ and $t^{'} \in \Phi^r_B$, then $tt^{'} \in \Phi^r_B$.
(3) $t \in \Phi^l_B \Leftrightarrow -t \in \Phi^l_B \Leftrightarrow t^n \in \Phi^l_B$ for all $n \in \mathbb{N}$.
(4) $t \in \Phi^r_B \Leftrightarrow -t \in \Phi^r_B \Leftrightarrow t^n \in \Phi^r_B$ for all $n \in \mathbb{N}$.
(5) If $tt^{'} \in \Phi^l_B$ (resp. $tt^{'} \in \Phi^r_B$), then $t^{'} \in \Phi^l_B$ (resp. $t \in \Phi^r_B$).

Proof. Let prove (1) and (3), then properties (2) and (4) can be done in the same way .
(1) Let $t, t^{'} \in \Phi^l_B$. Then there exist $p, q \in B$ and $j, k \in K_B$ such that $pt = 1 - j$ and $qt^{'} = 1 - k$. It follows that

$$\begin{aligned} qptt^{'} &= q(1 - j)t^{'} \\ &= 1 - k - qjt^{'}. \end{aligned}$$

Thus, we have $btt^{'} = 1 - m$, where $b = qp \in B$ and $m = k + qjt^{'} \in K_B$, this proves that $tt^{'} \in \Phi^l_B$.
(3) Obviously, $t \in \Phi^l_B \Leftrightarrow -t \in \Phi^l_B$. Now, we show that $t \in \Phi^l_B \Leftrightarrow t^n \in \Phi^l_B$ for all $n \in \mathbb{N}$. Let $t \in \Phi^l_B$. By induction, it comes from (i) that $t^n \in \Phi^l_B$. Conversely, if $t^n \in \Phi^l_B$, then there exists $p \in B$ and $k \in K_B$ such that $pt^n = 1 - k$. Thus, $bt = 1 - k$, where $b = pt^{n-1} \in B$ and $k \in K_B$ which proves that $t \in \Phi^l_B$.
(5) This property is easily checked by using definition. Q.E.D.

Remark 2.2. Results of Proposition 2.6 are a generalization of some depicted findings (see [7, Theorem 1]).

The concept of relative demicompactness (or demicompactness in particular) on a Banach algebra will now be presented. This class is key in refining several established properties investigated in the theory of linear operators acting on a Banach spaces (see [7]).

Definition 2.7. Let $s_0 \in A$. An element $t \in A$ is said to be s_0-demicompact (or relative demicompact with respect to s_0), if there exist $p \in B$ and $k \in K_B$ such that $p(s_0 - t) = 1 - k$. We denote by $\mathcal{DC}_{s_0}(B)$, the class of s_0-demicompact elements of A with respect to B.
When $s_0 = 1$, t is simply said to be demicompact and by $\mathcal{DC}(B)$, we denote the class of demicompact elements of A with respect to B.

Example 2.8. (i) Let $t, s_0 \in A$. Obviously, if $s_0 - t$ has a left Fredholm inverse modulo K_B, then t is s_0-demicompact. This example is often viewed as a generalization of [8, Proposition 2.3].
(ii) If $s_0 \in K_B$, then each $t \in Inv_B(A)$ is s_0-demicompact. Indeed, we have $Inv_B(A) \subset \Phi^l_B$, then if $t \in Inv_B(A)$ there exist $b \in B$ and $j \in K_B$ such that $bt = 1 - j$. Therefore, $b^{'}(s_0 - t) = 1 - m$ wherever $b^{'} = -b \in B$ and $m = bs_0 + j \in K_B$ which proves that t is s_0-demicompact.
(iii) If $s_0 \in Inv_B(A)$ and $t \in A$ such that $s_0^{-1}t \in K_B$ or nilpotent, i.e., there exists $n \in \mathbb{N} \setminus \{0\}$ such that $(s_0^{-1}t)^n = 0$, then t is s_0-demicompact.
(iv) Let X be a real Banach space, and set $A = B = \mathcal{L}(X)$ and take $K_B = \mathcal{K}(X)$. One can easily shake that $\mathcal{K}(X) \subset \mathcal{DC}\big(\mathcal{L}(X)\big)$. Then, compact operators play a motivating role once study perturbations of Fredholm operators.

By referring to the usual definition of an element relative demicompact with respect to s_0, we can check the following properties.

Proposition 2.9. Let $s_0, t \in A$. Then the following assertions hold.
(i) If $s_0 \in Inv_B(A)$, then $K_B \subset \mathcal{DC}_{s_0}(B)$.
(ii) $t \in \mathcal{DC}_{s_0}(B)$ if, and only if, $s_0 - t \in \Phi^l_B$.
(iii) $\mathcal{DC}_{s_0}(B) + K_B \subset \mathcal{DC}_{s_0}(B)$.
(iv) Let $t \in B \cap Inv_B(A)$. Then, $t \in \mathcal{DC}_{s_0}(B)$ if, and only if, $s_0 t^{-1} \in \mathcal{DC}(B)$.

Remark 2.3. (i) An element $t \in K_B$ cannot be necessarily s_0-demicompact when s_0 is not invertible, then K_B is strictly included in $\mathcal{DC}_{s_0}(B)$. However, $\mathcal{DC}_{s_0}(B)$ cannot be an ideal of B in general. Indeed, let X be a Banach space, take $B = \mathcal{L}(X)$ and let $T \in \mathcal{L}(X)$. Obviously, if T^n is demicompact for all $n \in \mathbb{N} \setminus \{0\}$, thereby T is demicompact. The converse could not be valid for instance when X is not reflexive and $T \in \mathcal{DC}(X) \cap \mathcal{L}(X)$ which satisfies $T^2 = I_X$, then T^2 is not demicompact. Therefore, in general, $\mathcal{DC}\big(\mathcal{L}(X)\big)$ is not an ideal of $\mathcal{L}(X)$.
(ii) Properties of Proposition 2.9 stand valid and we find the analogousness outcomes in Fredholm theory if we substitute the subalgebra B by $\mathcal{L}(X)$, K_B by $\mathcal{K}(X)$ and Φ^l_B by $\Phi_+(X)$, where $\Phi_+(X)$ stands for the upper semi-Fredholm operators set from X into X. For instance, it was shown in [8, Theorem 2.6] that if T is a closed linear operator, then T is s_0-demicompact if, and only if, $s_0 - T \in \Phi_+(X)$. The assertion (ii) seems to be the analogous of this theorem relative to a Banach subalgebra B. Although such resemblance, it is not possible to identify the set $\Phi^l_{\mathcal{L}(X)} := \{T \in \mathcal{L}(X) : I_X - T_l T \in \mathcal{K}(X) \text{ for some } T_l \in \mathcal{L}(X)\}$ with $\Phi_+(X)$, so we have just $\Phi^l_{\mathcal{L}(X)} \subsetneq \Phi_+(X)$ (see [12, page. 15]).

Lemma 2.10. Let $s_0, t \in A$. If $t \in \mathcal{DC}_{s_0}(B)$, then there exists $\tau > 0$ such that for all $t^{'} \in B$, with $\|t^{'}\|_B < \tau$ we have $t + t^{'} \in \mathcal{DC}_{s_0}(B)$.

Proof. We have $t \in \mathcal{DC}_{s_0}(B)$, then there exist $p \in B$ and $k \in K_B$ such that for all $t^{'} \in B$, we can write

$$p[s_0 - (t + t^{'})] = 1 - k - pt^{'}.$$

For $\tau = \|p\|_B^{-1}$, we get $\|pt^{'}\|_B < 1$ and so, $1 - pt^{'} \in Inv_B(A)$. Therefore,

$$(1 - pt^{'})^{-1}p[s_0 - (t + t^{'})] = 1 - (1 - pt^{'})^{-1}k.$$

Hence, $b[s_0 - (t + t^{'})] = 1 - j$, where $b = (1 - pt^{'})^{-1}p \in B$ and $j = (1 - pt^{'})^{-1}k \in K_B$. This proves that $t + t^{'} \in \mathcal{DC}_{s_0}(B)$. Q.E.D.

Remark 2.4. From Lemma 2.10, we can easily check that $\mathcal{DC}_{s_0}(B)$ is an open set of B.

Proposition 2.11. 1) Assume that A is commutative. Let $n \in \mathbb{N} \setminus \{0\}$ and $t, s_0 \in A$. Then the following assertions hold.

(*i*) If $s_0t^n \in \mathcal{DC}_{s_0}(B)$, then $s_0t \in \mathcal{DC}_{s_0}(B)$.

(*ii*) $s_0t^2 \in \mathcal{DC}_{s_0}(B)$ if, and only if, $s_0t \in \mathcal{DC}_{s_0}(B)$ and $-s_0t \in \mathcal{DC}_{s_0}(B)$.

2) Let $t, s_0 \in A$ (A not necessarily commutative). Then we have:

(*i*) $s_0 - t \in \mathcal{DC}_{s_0}(B)$ if, and only if, $s_0 - t^n \in \mathcal{DC}_{s_0}(B)$, for all $n \in \mathbb{N} \setminus \{0\}$.

(*ii*) If $t, s_0 \in B$ and $\lim\limits_{n\to 0}\Big(\|(1 + t - s_0)^n\|_B\Big)^{\frac{1}{n}} = 0$, then there exists $n_0 \in \mathbb{N}$ such that $t^n \in \mathcal{DC}_{s_0}(B)$ for all $n \geq n_0$.

Proof. 1) (*i*) If $s_0t^n \in \mathcal{DC}(B)$, there exist $p \in B$ and $k \in K_B$ such that $ps_0(1 - t^n) = 1 - k$.
Then,

$$ps_0\Big(\sum_{i=0}^{n-1} t^i\Big)\Big(1 - t\Big) = 1 - k.$$

Therefore, $b(s_0 - s_0t) = 1 - k$, where $b = p\Big(\sum_{i=0}^{n-1} t^i\Big) \in B$ and $k \in K_B$. This shows that $s_0t \in \mathcal{DC}_{s_0}(B)$.

(*ii*) Using assertion (*i*) of 1) and according to $s_0t^2 \in \mathcal{DC}s_0(B)$ we get $s_0t \in \mathcal{DC}_{s_0}(B)$. In the other hand, since $s_0t^2 \in \mathcal{DC}s_0(B)$, there exist $b \in B$ and $j \in K_B$ such that $b^{'}(s_0 + s_0t) = 1 - j$, where $b^{'} = b(1 - t)$. This proves that $-s_0t \in \mathcal{DC}(B)$.

Conversely, if $s_0t \in \mathcal{DC}_{s_0}(B)$ and $-s_0t \in \mathcal{DC}_{s_0}(B)$ there exist $b, b^{'} \in B$ and $j, j^{'} \in K_B$ such that

$$\begin{cases} b(s_0 - s_0t) = 1 - j \\ b^{'}(s_0 + s_0t) = 1 - j^{'}. \end{cases}$$

Then,

$$b(s_0 - s_0t)b^{'}(s_0 - s_0t^2) = (1 - j)(1 - j^{'})(1 - t).$$

From which, we infer that

$$c(s_0 - s_0t^2) = 1 - m,$$

where $c = b(s_0 - s_0t)b^{'} \in B$ and $m = j + b[j^{'} + j(1 - j^{'})](1 - t) \in K_B$. This proves that $s_0t^2 \in \mathcal{DC}s_0(B)$.

2) (*i*) Using Proposition 2.6, we get $s_0 - t \in \mathcal{DC}(B) \Leftrightarrow t \in \Phi_B^l \Leftrightarrow t^n \in \Phi_B^l \Leftrightarrow s_0 - t^n \in \mathcal{DC}(B)$.

(*ii*) By dint of $\lim\limits_{n\to 0}\Big(\|(1 + t - s_0)^n\|_B\Big)^{\frac{1}{n}} = 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ we get $\|(1 + t - s_0)^n\|_B < 1$, thus $(s_0 - t) \in Inv_B(A) \subset \Phi_B^l$. Hence, $t \in \mathcal{DC}_{s_0}(B)$. Q.E.D.

Remark 2.5. The property (*ii*) of 2) could be perceived of as a generalization of the examined results in spectral, Fredholm and relative demicompactness theories, exactly, if X is a Banach space, $A = B = \mathcal{L}(X)$ and $T \in \mathcal{L}(X)$. Recall (see [16]) that $T_{S_0} = S_0 - T \in \mathcal{L}(X)$ is said to be quasi-nilpotent if spr $T_{S_0} = 0$, where spr T_{S_0} denotes the spectral radius of T_{S_0} defined by:

$$\text{spr } T_{S_0} = \lim_{n\to 0}\Big(\|T_{S_0}^n\|_X\Big)^{\frac{1}{n}}.$$

If T_{S_0} is quasi-nilpotent ($\lim\limits_{n\to 0}\left(\|T^n_{S_0}\|_X\right)^{\frac{1}{n}} = 0$), then there exists $n_0 \in \mathbb{N}$ such that $\|(S_0 - T)^n\|_X < 1$. Therefore, $I + t - S_0$ is invertible which is equivalent to $S_0 - T \in \Phi_+(X)$. Hence, by using [8, Theorem 2.6], we deduce that T is S_0-demicompact.

3. Relative demicompactness perturbation in a Banach subalgebra

Here, we bring in the set of relative demicompact perturbations of B denoted by $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ and we show that is a two-sided closed ideal of B containing $Pr(\Phi^l_B)$. The purpose is to achieve some perturbation results in Section 4.

Definition 3.1. Let $s_0 \in A$. An element $p \in B$ is called s_0-demicompact perturbation if $p + t \in \mathcal{DC}_{s_0}(B)$ for all $t \in \mathcal{DC}_{s_0}(B)$. We denote by $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ the set of s_0-demicompact perturbations of B. When $s_0 = 1$, t is simply said to be a demicompact perturbation and by $Pr\Big(\mathcal{DC}(B)\Big)$, we denote the class of demicompact perturbations of B.

Remark 3.1. By referring to Proposition 2.9, we can easily show that:

(i) $F_B \subset K_B \subset Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

(ii) $K_B + Pr\Big(\mathcal{DC}_{s_0}(B)\Big) \subset \mathcal{DC}_{s_0}(B)$.

Example 3.2. (i) Suppose that F_B is a left ideal of A and let $t \in A$. An element $a \in A$ (see [2, Section 4]) is said to be t-inessential if a is written as the form $a = tj + j'$, where $j, j' \in K_B$ (when F_B is a right ideal of A, t-inessential elements have the form $jt + j'$). Then from Remark 3.1, all t-inessential elements of A are s_0-demicompact perturbations of B. In addition, by [2, Note 14], if $a \in A$ is t-inessential then a and $-a$ are s_0-demicompact perturbations of B.

(ii) Let X be a Hilbert space, and assume that $\{X_i\}$, $i \geq 1$, could be a sequence of closed subspaces of X such that $X_i \perp X_j$ once $i \neq j$, by possessing the condition $X = \oplus\sum\limits_{I=1}^{\infty} X_i$. Consider A the algebra of all sequences $T = \{T_i\}_{i\geq 1}$, wherever $\{T_i\} \in \mathcal{L}(X_i)$, for $i \geq 1$. Set $B = \Big\{T = \{T_i\}_{i\geq 1} \in A : \text{ the sequence of norms } \{\|T_i\|\} \text{ is bounded}\Big\}$. If we take $\|T_B\| = \sup\Big\{\|T_i\| : \ i \geq 1\Big\}$, then, $(B, \|\cdot\|_B)$ is a C^*-algebra (see [2]). Consider the two following sets:

$K_B = \Big\{\{T_i\}_{i\geq 1} \in B$: every T_i belongs to $\mathcal{K}(X_i)$, and $\lim\limits_{i\to 0}\|T_i\| = 0\ \Big\}$,

$F_B = \Big\{\{T_i\}_{i\geq 1} \in A$: each T_i has finite-dimensional range, and $T_i = 0$ nearly at the most a finite number of indices $i\Big\}$.

During this case, F_B could be a two-sided ideal of A, and since the closure of F_B within the B-norm is K_B, F_B/K_B becomes a radical algebra. Hence, all operators $\{T_i\}_{i\geq 1}$ of F_B or also K_B belong to $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

Proposition 3.3. Let $t \in A$.

(1) If $t \in \Phi_B \cap B$ and $p \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$, then

(i) $tp \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

(ii) $pt \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

(2) If $t \in \Phi^0_B \cap B$ and $p \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$, then

(i) $tp \in \mathcal{DC}_{s_0}(B) \cap Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

(ii) $pt \in \mathcal{DC}_{s_0}(B) \cap Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$.

(3) $\Big(Pr\Big(\mathcal{DC}_{s_0}(B)\Big), +\Big)$ is a subgroup of $(B, +)$.

Proof. (1) (i) Let $t^{'} \in \mathcal{DC}_{s_0}(B)$. We have to show that $t^{'} + tp \in \mathcal{DC}_{s_0}(B)$. By using both Proposition 2.3 and the fact that $t \in \Phi_B \cap B$, we deduce the existence of $b \in B$ and $k, j \in F_B$ such that $tb = 1 - k$ and $bt = 1 - j$. In addition, since $t^{'} \in \mathcal{DC}_{s_0}(B)$ there exist $b^{'} \in B$ and $j^{'} \in K_B$ such that $b^{'}(s_0 - t^{'}) = 1 - j^{'}$.
Furthermore, we can see that

$$\begin{aligned} b^{'}tb(s_0 - t^{'}) &= b^{'}(1-k)(s_0 - t^{'}) \\ &= b^{'}(s_0 - t^{'}) - b^{'}k(s_0 - t^{'}) \\ &= 1 - j^{'} - b^{'}k(s_0 - t^{'}) \\ &= 1 - m, \end{aligned}$$

where $b^{'}t \in B$ and $m \in K_B$. Hence, $b(s_0 - t^{'}) \in \Phi^l_B$ and so, $s_0 - b(s_0 - t^{'}) \in \mathcal{DC}_{s_0}(B)$.
Moreover, $p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, then $p + s_0 - b(1 - t^{'}) \in \mathcal{DC}_{s_0}(B)$ and it follows that $[-p + b(s_0 - t^{'})] \in \Phi^l_B$. By dint of Proposition 2.6, we have $t[-p + b(s_0 - t^{'})] \in \Phi^l_B$, so

$$s_0 + tp - tb(1 - t^{'}) = tp + t^{'} + k(s_0 - t^{'}) \in \mathcal{DC}_{s_0}(B).$$

This implies that $t^{'} + tp \in \mathcal{DC}_{s_0}(B)$, so $tp \in Pr\big(\mathcal{DC}(B)\big)$.
(ii) To prove that $pt \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, we can write $(s_0 - t^{'})b \in \Phi^l_B$. Indeed:

$$\begin{aligned} tb^{'}(s_0 - t^{'})b &= t(1 - j^{'})b \\ &= tb - tj^{'}b \\ &= 1 - k - tj^{'}b \\ &= 1 - m^{'}, \end{aligned}$$

where $tb^{'} \in B$ and $m^{'} \in K_B$. Therefore, $(s_0 - t^{'})b \in \Phi^l_B$, so $s_0 - (s_0 - t^{'})b \in \mathcal{DC}_{s_0}(B)$. Moreover, $p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, then $p + s_0 - (s_0 - t^{'})b \in \mathcal{DC}_{s_0}(B)$ which implies that $[-p + (s_0 - t^{'})b] \in \Phi^l_B$. Thus, by using Proposition 2.6, we infer that $[-p + (s_0 - t^{'})b]t \in \Phi^l_B$, so

$$s_0 + pt - (s_0 - t^{'})bt = pt + t^{'} + (s_0 - t^{'})j \in \mathcal{DC}_{s_0}(B).$$

In addition, $-(1 - t^{'})j \in K_B \subset Pr\big(\mathcal{DC}(B)\big)$ implies that $t^{'} + pt \in \mathcal{DC}_{s_0}(B)$. Hence, $pt \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$.
(2) (i) Let $t^{'} \in \mathcal{DC}_{s_0}(B)$, we prove that $t^{'} + tp \in \mathcal{DC}_{s_0}(B)$. If $t \in \Phi^0_B$, then there exist $a \in F_B$ and $b \in Inv_B(A)$ such that $t = a + b$. Knowing that $b \in \Phi_B \cap B$, it follows that $bp \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ and using $ap \in K_B$ we deduce, by using Remark 3.1, that $tp = ap + bp \in \mathcal{DC}_{s_0}(B)$. Moreover, we have $bp \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ leads to $t^{'} + tp = t^{'} + ap + bp \in \mathcal{DC}_{s_0}(B)$, it follows that $tp \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Thus, $tp \in \mathcal{DC}_{s_0}(B) \cap Pr\big(\mathcal{DC}_{s_0}(B)\big)$.
(ii) Let $t \in \Phi^0_B \cap B$. We can write that $pt = pa + pb$, where $pa \in K_B$ and $pb \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Proceeding as in ($i$), we obtain $pt \in Pr\big(\mathcal{DC}_{s_0}(B)\big) \cap \mathcal{DC}_{s_0}(B)$.
(3) Obviously, $0 \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Let $p, p^{'} \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ and let $t \in \mathcal{DC}_{s_0}(B)$. By referring to assertion (1), we deduce that $-p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Furthermore, $p + p^{'} + t \in \mathcal{DC}_{s_0}(B)$ because $p^{'} + t \in \mathcal{DC}_{s_0}(B)$. Hence, $p + p^{'} \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. So, $p - p^{'} \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, which achieves the proof. Q.E.D.

Corollary 3.4. $Pr\big(\mathcal{DC}_{s_0}(B)\big)$ is a two-sided closed ideal of B.

Proof. It's ample to show that $tp \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ for all $p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ and $t \in B$. Let $t^{'} \in \Phi_B \cap B$. By referring to Lemmas 2.4 and 2.5, we can write $t = (t - \mu t^{'}) + \mu t^{'}$, wherever $(t - \mu t^{'})$ and $\mu t^{'}$ belong to

$\Phi_B \cap B$. It follows, by using Proposition 3.3, that $tp = (t - \mu t')p + \mu t' p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Similarly, show that $pt \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Hence, $Pr\big(\mathcal{DC}_{s_0}(B)\big)$ is a two-sided ideal of B. It remains to show that $Pr\big(\mathcal{DC}_{s_0}(B)\big)$ is a closed subset of B. Let $t \in \mathcal{DC}_{s_0}(B)$ and set $(a_n)_{n\in\mathbb{N}}$ any sequence in $Pr\big(\mathcal{DC}_{s_0}(B)\big)$ with $\|a_n - a\|_B \to 0$. By virtue of Lemma 2.10, there exists $\lambda > 0$ such that for every $t' \in B$ with $\|t'\|_B < \lambda$, we have $t + t' \in \mathcal{DC}_{s_0}(B)$. Moreover, there exists $m \in \mathbb{N}$ such that $\|a - a_m\|_B < \lambda$, thus $t + a - a_m \in \mathcal{DC}(B)$. By means of $a_k \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, it follows that $t + a \in \mathcal{DC}_{s_0}(B)$. Hence, $a \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$ which achieve the proof. Q.E.D.

Corollary 3.5. $F_B \subset K_B \subset Pr\big(\Phi_B\big) \subset Pr\big(\Phi^l_B\big) \subset Pr\big(\mathcal{DC}_{s_0}(B)\big)$.

Proof. In [6, Remark 2.3] the authors showed that $Pr\big(\Phi_B\big) \subset Pr\big(\Phi^l_B\big)$, then it solely remains to show that $Pr\big(\Phi^l_B\big) \subset Pr\big(\mathcal{DC}_{s_0}(B)\big)$. To do so, let $p \in Pr\big(\Phi^l_B\big)$ and let $t \in \mathcal{DC}_{s_0}(B)$. Obviously, we have $-p + s_0 - t \in \Phi^l_B$ this implies that $p + t \in \mathcal{DC}_{s_0}(B)$. Hence, $p \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$. Q.E.D.

Remark 3.2. In [10], it was shown that $\mathcal{F}^b(X)$ is a two-sided ideal of $\mathcal{L}(X)$, where $\mathcal{F}^b(X)$ is the upper semi-Fredholm perturbations set in $\mathcal{L}(X)$. If we set $A = B = \mathcal{L}(X)$, then $Pr(\Phi_B) = \mathcal{F}^b(X)$ (see [6]), this implies that $Pr\big(\mathcal{DC}_{s_0}(\mathcal{L}(X))\big)$ include all upper semi-Fredholm perturbations in $\mathcal{L}(X)$. Despite, it is worth indicating that the structure ideal of $\mathcal{L}(X)$ is very complex to investigate. Several results used and described on ideal structure run the prominent closed ideals stemming from operator or Fredholm theories and a few applied works such as Fredholm perturbations, weakly compact or compact operators and other fields.

4. Some perturbation and stability results

Let B be a Banach subalgebra of a given algebra A. Here, the purpose is to show some perturbation and essential spectra results for an element $t \in B$ by including the demicompactness concept on A with respect to B and to examine their stability. Foremost, we remind some essential spectra definitions and outcomes.

Definition 4.1. For $t \in A$ and $s_0 \in A \setminus \{0\}$, define the following s_0-essential spectra:

$\Phi_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \in \Phi_B\}$,
$\Phi^l_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \in \Phi^l_B\}$,
$\Phi^r_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \in \Phi^r_B\}$,
$F\sigma_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \notin \Phi_B\} := \mathbb{C} \setminus \Phi_{B,\,s_0}(t)$,
$F\sigma^l_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \notin \Phi^l_B\} := \mathbb{C} \setminus \Phi^l_{B,\,s_0}(t)$,
$F\sigma^r_{B,\,s_0}(t) = \{\lambda \in \mathbb{C} : (\lambda s_0 - t) \notin \Phi^r_B\} := \mathbb{C} \setminus \Phi^r_{B,\,s_0}(t)$,
$W\sigma_B(t) = \bigcap_{k\in K_B} \sigma_B(t + k)$.

Note that if $s_0 = 1$, we recover the usual definitions of the Fredholm spectrums of t defined in [2, 6] by:

$F\sigma_B(t) = \{\lambda \in \mathbb{C} : (\lambda - t) \notin \Phi_B\} := \mathbb{C} \setminus \Phi_B(t)$,
$F\sigma^l_B(t) = \{\lambda \in \mathbb{C} : (\lambda - t) \notin \Phi^l_B\} := \mathbb{C} \setminus \Phi^l_B(t)$,
$F\sigma^r_B(t) = \{\lambda \in \mathbb{C} : (\lambda - t) \notin \Phi^r_B\} := \mathbb{C} \setminus \Phi^r_B(t)$.

The sets $F\sigma_{B,\,s_0}(t)$, $F\sigma^l_{B,\,s_0}(t)$ and $F\sigma^r_{B,\,s_0}(t)$ are called the Fredholm spectrum, the left Fredholm spectrum and the right Fredholm spectrum of t relative to s_0, respectively.
In addition, for $t \in A$, $s_0 \in A \setminus \{0\}$ define the following sets:

$$I^l_t(A) := \{t_l \in A : \ t_l \text{ is a left Fredholm inverse of t}\};$$

$$\Gamma_{l,\,s_0,\,t} := \{a \in A : \ \forall \lambda \in \Phi^l_{B,\,s_0}(t),\ \exists\, t_{\lambda l} \in I^l_{\lambda s_0 - t}(A),\ (a t_{\lambda l}) \in \mathcal{DC}(B)\}.$$

Theorem 4.2. Let $t, t^{'} \in A$ and $s_0 \in A \setminus \{0\}$. If for every $\lambda \in \Phi^l_{B,\, s_0}(t)$ there exists $t_{\lambda_l} \in I^l_{\lambda s_0 - t}(A)$ such that $t^{'} t_{\lambda_l}$ is demicompact, then

$$F\sigma^l_{B,\, s_0}(t + t^{'}) \subseteq F\sigma^l_{B,\, s_0}(t).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$, and $t_{\lambda_l} \in I^l_{\lambda s_0 - t}(A)$, then there exists $k \in K_B$ such that

$$t_{\lambda l}(\lambda s_0 - t) = 1 - k.$$

Therefore, we can write

$$\lambda s_0 - t - t^{'} \quad = \quad (1 - t^{'} t_{\lambda l})(\lambda s_0 - t) - t^{'} k. \tag{1}$$

Suppose that $\lambda \notin F\sigma^l_{B,\, s_0}(t)$, then $(\lambda s_0 - t) \in \Phi^l_B$ and since $t^{'} t_{\lambda_l}$ is demicompact, it follows from Proposition 2.11 that $1 - t^{'} t_{\lambda_l} \in \Phi^l_B$. Consequently, by using Proposition 2.6, we get $(1 - t^{'} t_{\lambda_l})(\lambda s_0 - t) \in \Phi^l_B$. Furthermore, knowing that $t^{'} k \in K_B \subset Pr\big(\Phi^l_B\big)$, we deduce from Eq (1) that $\lambda s_0 - t - t^{'} \in \Phi^l_B$. Hence, $\lambda \notin F\sigma^l_{B,\, s_0}(t + t^{'})$ and this proves that $F\sigma^l_{B,\, s_0}(t + t^{'}) \subseteq F\sigma^l_{B,\, s_0}(t)$. Q.E.D.

Remark 4.1. Theorem 4.2 could be seen as a generalization of [17, Theorem 3.1] whereby $\sigma_{e_{1,S}}(\cdot)$: the Gustafson S-essential spectrum relative to Fredholm theory in a Banach space is expanded by $F\sigma^l_{B,\, s_0}(\cdot)$: the left Fredholm spectrum relative to Fredholm theory in a Banach subalgebra B of A.

The object of our study here is to present some perturbation results of the s_0-essential spectrum of elements of the A involving the relative demicompactness concept.

Theorem 4.3. Let A be a Banach alegbra, let $t_1, t_2 \in A$, and $s_0 \in Inv_B(A)$. Assume that $res_{B,\, s_0}(t_1) \cap res_{B,\, s_0}(t_2) \neq \emptyset$. If for some $\lambda \in res_{B,\, s_0}(t_1) \cap res_{B,\, s_0}(t_2)$, we have $(\lambda s_0 - t_1)^{-1} - (\lambda s_0 - t_2)^{-1} \in \Gamma_{l,\, s_0^{-1},\, -(\lambda s_0 - t_1)^{-1}}$, then

$$F\sigma^l_{B,\, s_0}(t_1) \subseteq F\sigma^l_{B,\, s_0}(t_2).$$

Proof. Let $\lambda \in res_{B,\, s_0}(t_1) \cap res_{B,\, s_0}(t_2)$. Assume, without loss of generality, that $\lambda = 0$ (i.e., $(-t_1)^{-1} - (-t_2)^{-1} \in \Gamma_{l,\, s_0^{-1},\, t_1^{-1}}$). For every $\mu \neq 0$ and $k \in \{1, 2\}$, we can write

$$\mu s_0 - t_k = -\mu s_0(\mu^{-1} s_0^{-1} - t_k^{-1}) t_k.$$

Under the assumption $0 \in res_{B,\, s_0}(t_k)$, we infer, by both Proposition 2.6 and Lemma 2.5, that

$$\mu \in \Phi^l_{B,\, s_0}(t_k) \text{ if, and only if, } \ \mu^{-1} \in \Phi^l_{B,\, s_0^{-1}}(t_k^{-1}).$$

Basing on $(-t_1)^{-1} - (-t_2)^{-1} \in \Gamma_{l,\, s_0^{-1},\, t_1^{-1}}$ and by using Theorem 4.2, we infer that $\Phi^l_{B,\, s_0^{-1}}(t_2) \subset \Phi^l_{B,\, s_0^{-1}}(t_1)$. Hence, $F\sigma^l_{B,\, s_0}(t_1) \subseteq F\sigma^l_{B,\, s_0}(t_2)$. Q.E.D.

Theorem 4.4. Let $t, t^{'}, s_0 \in A$. Assume that $res_B(t) \cap res_B(t^{'}) \neq \emptyset$ and for some $\mu \in res_B(t) \cap res_B(t^{'})$, we have $(\mu - t)^{-1} - (\mu - t^{'})^{-1} \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, then

$$F\sigma^l_B(t) = F\sigma^l_B(t^{'}).$$

Proof. Let $\lambda \notin F\sigma^l_B(t)$. Then, $(\lambda - t) \in \Phi^l_B$, it follows, by referring to [6, Theorem 3.2], that for $\mu \neq \lambda$ we have $s_0 - \big((\mu - \lambda)^{-1} - (\mu - t)^{-1}\big) \in \mathcal{DC}_{s_0}(B)$. According to $(\mu - t)^{-1} - (\mu - t^{'})^{-1} \in Pr\big(\mathcal{DC}_{s_0}(B)\big)$, we deduce that $s_0 - \big((\mu - \lambda)^{-1} - (\mu - t^{'})^{-1}\big) \in \mathcal{DC}_{s_0}(B)$. Hence, $\big((\mu - \lambda)^{-1} - (\mu - t^{'})^{-1}\big) \in \Phi^l_B$. Again, by using [6, Theorem 3.2], we infer that $(\lambda - t^{'}) \in \Phi^l_B$ which implies that $F\sigma^l_B(t^{'}) \subset F\sigma^l_B(t)$. Following the same reasoning as in this proof, the opposite inclusion is evidenced. Consequently, $F\sigma^l_B(t) = F\sigma^l_B(t^{'})$. Q.E.D.

Corollary 4.5. Let $t, s_0 \in A$ and $h = tb + c$ be a t-bounded element of A, where $b, c \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$. Suppose that $r_{\sigma_B}\Big((\lambda - t)^{-1})h\Big) < 1$ for some $\lambda \in res_B(t)$, then:

$$F\sigma^l_B(t + h) = F\sigma^l_B(t).$$

Proof. Set $t' = t+h$ and $(\lambda-t')\ell = s$, where $\ell, s \in B$. Thus, $(\lambda-t-h)\ell = s$, it follows that $(\lambda-t)\Big(1-(\lambda-t)^{-1}h\Big)\ell = s$, this implies $\Big(1-(\lambda-t)^{-1}h\Big)\ell = (\lambda-t)^{-1}s$. According to $r_{\sigma_B}\Big((\lambda-t)^{-1})h\Big) < 1$, we obtain $\ell = \sum\limits_{n\geq 0}\Big((\lambda-t)^{-1}h\Big)^n(\lambda-t)^{-1}s$. Moreover, from $(\lambda - t) \in Inv_B(A)$, we infer that $(\lambda - t)^{-1}h \in B$, then we deduce, by using [4, Proposition 7], that $(\lambda - t') \in Inv_B(A)$ and $(\lambda - t')^{-1} = \sum\limits_{n\geq 0}\Big((\lambda - t)^{-1}h\Big)^n(\lambda - t)^{-1}$. Consequently,

$$\begin{aligned}(\lambda - t')^{-1} - (\lambda - t)^{-1} &= \sum_{n\geq 1}\Big((\lambda - t)^{-1}h\Big)^n(\lambda - t)^{-1}\\ &= (\lambda - t)^{-1}h\;[1 + \sum_{n\geq 2}\Big((\lambda - t)^{-1}h]^{n-1}\Big)(\lambda - t)^{-1}.\end{aligned}$$

By virtue of $Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$ is a two-sided ideal of B, we deduce that h, $(\lambda - t)^{-1}h \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$, then $(\lambda - t')^{-1} - (\lambda - t)^{-1} \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)$. Hence, Theorem 4.4 easily achieves the proof. Q.E.D.

Let $\lambda \in \mathbb{C}$. If t_{λ_l} is a left Fredholm inverse of $\lambda - t$ modulo K_B, then there exists $k \in K_B$ such that

$$t_{\lambda_l}(\lambda - t) = 1 - k.$$

Therefore, we can write

$$\lambda - t - t' \;=\; (1 - t't_{\lambda_l})(\lambda - t) - t'k. \tag{2}$$

In the same way, if there exists t_{λ_r} a right Fredholm inverse of $\lambda - t$ modulo K_B, we can write

$$\lambda - t - t' \;=\; (\lambda - t)(1 - t_{\lambda_r}t') - k't', \tag{3}$$

where $k' \in K_B$.

Theorem 4.6. Let $t, t', s_0 \in A$. Assume that the following assertions hold:
(i) For every $\lambda \in \Phi^l_B(t + t') \setminus \{0\}$, there exists $F_{\lambda l}$ (resp. $F_{\lambda r}$) a left (resp. a right) Fredholm inverse of $(\lambda - t - t')$ modulo K_B such that $-\lambda^{-1}tt'F_{\lambda l}$ (resp. $-\lambda^{-1}F_{\lambda r}tt'$) is demicompact.
(ii) For every $\lambda \in \Phi^l_B(t + t') \setminus \{0\}$, there exists $G_{\lambda l}$ (resp. $G_{\lambda r}$) a left (resp. a right) Fredholm inverse of $(\lambda - t - t')$ modulo K_B such that $-\lambda^{-1}t'tG_{\lambda l}$ (resp. $-\lambda^{-1}G_{\lambda r}t't$) is demicompact.
Then we have

$$\Big(F\sigma^l_B(t) \cup F\sigma^l_B(t')\Big) \setminus \{0\} \subseteq F\sigma^l_B(t + t') \setminus \{0\}.$$

Proof. Set $\alpha \in \mathbb{C} \setminus \{0\}$. Assume that $k \in K_B$ such that $F_{\alpha l}(\alpha - t - t') = 1 - k$, then we can write

$$\begin{aligned}(\alpha - t)(\alpha - t') &= \alpha(\alpha - t - t') + tt'\\ &= \alpha(\alpha - t - t') + tt'F_{\alpha l}(\alpha - t - t') + tt'k,\end{aligned}$$

which implies that

$$(\alpha - t)(\alpha - t') \;=\; \alpha(1 + \alpha^{-1}tt'F_{\alpha l})(\alpha - t - t') + tt'k. \tag{4}$$

Similarly,, we have

$$(\alpha - t')(\alpha - t) \;=\; \alpha(1 + \alpha^{-1}t'tF_{\alpha l})(\alpha - t - t') + t'tk. \tag{5}$$

Now assume that $k^{'} \in K_B$ such that $(\alpha - t - t^{'})F_{\alpha r} = 1 - k^{'}$, then we can write

$$(\alpha - t)(\alpha - t^{'}) = \alpha(\alpha - t - t^{'})(1 + \alpha^{-1}F_{\alpha r}tt') + k^{'}tt', \quad (6)$$

and

$$(\alpha - t^{'})(\alpha - t) = \alpha(\alpha - t - t^{'})(1 + \alpha^{-1}F_{\alpha r}t^{'}t) + k^{'}t^{'}t. \quad (7)$$

Furthermore, if $\alpha \in \Phi^l_{B,\,(t+t^{'})} \setminus \{0\}$, we deduce from the assumptions of this theorem that $1+\alpha^{-1}tt'F_{\alpha l} \in \Phi^l_B$ and $1+\alpha^{-1}t^{'}tG_{\alpha l} \in \Phi^l_B$. Then, it follows, by using both Proposition 2.6, Eq. (4) and Eq. (5), that $(\alpha-t)(\alpha-t^{'}) \in \Phi^l_B$ and $(\alpha - t^{'})(\alpha - t) \in \Phi^l_B$, which implies that $t \in [\Phi^l_B(t) \cap \Phi^l_B(t^{'})] \setminus \{0\}$. For the rest of the other cases, arguing as in this proof using the same arguments by replacing Eq. (4) by Eq. (6) and Eq. (5) by Eq. (7). Q.E.D. Le spectre essentiel de Weyl d'un ment $t \in A$ affili B not $W\sigma_B(t)$ est dfinit par $W\sigma_B(t) = \bigcap\limits_{k \in K_B} \sigma_B(t + k)$.

Theorem 4.7. Let $t \in A$. If t is affiliated with B, then $W\sigma_B(t) = \bigcap\limits_{a \in \Omega_B} \sigma_B(t + a)$,

where $\Omega_B = \Big\{a = tk + j; k, j \in Pr\Big(\mathcal{DC}_{s_0}(B)\Big)\Big\}$.

Proof. Let $\mathcal{M} = \bigcap\limits_{a \in \Omega_B} \sigma_B(t + a)$. Clearly, we have $K_B \subset Pr\Big(\mathcal{DC}_{s_0}(B)\Big) \subset \Omega_B$, thus $\mathcal{M} \subset W\sigma_B(t)$. Reciprocally, assume that $a = tk + j \in \Omega_B$ such that $(\lambda - t - a) \in Inv_B(A) \subset \Phi^0_B$. Then, by referring to [2, Theorem 16], we deduce that $(\lambda - t) \in \Phi^0_B$ and so [2, Theorem 12] leads to $\lambda \notin W\sigma_B(t)$, which achieves the reverse inclusion. Q.E.D.

Remark 4.2. (i) Obviously, we can prove that Ω_B is a two-sided ideal of B.

(ii) If $\mathcal{D} \subset B$ such that $Pr\Big(\mathcal{DC}_{s_0}(B)\Big) \subseteq \mathcal{D} \subseteq \Omega_B$, we deduce that

$$W\sigma_B(t) = \bigcap_{a \in \mathcal{D}} \sigma_B(t + a).$$

Question: It would be interesting to study the stability of the Gustafson spectrum coming from an ambiguity of the continuity of the index i in $\Phi^l_B(t)$, where $t \in A$. Here, JJ. Grobler, in [13] suggests a worthwhile approach to establish the abstract index theory of Fredholm elements in the algebra. He proves that the continuity of the index i in $\Phi(A, I)$, where I is a proper ideal of A and $\Phi(A, I)$ being the set of all Fredholm elements relative to I. Nonetheless, when it comes to study the stability of other essential spectra, there is a need to identify the additional or sufficient conditions so that to ensure the continuity of the index i in $\Phi^l_B(t)$.

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