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Maximum principle for forward-backward partially observed optimal control of stochastic systems with delay

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Abstract. In this paper, we consider partially observed optimal control for forward-backward stochastic delay differential equations (FBSDDEs) where the control domain is non-convex and the control variable is allowed to enter into both diffusion and observation terms. We obtain a general stochastic maximum principle of these optimal control problems by using Girsanov's theorem, the spike variational method and the filtering technique. We also derive the adjoint equations to the problem. Finally, we apply our results to study a linear-quadratic (LQ) optimal control with delay.

1. Introduction

The stochastic maximum principle is one of the most important approaches to solve optimal control problems of stochastic systems. In the literature, control domains of these problems are considered in two approaches, convex and non-convex. We refer the reader to [1–3], and references therein for more information on maximum principle under convexity control domain. When the control region is non-convex and the diffusion term involves the control variable, the first-order Taylor expansion is not sufficient to obtain the maximum principle. Peng [4] introduced the second-order adjoint equations and used the spike variation to overcome this difficulty in SDEs. Since then, investigation of this kind of optimal control problems has been developed, for more information refer to [5, 6].
Forward-backward stochastic differential equations (FBSDEs) play an important role in lots of fields such as finance, economics and, so on. They have become a powerful tool to study the maximum principle of FBSDEs under the convexity assumption of the control domain, see e.g. [7–10]. It seems that obtaining the maximum principle for forward-backward stochastic systems in a non-convex control domain would be difficult. Due to the presence of the process $z(t)$ in the backward equation (see the equation (4) in the context), we cannot obtain the second-order variation and the second-order adjoint equations similar to [4]. This problem is solved by taking the martingale process $z(t)$ in the backward equation as a control in [11–13]. The authors introduced a new stochastic control problem which is a forward equation for the case with terminal state constraints. Also, Yong in [14] obtained necessary conditions of maximum principle for

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a controlled FBSDE with mixed initial-terminal conditions, which the control domain is not assumed to be convex, and the control enters in the diffusion coefficient of the forward equation.

Maximum principle for partially observed stochastic optimal control problems has been discussed by many authors as well. Li and Fu [15] established a maximum principle for partially observed optimal control problems of mean-field FBSDEs without the convexity control domain. Tang [16] and Wang et al. [17] investigated a partial information optimal control problem derived by stochastic systems with correlated noises between the system and the observation in convex and non-convex domains, respectively. In a non-convex domain control, a maximum principle of FBSDEs under partially observed noise and of fully-coupled FBSDEs with partial noise is presented in [18] and [19], respectively. Be aware that, their results are established when the forward diffusion coefficient does not depend on the control. So, how to reproduce a maximum principle, in this case, is a new problem. We are interested to remedy this issue in this research.

It is quite natural to investigate the controlled delay systems which are much closer to reality. It is well-known that a stochastic optimal control of stochastic delay differential equations (SDDEs) in the case of the convex region has been studied in several articles, for example, see [20–26]. In the case of non-convex, Meng and Shi [27] studied on optimal control of SDDEs where the diffusion term contains both control and its delayed term. Also, Hao, and Meng [7] derived a second-order maximum principle for delay stochastic optimal controls with recursive utilities as the backward diffusion term does not contain the control variables.

To the best of our knowledge, there is no paper analyzing partially observed FBSDEs involving delays in case that the control domain is not convex. In this manuscript, inspiring the results in [13] and [18] we obtain a general maximum principle for these problems. In spite of the FBSDE discussed in these papers the main contribution of our paper is to study the partial observed FBSDEs with delay when the forward diffusion coefficient contains control variables. Indeed, we solve the problem by taking $z(.)$ as a control process and the terminal condition $y(T) = \psi(x(T))$ as a terminal state constraint and utiliting the Ekeland's variational principle. However, what we should pay special attention to is that the adjoint equation, corresponding to these problems with observation and delay is totally different from systems without these conditions.

The rest of this paper is organized as follows. In Section 2, we formulate the problem. We prove some lemmas that are necessary to derive the main result in Section 3. In Section 4, we establish the general maximum principle for FBSDDEs with observed noise. In Section 5, a linear-quadratic partially observed optimal control problem with delay is studied.

2. Statement of the problem

In this paper, we consider $(\Omega, \mathcal{F}, P)$ as a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $(W(.), Y(.))$ as an $\mathbb{R}^{d+r}$-valued standard Brownian motion defined in this space. Throughout this section, $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space and $|.|$ denotes the norm in a Euclidean space. We assume that $\mathcal{F}_t = \sigma\{W(s), Y(s); 0 \leq s \leq t\}$. Let $\mathcal{F} = \mathcal{F}_T$ and $T > 0$ is a fixed time horizon.

2.1. Motivation

Consider a financial market with two investment possibilities: a risk-free asset (bond) and a risky asset (stock). The price dynamic of the risk-free asset is given by $dS_0(t) = a(t)S_0(t)dt$, where $a(t) > 0$ is a bounded deterministic function, and the price dynamic of the risky assets is given by $dS_1(t) = S_1(t)[b(t)dt + \sigma(t)dW(t)]$, where $b(t) > a(t)$ is the appreciation rate process, and $\sigma(t)$ is the volatility. As in [28], the wealth dynamics with delay is

$$\begin{cases} dx(t) = \Big[a(t)x(t-\tau) + (b(t) - a(t))\nu(t)\Big]dt + \sigma(t)\nu(t)dW(t), \\ x(0) = x_0(t), \quad t \in [-\delta, 0], \end{cases} \tag{1}$$

where $\nu(t)$ represents the portfolio strategy of a policymaker. In fact, it is possible for the policymaker to partially observe the wealth. See, e.g., [29] and [30]. Thus, the factor model is described by the following

$$\begin{cases} dY(t) = \Big(\frac{1}{\sigma(t)}b(t) - \frac{1}{2}\sigma(t)\Big)dt + d\tilde{W}(t), \\ Y(0) = 0, \end{cases} \tag{2}$$

where $Y(t) = \frac{1}{\sigma(t)} \log S_1(t)$, and $\tilde{W}(t)$ is a stochastic process with $\tilde{W}(0) = 0$. Note that the above factor model is similar to [31] and [30]. The objective of the policymaker is to find a control strategy $v(.) \in \mathcal{U}_{ad}$ so that

$$J(v(.)) = \min_{v(.)\in\mathcal{U}_{ad}} \frac{1}{2} E^{v}\Big[(x(T) - M)^2 + (y(0) - N)^2\Big],$$

subject to (1), (2) and (3), where y is a recursive utility from wealth $x(t)$ with the following backward equation

$$\begin{cases} dy(t) = \Big(-\alpha(t)y(t) + \gamma(t)v(t)\Big)dt - z(t)dW(t), \\ y(T) = Kx(T), \end{cases} \tag{3}$$

where K is a constant. See, [32] for more details about a recursive utility. Clearly, this is an LQ optimal control problem of FBSDE systems.

2.2. Problem

We consider the following controlled FBSDDE

$$\begin{cases} dx(t) = b(t, x(t), x(t-\tau), v(t))dt + \sigma(t, x(t), x(t-\tau), v(t))dW(t), \\ dy(t) = a(t, x(t), x(t-\tau), y(t), z(t), v(t))dt + z(t)dW(t), \quad t \in [0, T] \\ x(t) = x_0(t), \quad t \in [-\tau, 0], \quad y(T) = \psi(x(T)), \end{cases} \tag{4}$$

where $\tau > 0$ is a constant regarding the time delay with $\tau < T \leq 2\tau$, so the system is called a system with one pointwise delay; $(x(.), y(.), z(.)) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is the state process with initial sate $x_0 \in C([-\tau, 0]; \mathbb{R}^n)$; $v(.)$ is the control process taking values in $U \subset \mathbb{R}^k$ which is not necessary convex, $0 \leq t \leq T$, $b : [0, T]\times\mathbb{R}^n\times\mathbb{R}^n\times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$, $a : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times U \to \mathbb{R}^m$, $\psi : \mathbb{R}^n \to \mathbb{R}^m$ are given continuous mappings.
Also, we consider the observation

$$\begin{cases} dY(t) = h(t, x(t), x(t-\tau), v(t))dt + d\tilde{W}(t), \\ Y(0) = 0, \end{cases} \tag{5}$$

where $\tilde{W}(.)$ is a stochastic process depending $v(.)$ and $h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^r$.
Let $\mathcal{Y}_t = \sigma\{Y(s); 0 \leq t \leq t\}$. An admissible control $v(.) : [0, T] \times \Omega \to U$ is an $\mathcal{Y}_t$- adapted process such that

$$\sup_{0\leq t\leq T} E|v(t)|^i < \infty, \quad \forall i = 1, 2, ...,$$

the set of all admissible control variables is denoted by $\mathcal{U}_{ad}$. We consider the following assumption:
Hypothesis 1. The functions a, b, σ, h and ϕ are twice continuously differentiable with respect to (x, x', y, z), and for some constant C,

$$(1 + |x| + |x'| + |v|)^{-1}|f(t, x, x', v)| + |f_q(t, x, x', v)| \leq C,$$
$$(1 + |x| + |x'| + |y| + |z| + |v|)^{-1}|a(t, x, x', y, z, v)| + |a_k(t, x, x', y, z, v)| + |a_{rs}(t, x, x', y, z, v)| \leq C,$$
$$|h(t, x, x', v)| + |h_q(t, x, x', v)| \leq C,$$

where $f = b, \sigma$, $q = x, x', xx, xx', x'x'$, $k = x, x', y, z$, and $r = x, x', y, z$, $s = x, x', y, z$. Under the hypothesis 1, for each $v \in \mathcal{U}_{ad}$, the controlled FBSDDE (4) admits a unique solution denoted by $(x^v(.), y^v(.), z^v(.))$. Define

$$Z^v(t) = \exp\left\{\int_0^t h(s, x^v(s), x^v(s-\tau), v(s))dY(s) - \frac{1}{2}\int_0^t h^2(s, x^v(s), x^v(s-\tau), v(s))ds\right\},$$

which is the solution of

$$\begin{cases} dZ^v(t) = Z^v(t)h(t, x^v(t), x^v(t-\tau), v(t))dY(t), \\ Z^v(0) = 1. \end{cases} \tag{6}$$

Using Ito's formula, we have for any $l = 1, 2, ..., \sup_{0\le t\le T} E|Z(t)|^l < +\infty$ and define a new probability measure P^ν such that $dP^\nu = Z^\nu(t)dP$. According to Girsanov's theorem and (5), we have $(W(.), \tilde{W}(.))$ is a $\mathbb{R}^{d+r}$-valued standard Brownian motion defined in the new probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^\nu)$.
The cost functional is in the form of

$$J(\nu(.)) = E^\nu\left[\int_0^T l(t, x^\nu(t), x^\nu(t-\tau), y^\nu(t), z^\nu(t), \nu(t))dt + \phi(x^\nu(T)) + \gamma(y^\nu(0))\right], \tag{7}$$

where $l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times U \to \mathbb{R}$, $\phi : \mathbb{R}^n \to \mathbb{R}$, $\gamma : \mathbb{R}^m \to \mathbb{R}$ and E^ν denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^\nu)$. We need the following assumption:
Hypothesis 2. The functions l, ϕ, γ are twice continuously differentiable with respect to (x, y, z), x and y, respectively and there exists a constant C such that

$$\begin{aligned}
&(1 + |x|^2 + |x'|^2 + |y|^2 + |z|^2 + |\nu|^2)^{-1}|l(t, x, x', y, z, \nu)| \\
&\qquad + (1 + |x| + |x'| + |y| + |z| + |\nu|)^{-1}|l_k(t, x, x', y, z, \nu)| + |l_{rs}(t, x, x', y, z, \nu)| \le C, \\
&\qquad (1 + |x|^2)^{-1}|\phi(x)| + (1 + |x|)^{-1}|\phi_x(x)| + |\phi_{xx}(x)| \le C, \\
&\qquad (1 + |y|^2)^{-1}|\gamma(y)| + (1 + |y|)^{-1}|\gamma_y(y)| + |\gamma_{yy}(y)| \le C,
\end{aligned}$$

where $k = x,\ x',\ y,\ z$, and $r = x, x', y, z$, $s = x, x', y, z$.
The cost functional (7) can be rewritten as

$$J(\nu(.)) = E\left[\int_0^T Z^\nu(t)l(t, x^\nu(t), x^\nu(t-\tau), y^\nu(t), z^\nu(t), \nu(t))dt + Z^\nu(T)\phi(x^\nu(T)) + \gamma(y^\nu(0))\right]. \tag{8}$$

If there exists an admissible $\nu^*(.)$ such that

$$J(\nu^*(.)) = \inf_{\nu(.)\in\mathcal{U}_{ad}} J(\nu(.)),$$

subject to (4) and (6), the process $\nu^*(.)$ is called optimal control. The process $(x^*(.), y^*(.), z^*(.))$, which is the solution of state equation (4) corresponding to $\nu^*(.)$, is called an optimal trajectory. The purpose of this paper is to derive a general maximum principle of FBSDDE (4), (6) with the cost functional (8).
We take z as a control variable while $y(T) = \psi(x(T))$ in (4) is a terminal state constraint, which this method is used in [13], [11] and [12]. So, we formulate a variation of problems (4), (6) and (8) as follows.
Consider the forward stochastic delay control system

$$\begin{cases} dx(t) = b(t, x(t), x(t-\tau), \nu(t))dt + \sigma(t, x(t), x(t-\tau), \nu(t))dW(t), \\ dy(t) = a(t, x(t), x(t-\tau), y(t), u(t), \nu(t))dt + u(t)dW(t), \quad t \in [0, T], \\ x(t) = x_0(t), \quad t \in [-\tau, 0], \quad y(0) = y_0, \end{cases} \tag{9}$$

with a terminal state constraint

$$E|y(T) - \psi(x(T))|^2 = 0. \tag{10}$$

Problem is to minimize

$$J(y_0, u(.), \nu(.)) = E\left[\int_0^T Z^\nu(t)l(t, x(t), x(t-\tau), y(t), u(t), \nu(t))dt + Z^\nu(T)\phi(x(T)) + \gamma(y(0))\right], \tag{11}$$

subject to (6), (9) and (10), where $y_0 \in \mathbb{R}^m$, $u(.) \in \mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$, $\nu(.) \in \mathcal{U}_{ad}$.

Remark 2.1. *For simplicity, we only study the one-dimensional case, i.e., $n = m = k = r = d = 1$, which is similar to the case of $n-$dimensional.*

3. Preliminary Lemmas

For any $v_1(.), v_2(.) \in \mathcal{U}_{ad}$ or $\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R})$, we choose the metric in $\mathcal{U}_{ad}$ and $\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R})$ as

$$d(v_1(.), v_2(.)) = E[mes\{0 \leq t \leq T; v_1(t) \neq v_2(t)\}],$$

where *mes* denotes the Lebesgue measure. It is easy to check that $(\mathcal{U}_{ad}, d(.,.))$ and $(\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}), d(.,.))$ are complete spaces.
Let $(y^*_0, u^*(.), v^*(.))$ be an optimal control of the problem (11) and $(x^*(.), y^*(.))$ and $Z^*(.)$ be the corresponding optimal state processes of (9) which satisfy $y^*(T) = \psi(x^*(T))$.
Define a new following cost functional

$$J_\lambda(y_0, u(.), v(.)) = \Big\{\Big[J(y_0, u(.), v(.)) - J(y^*_0, u^*(.), v^*(.)) + \lambda\Big]^2 + \Big[E^v|y^v(T) - \psi(x^v(T))|^2\Big]^2\Big\}^{\frac{1}{2}}, \tag{12}$$

where $(x^v(.), y^v(.))$ is the trajectory corresponding to $v(.)$ and $\lambda > 0$ is a constant. We have to take note that the unboundedness $\mathbb{R}$, $L^2_{\mathcal{F}}(0, T; \mathbb{R})$ and U_{ad} cannot assure lower semi-continuity of $J_\lambda(y_0, u(.), v(.))$. Thus Ekeland's variational principle ([33]) cannot directly be used to solve the optimal control of problem (9) and (12). So we consider two steps.
Step 1. Assume that $y_0, u(.)$ take values in $\mathcal{M}, \mathcal{N} \subset \mathbb{R}$ and $\mathcal{M}$ be convex and $\mathcal{M}, \mathcal{N}, \mathcal{U}$ are bounded.
Thus $J_\lambda(y_0, u(.), v(.)) : \mathcal{M} \times L^2_{\mathcal{F}}(0, T; \mathcal{N}) \times \mathcal{U}_{ad} \to \mathbb{R}$ is a semi-lower continuous function and it is easy to check that

$$\begin{aligned} &J_\lambda(y_0, u(.), v(.)) \geq 0, \quad \forall (y_0, u(.), v(.)) \in (\mathcal{M} \times L^2_{\mathcal{F}}(0, T; \mathcal{N}) \times \mathcal{U}_{ad}),\\ &J_\lambda(y^*_0, u^*(.), v^*(.)) = \lambda,\\ &J_\lambda(y^*_0, u^*(.), v^*(.)) \leq \inf_{(y_0, u(.), v(.))} J_\lambda(y_0, u(.), v(.)) + \lambda. \end{aligned}$$

Therefore we can apply Ekeland's variational principle, [33], that there exists $(y_{\lambda 0}, u_\lambda(.), v_\lambda(.)) \in \mathcal{M} \times L^2_{\mathcal{F}}(0, T; \mathcal{N}) \times \mathcal{U}_{ad}$ such that

$$\begin{aligned} &(a)\, J_\lambda(y_{\lambda 0}, u_\lambda(.), v_\lambda(.)) \leq J_\lambda(y^*_0, u^*(.), v^*(.)) = \lambda,\\ &(b)\, d\Big((y_{\lambda 0}, u_\lambda(.), v_\lambda(.)), (y^*_0, u^*(.), v^*(.))\Big) \leq \sqrt{\lambda},\\ &(c) \text{ for any } (y_0, u(.), v(.)) \in (\mathcal{M} \times L^2_{\mathcal{F}}(0, T; \mathcal{N}) \times \mathcal{U}_{ad}),\\ &\quad J_\lambda(y_0, u(.), v(.)) \geq J_\lambda(y_{\lambda 0}, u_\lambda(.), v_\lambda(.)) - \sqrt{\lambda} d\Big((y_0, u(.), v(.)), (y_{\lambda 0}, u_\lambda(.), v_\lambda(.))\Big). \end{aligned} \tag{13}$$

Then $(y_{\lambda 0}, u_\lambda(.), v_\lambda(.))$ is the optimal control of problem (9) and (12). We can make the spike variations

$$u^\varepsilon_\lambda(t) = \begin{cases} u, & d \leq t \leq d + \varepsilon \\ u_\lambda(t), & \text{otherwise,} \end{cases}$$

$$v^\varepsilon_\lambda(t) = \begin{cases} v, & d \leq t \leq d + \varepsilon \\ v_\lambda(t), & \text{otherwise,} \end{cases}$$

where $0 \leq d < T$ is fixed, $\varepsilon > 0$ is sufficiently small and $u \in \mathcal{N}, v \in \mathcal{U}$ are arbitrary $\mathcal{Y}_d$- measurable random variables such that

$$\sup_{\omega \in \Omega} |u(\omega)| < +\infty, \qquad \sup_{\omega \in \Omega} |v(\omega)| < +\infty.$$

Let $y_0 \in \mathcal{M}$ be a control variable such that $y_{\lambda 0} + y_0 \in \mathcal{M}$, because the set $\mathcal{M}$ is convex, then we consider the following perturbed control process of $\mathcal{M}$

$$y^\varepsilon_{\lambda 0} = y_{\lambda 0} + \varepsilon y_0, \quad \forall 0 \leq \varepsilon \leq 1.$$

Let $(x^{\varepsilon}_{\lambda}(.), y^{\varepsilon}_{\lambda}(.))$ and $Z^{\varepsilon}_{\lambda}(.)$ be the trajectories corresponding to $(y^{\varepsilon}_{\lambda 0}, u^{\varepsilon}_{\lambda}(.), \nu^{\varepsilon}_{\lambda}(.))$ and $(x_{\lambda}(.), y_{\lambda}(.))$ and $Z_{\lambda}(.)$ be the trajectories corresponding to $(y_{\lambda 0}, u_{\lambda}(.), \nu_{\lambda}(.))$.
For convenience, we use the following notations throughout the paper.
Set for $f = a, a_x, a_y, a_{xx}, a_{x'x'}, a_{xx'}, a_{yy}, a_{xy}, a_{x'y}, l, l_x, l_y, l_{xx}, l_{x'x'}, l_{xx'}, l_{yy}, l_{xy}, l_{x'y}$,

$$f(t) = f(t, x_{\lambda}(t), x_{\lambda}(t-\tau), y_{\lambda}(t), u_{\lambda}(t), \nu_{\lambda}(t)), \quad f(\nu^{\varepsilon}_{\lambda}(t)) = f(t, x_{\lambda}(t), x_{\lambda}(t-\tau), y_{\lambda}(t), u^{\varepsilon}_{\lambda}(t), \nu^{\varepsilon}_{\lambda}(t)),$$

and for $g = b, b_x, b_{xx}, b_{x'x'}, b_{xx'}, \sigma, \sigma_x, \sigma_{xx}, \sigma_{x'x'}, \sigma_{xx'}, h, h_x, h_{xx}, h_{x'x'}, h_{xx'}$,

$$g(t) = g(t, x_{\lambda}(t), x_{\lambda}(t-\tau), \nu_{\lambda}(t)), \qquad g(\nu^{\varepsilon}_{\lambda}(t)) = g(t, x_{\lambda}(t), x_{\lambda}(t-\tau), \nu^{\varepsilon}_{\lambda}(t)),$$

and

$$\Delta k(t) = k(\nu^{\varepsilon}_{\lambda}(t)) - k(t), \quad \text{for } k = b, \sigma, h, a, l, b_x, \sigma_x, h_x, a_x, a_y, b_{x'}, \sigma_{x'}, h_{x'}, a_{x'}.$$

We introduce the first-order and second-order variational equations

$$\begin{cases} dx^1_{\lambda}(t) = \Big[b_x(t)x^1_{\lambda}(t) + b_{x'}(t)x^1_{\lambda}(t-\tau) + \Delta b(t)\Big]dt \\ \qquad + \Big[\sigma_x(t)x^1_{\lambda}(t) + \sigma_{x'}(t)x^1_{\lambda}(t-\tau) + \Delta\sigma(t)\Big]dW(t), \quad t \in [0,T], \\ dy^1_{\lambda}(t) = \Big[a_x(t)x^1_{\lambda}(t) + a_{x'}(t)x^1_{\lambda}(t-\tau) + a_y(t)y^1_{\lambda}(t) + \Delta a(t)\Big]dt + \Big[u^{\varepsilon}_{\lambda}(t) - u_{\lambda}(t)\Big]dW(t), \\ dZ^1_{\lambda}(t) = \Big[h(t)Z^1_{\lambda}(t) + Z_{\lambda}(t)h_x(t)x^1_{\lambda}(t) + Z_{\lambda}(t)h_{x'}(t)x^1_{\lambda}(t-\tau) + Z_{\lambda}(t)\Delta h(t)\Big]dY(t), \\ x^1_{\lambda}(t) = 0, \ t \in [-\tau, 0], \quad y^1_{\lambda}(0) = \varepsilon y_0, \quad Z^1_{\lambda}(0) = 0, \end{cases} \tag{14}$$

$$\begin{cases} dx^2_{\lambda}(t) = \Big[b_x(t)x^2_{\lambda}(t) + b_{x'}(t)x^2_{\lambda}(t-\tau) + \frac{1}{2}b_{xx}(t)(x^1_{\lambda}(t))^2 + \frac{1}{2}b_{x'x'}(t)(x^1_{\lambda}(t-\tau))^2 \\ \qquad + b_{xx'}(t)x^1_{\lambda}(t)x^1_{\lambda}(t-\tau) + \Delta b_x(t)x^1_{\lambda}(t) + \Delta b_{x'}(t)x^1_{\lambda}(t-\tau)\Big]dt \\ \qquad + \Big[\sigma_x(t)x^2_{\lambda}(t) + \sigma_{x'}(t)x^2_{\lambda}(t-\tau) + \frac{1}{2}\sigma_{xx}(t)(x^1_{\lambda}(t))^2 + \frac{1}{2}\sigma_{x'x'}(t)(x^1_{\lambda}(t-\tau))^2 \\ \qquad + \sigma_{xx'}(t)x^1_{\lambda}(t)x^1_{\lambda}(t-\tau) + \Delta\sigma_x(t)x^1_{\lambda}(t)\Big]dW(t), \quad t \in [0,T], \\ dy^2_{\lambda}(t) = \Big[a_x(t)x^2_{\lambda}(t) + a_{x'}(t)x^2_{\lambda}(t-\tau) + \frac{1}{2}a_{xx}(t)(x^1_{\lambda}(t))^2 + \frac{1}{2}a_{x'x'}(t)(x^1_{\lambda}(t-\tau))^2 \\ \qquad + a_{xx'}(t)x^1_{\lambda}(t)x^1_{\lambda}(t-\tau) + a_y(t)y^2_{\lambda}(t) + \frac{1}{2}a_{yy}(t)(y^1_{\lambda}(t))^2 + a_{xy}(t)x^1_{\lambda}(t)y^1_{\lambda}(t) \\ \qquad + a_{x'y}(t)x^1_{\lambda}(t-\tau)y^1_{\lambda}(t) + \Delta a_x(t)x^1_{\lambda}(t) + \Delta a_{x'}(t)x^1_{\lambda}(t-\tau) + \Delta a_y(t)y^1_{\lambda}(t)\Big]dt, \end{cases} \tag{15}$$

$$\begin{cases} dZ^2_{\lambda}(t) = \Big[h(t)Z^2_{\lambda}(t) + Z_{\lambda}(t)(h_x(t)x^2_{\lambda}(t) + h_{x'}(t)x^2_{\lambda}(t-\tau) + \frac{1}{2}h_{xx}(t)(x^1_{\lambda}(t))^2 \\ \qquad + \frac{1}{2}h_{x'x'}(t)(x^1_{\lambda}(t-\tau))^2 + h_{xx'}(t)x^1_{\lambda}(t)x^1_{\lambda}(t-\tau) + \Delta h_x(t)x^1_{\lambda}(t) \\ \qquad + \Delta h_{x'}(t)x^1_{\lambda}(t-\tau)) + Z^1_{\lambda}\Delta h(t)\Big]dY(t), \\ x^2_{\lambda}(t) = 0, \ t \in [-\tau, 0], \quad y^2_{\lambda}(0) = 0, \quad Z^2_{\lambda}(0) = 0. \end{cases}$$

Under hypothesis 1, (14) and (15) have unique solutions.

Lemma 3.1. *Suppose hypothesis 1 holds, then for some constant C*

1. $\sup_{-\tau\le t\le T} E|x^1_{\lambda}(t)|^4 \le C\varepsilon^2$, 2. $\sup_{-\tau\le t\le T} E|x^1_{\lambda}(t)|^8 \le C\varepsilon^4$, 3. $\sup_{-\tau\le t\le T} E|x^1_{\lambda}(t)|^{16} \le C\varepsilon^8$,
4. $\sup_{-\tau\le t\le T} E|x^2_{\lambda}(t)|^2 \le C\varepsilon^2$, 5. $\sup_{-\tau\le t\le T} E|x^2_{\lambda}(t)|^4 \le C\varepsilon^4$, 6. $\sup_{-\tau\le t\le T} E|x^2_{\lambda}(t)|^8 \le C\varepsilon^8$,
7. $\sup_{-\tau\le t\le T} E|x^{\varepsilon}_{\lambda}(t) - x_{\lambda}(t) - x^1_{\lambda}(t) - x^2_{\lambda}(t)|^2 \le C_{\varepsilon}\varepsilon^2$,
8. $\sup_{-\tau\le t\le T} E|x^{\varepsilon}_{\lambda}(t) - x_{\lambda}(t) - x^1_{\lambda}(t) - x^2_{\lambda}(t)|^4 \le C_{\varepsilon}\varepsilon^4$,

where C_{ε} *is a constant which* $\lim_{\varepsilon\to 0} C_{\varepsilon} = 0$.

Proof. See the Appendix. □

Lemma 3.2. *Suppose hypothesis 1 holds, then for some constant C*

1. $\sup_{0\le t\le T} E|y^1_\lambda(t)|^4 \le C\varepsilon^2$, 2. $\sup_{0\le t\le T} E|y^1_\lambda(t)|^8 \le C\varepsilon^4$,
3. $\sup_{0\le t\le T} E|y^2_\lambda(t)|^2 \le C\varepsilon^2$, 4. $\sup_{0\le t\le T} E|y^2_\lambda(t)|^4 \le C\varepsilon^4$,
5. $\sup_{0\le t\le T} E|y^\varepsilon_\lambda(t) - y_\lambda(t) - x^1_\lambda(t) - y^2_\lambda(t)|^2 \le C_\varepsilon \varepsilon^2$,

where C_ε *is a constant which* $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

Proof. The proof is similar to Lemma 3.2 in [13]. □

Lemma 3.3. *Suppose hypothesis 1 holds, then for some constant C*

1. $\sup_{0\le t\le T} E|Z^1_\lambda(t)|^2 \le C\varepsilon$, 2. $\sup_{0\le t\le T} E|Z^1_\lambda(t)|^4 \le C\varepsilon^2$,
3. $\sup_{0\le t\le T} E|Z^2_\lambda(t)|^2 \le C\varepsilon^2$, 4. $\sup_{0\le t\le T} E|Z^2_\lambda(t)|^4 \le C\varepsilon^4$,
5. $\sup_{0\le t\le T} E|Z^\varepsilon_\lambda(t) - Z_\lambda(t) - Z^1_\lambda(t) - Z^2_\lambda(t)|^2 \le C_\varepsilon \varepsilon^2$,

where C_ε *is a constant which* $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

Proof. See the Appendix. □

Define the Hamiltonian function H as follows:

$$\begin{aligned} H(t,x,x',y,u,v,p,q,k,r,N,\beta) =& pb(t,x,x',v) + qa(t,x,x',y,u,v) + k\sigma(t,x,x',v) \\ &+ ru(t) + Nh(t,x,x',v) + \beta l(t,x,x',y,u,v), \end{aligned} \tag{16}$$

and we formulate the first order adjoint equations as follows:

$$\begin{cases} dm(t) = -\Big(N(t)h(t) + \beta l(t)\Big)dt + N(t)dY(t), \\ dp(t) = -\Big[p(t)b_x(t) + q(t)a_x(t) + k(t)\sigma_x(t) + N(t)h_x(t) + \beta l_x(t) \\ \qquad + E^{\mathcal{F}_t}(p(t+\tau)b_{x'}(t+\tau)) + E^{\mathcal{F}_t}(q(t+\tau)a_{x'}(t+\tau)) + E^{\mathcal{F}_t}(k(t+\tau)\sigma_{x'}(t+\tau)) \\ \qquad + E^{\mathcal{F}_t}(N(t+\tau)h_{x'}(t+\tau)) + \beta E^{\mathcal{F}_t}(l_{x'}(t+\tau))\Big]dt + k(t)dW(t), \\ dq(t) = -\Big(q(t)a_y(t) + \beta l_y(t)\Big)dt + r(t)dW(t), \\ m(T) = \beta\phi(x(T)), \quad p(T) = \beta\phi_x(x(T)), \quad p(t) = 0, \quad t\in(T, T+\tau], \quad q(T) = 0, \end{cases} \tag{17}$$

and we have the following second order adjoint equation:

$$\begin{cases} dP_1(t) = -\Big[2P_1(t)b_x(t) + \sigma^2_x(t)P_1(t) + 2\sigma_x(t)Q_1(t) + P_3(t)a_x(t) + H_{xx}(t) \\ \qquad + E^{\mathcal{F}_t}(P'_1(t+\tau)b_{x'}(t+\tau)) + E^{\mathcal{F}_t}(P_1(t+\tau)\sigma^2_{x'}(t+\tau)) \\ \qquad + E^{\mathcal{F}_t}(P'_1(t+\tau)\sigma_x(t)\sigma_{x'}(t+\tau)) + E^{\mathcal{F}_t}(Q'_1(t+\tau)\sigma_{x'}(t+\tau)) \\ \qquad + E^{\mathcal{F}_t}(P'_3(t+\tau)a_{x'}(t+\tau)) + H_{x'x'}(t+\tau)\Big]dt + Q_1(t)dW(t), \\ P_1(T) = \beta\phi_{xx}(T) + 2\mu(\psi_x(x(T)))^2, \quad P_1(t) = 0, \quad t\in(T, T+\tau], \end{cases} \tag{18}$$

$$\begin{cases} dP_2(t) = -\Big[2P_2(t)a_y(t) + H_{yy}(t)\Big]dt + Q_2(t)dW(t), \\ P_2(T) = 2\mu, \end{cases} \tag{19}$$

$$\begin{cases} dP_3(t) = -\Big[P_3(t)a_y(t) + P_3(t)b_x(t) + 2a_x(t)P_2(t) + \sigma_x(t)Q_3(t) + H_{xy}(t) \\ \qquad + E^{\mathcal{F}_t}(P_3(t+\tau)b_{x'}(t+\tau)) + E^{\mathcal{F}_t}(Q_3(t+\tau)\sigma_{x'}(t+\tau)) \\ \qquad + 2E^{\mathcal{F}_t}(P_2(t+\tau)a_{x'}(t+\tau))\Big]dt + Q_3(t)dW(t), \\ P_3(t) = 0, \quad t \in [T, T+\tau], \end{cases} \tag{20}$$

$$\begin{cases} dP_1'(t) = -\Big[P_1'(t)b_x(t) + 2P_1(t)b_{x'}(t) + \sigma_x(t)Q_1'(t) + P_1'(t)b_x(t-\tau) \\ \qquad + 2Q_1(t)\sigma_{x'}(t) + Q_1'(t)\sigma_x(t-\tau) + P_1'(t)\sigma_x(t)\sigma_x(t-\tau) \\ \qquad + P_3(t)a_{x'}(t) + P_3'(t)a_x(t) + 2P_1(t)\sigma_{x'}(t)\sigma_x(t) + H_{xx'}(t)\Big]dt + Q_1'(t)dW(t), \\ P_1'(T) = 0 \;\; t \in [T, T+\tau], \end{cases} \tag{21}$$

$$\begin{cases} dP_3'(t) = -\Big[P_3'(t)a_y(t) + P_3'(t)b_x(t-\tau) + Q_3'(t)\sigma_x(t-\tau) + H_{x'y}(t)\Big]dt + Q_3'(t)dW(t), \\ P_3'(t) = 0, \;\; t \in [T, T+\tau]. \end{cases} \tag{22}$$

Here $\phi(t) = \phi(t, x, x', \nu)$ for $\phi = b, \sigma, h, b_x, \sigma_x, h_x$, $\psi(t) = \psi(t, x, x', y, u, \nu)$ for $\psi = a, a_x, a_y, l, l_x, l_y$ and $L(t) = L(t, x, x', y, u, \nu)$ for $L = H_{xx}, H_{yy}, H_{xy}, H_{x'x'}, H_{xx'}, H_{x'y}$. Now, we prove the following Lemma which is necessary to prove the main theorem in the last section.

Lemma 3.4. *Let hypothesis 1 and hypothesis 2 hold, then we have two following terms*

$$\begin{aligned} J(y^\varepsilon_{\lambda 0}, u^\varepsilon_\lambda(.), \nu^\varepsilon_\lambda(.)) - J(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.)) = E\int_0^T \Big[&(Z^1_\lambda(t) + Z^2_\lambda(t))l(t) \\ &+ Z_\lambda(t)\Big[l_x(t)(x^1_\lambda(t) + x^2_\lambda(t)) + l_{x'}(t)(x^1_\lambda(t-\tau) + x^2_\lambda(t-\tau)) \\ &+ l_y(t)(y^1_\lambda(t) + y^2_\lambda(t)) + \frac{1}{2}l_{xx}(t)(x^1_\lambda(t))^2 + \frac{1}{2}l_{x'x'}(t)(x^1_\lambda(t-\tau))^2 \\ &+ l_{xx'}(t)(x^1_\lambda(t-\tau)x^1_\lambda(t)) + \frac{1}{2}l_{yy}(t)(y^1_\lambda(t))^2 + l_{x'y}(t)(x^1_\lambda(t-\tau)y^1_\lambda(t)) + \Delta l(t)\Big]\Big]dt \\ &+ E\Big[(Z^1_\lambda(t) + Z^2_\lambda(t))\phi(x_\lambda(T)\Big] + E\Big[Z_\lambda(t)(\phi_x(x_\lambda(T))(x^1_\lambda(T) + x^2_\lambda(T)) \\ &+ \frac{1}{2}\phi_{xx}(x_\lambda(T))(x^1_\lambda(T))^2)\Big] + E\Big[\gamma(y_\lambda(0)\varepsilon y_0\Big] + o(\varepsilon), \end{aligned}$$

and

$$\begin{aligned} &\Big[E^{\nu^\varepsilon_\lambda}|y^\varepsilon_\lambda(T) - \psi(x^\varepsilon_\lambda(T))|^2\Big]^2 - \Big[E^{\nu_\lambda}|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big]^2 \\ &\quad = 2E\Big[Z_\lambda(T)|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big] \\ &\quad \times \Big\{Z_\lambda(T)(2E[(y_\lambda(T) - \psi(x_\lambda(T)))(y^1_\lambda(T) + y^2_\lambda(T))] + E((y^1_\lambda(T))^2) \\ &\quad + 2E[y_\lambda(T) - \psi(x_\lambda(T)) \times (-\psi_x(x_\lambda(T)(x^1_\lambda(T) + x^2_\lambda(T)))] \\ &\quad + E[y_\lambda(T) - \psi(x_\lambda(T)))(-\psi_{xx}(x_\lambda(T)(x^1_\lambda(T))^2] \\ &\quad + E[(\psi_x(x_\lambda(T))^2(x^1_\lambda(T))^2]) + E\Big[(Z^1_\lambda(T) + Z^2_\lambda(T))|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big]^2\Big\} + o(\varepsilon). \end{aligned}$$

Proof. The first term is similar to Lemma 2.3 in [18], so we only prove the second term.
Using Lemma 3.1, Lemma 3.2 and Lemma 3.3, we can derive

$$\begin{aligned} &\Big[E^{\nu^\varepsilon_\lambda}|y^\varepsilon_\lambda(T) - \psi(x^\varepsilon_\lambda(T))|^2\Big]^2 - \Big[E^{\nu_\lambda}|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big]^2 \\ &\quad = \Big[E(Z^\varepsilon_\lambda(T)|y^\varepsilon_\lambda(T) - \psi(x^\varepsilon_\lambda(T))|^2)\Big]^2 - \Big[E(Z_\lambda(T)|y_\lambda(T) - \psi(x_\lambda(T))|^2)\Big]^2 \end{aligned}$$

$$
\begin{aligned}
&= \Big[E\Big[(Z^{\varepsilon}_{\lambda}(T) - Z_{\lambda}(T) - Z^{1}_{\lambda}(T) - Z^{2}_{\lambda}(T))|y^{\varepsilon}_{\lambda}(T) - \psi(x^{\varepsilon}_{\lambda}(T))|^2\Big] \\
&\quad + E\Big[(Z_{\lambda}(T) + Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T)) \\
&\quad \times (|y^{\varepsilon}_{\lambda}(T) - \psi(x^{\varepsilon}_{\lambda}(T))|^2 - |y_{\lambda}(T) + y^{1}_{\lambda}(T) + y^{2}_{\lambda}(T) - \psi(x_{\lambda}(T) + x^{1}_{\lambda}(T) + x^{2}_{\lambda}(T))|^2))\Big] \\
&\quad + E\Big[(Z_{\lambda}(T) + Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T)) \\
&\quad \times (|y_{\lambda}(T) + y^{1}_{\lambda}(T) + y^{2}_{\lambda}(T) - \psi(x_{\lambda}(T) + x^{1}_{\lambda}(T) + x^{2}_{\lambda}(T))|^2 - |y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2)] \\
&\quad + E[(Z_{\lambda}(T) + Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T))|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2\Big]]^2 \\
&\quad - E[(Z_{\lambda}(T))|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2\Big]^2 \\
&= \Big[E\Big[(Z_{\lambda}(T) + Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T)) \\
&\quad \times \Big\{2(y_{\lambda}(T) - \psi(x_{\lambda}(T)))(y^{1}_{\lambda}(T) + y^{2}_{\lambda}(T)) + (y^{1}_{\lambda}(T))^2) + 2E[y_{\lambda}(T) - \psi(x_{\lambda}(T) \\
&\quad \times (\psi_x(x_{\lambda}(T)(x^{1}_{\lambda}(T) + x^{2}_{\lambda}(T)))] + (y_{\lambda}(T) - \psi(x_{\lambda}(T)))(-\psi_{xx}(x_{\lambda}(T)(x^{1}_{\lambda}(T))^2 \\
&\quad + (\psi_x(x_{\lambda}(T))^2(x^{1}_{\lambda}(T))^2) + (-\psi_x(x_{\lambda}(T))(x^{1}_{\lambda}(T)y^{1}_{\lambda}(T))\Big\}\Big] \\
&\quad + E\Big[(Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T))|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2\Big] + E\Big[(Z_{\lambda}(T))|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2]\Big]^2 \\
&\quad - [E(Z_{\lambda}(T)|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2)]^2 + o(\varepsilon) \\
&= 2E[Z_{\lambda}(T)|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2] \\
&\quad \times \Big\{Z_{\lambda}(T)(2E\Big[(y_{\lambda}(T) - \psi(x_{\lambda}(T)))(y^{1}_{\lambda}(T) + y^{2}_{\lambda}(T))\Big] + E((y^{1}_{\lambda}(T))^2) \\
&\quad + 2E\Big[y_{\lambda}(T) - \psi(x_{\lambda}(T)) \times (-\psi_x(x_{\lambda}(T)(x^{1}_{\lambda}(T) + x^{2}_{\lambda}(T)))\Big] \\
&\quad + E\Big[y_{\lambda}(T) - \psi(x_{\lambda}(T)))(-\psi_{xx}(x_{\lambda}(T)(x^{1}_{\lambda}(T))^2] + E[(\psi_x(x_{\lambda}(T))^2(x^{1}_{\lambda}(T))^2\Big]) \\
&\quad + E\Big[(Z^{1}_{\lambda}(T) + Z^{2}_{\lambda}(T))|y_{\lambda}(T) - \psi(x_{\lambda}(T))|^2\Big]^2\Big\} + o(\varepsilon).
\end{aligned}
$$

Thus the proof is completed. □

4. A general maximum principle

In this section, we derive a general maximum principle, which is the main result of this paper.

Theorem 4.1. *Let hypothesis 1 and hypothesis 2 hold, $\mathcal{M}, \mathcal{N}, \mathcal{U}$ are all bounded. If $(y^*_0(.), u^*(.), v^*(.))$ is an optimal solution of the optimal control problem (9), (11) with the final state constraint (10), then there are two parameters β and μ with $\beta^2 + \mu^2 = 1$, such that for any $y_0 \in \mathcal{M}, u \in \mathcal{N}, v \in \mathcal{U}$, we have*

$$
\begin{aligned}
E^{v} \int_0^T &\Big[H(t, x^*(t), x^*(t-\tau), y^*(t), u(t), v(t), p(t), q(t), k(t), r(t), N(t), \beta(t)) \\
&- H(t, x^*(t), x^*(t-\tau), y^*(t), u^*(t), v^*(t), p(t), q(t), k(t), r(t), N(t), \beta(t))\Big]dt \\
&+ \frac{1}{2}\Big[\sigma(t, x^*(t), x^*(t-\tau), v(t)) - (\sigma(t, x^*(t), x^*(t-\tau), v^*(t))\Big]^2 P_1(t) \\
&+ \Big[(\sigma(t, x^*(t), x^*(t-\tau), v(t)) - \sigma(t, x^*(t), x^*(t-\tau), v^*(t))) \\
&\times (\sigma(t-\tau, x^*(t-\tau), x^*(t-2\tau), v(t)) - \sigma(t-\tau, x^*(t-\tau), x^*(t-2\tau), v^*(t)))\Big]P'_1(t) \\
&+ \Big[(\sigma(t, x^*(t), x^*(t-\tau), v(t)) - \sigma(t, x^*(t), x^*(t-\tau), v^*(t)))(u(t) - u^*(t))\Big]P_3(t) \\
&+ \Big[(\sigma(t-\tau, x^*(t-\tau), x^*(t-2\tau), v(t)) - \sigma(t-\tau, x^*(t-\tau), x^*(t-2\tau), v^*(t)))
\end{aligned}
\tag{23}
$$

$$\times (u(t) - u^*(t))\Big]P_3'(t)$$
$$+ \frac{1}{2}\Big[(u(t) - u^*(t))^2\Big]P_2(t) + \beta\gamma_y(y_0^*)\varepsilon y_0 + q(0)y_0 \geq 0, \quad a.e., a.s.$$

Proof. From (c) in (13)

$$\begin{aligned} 0 &\leq J_\lambda(y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)) - J_\lambda(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.)) \\ &\quad + \sqrt{\lambda} d((y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)), (y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.))) \\ &= \frac{J_\lambda^2(y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)) - J_\lambda^2(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.))}{J_\lambda(y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)) + J_\lambda(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.))} + \varepsilon\sqrt{\lambda}\sqrt{2 + |y_0|^2}. \end{aligned} \tag{24}$$

Using Lemma 3.4, we have

$$\begin{aligned} &J_\lambda^2(y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)) - J_\lambda^2(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.)) \\ &\quad = \Big[J(y_{\lambda 0}^\varepsilon, u_\lambda^\varepsilon(.), \nu_\lambda^\varepsilon(.)) - J(y_0^*, u^*(.), \nu^*(.)) + \lambda\Big]^2 \\ &\quad - \Big[J(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.)) - J(y_0^*, u^*(.), \nu^*(.)) + \lambda\Big]^2 \\ &\quad + \Big[E^{\nu_\lambda^\varepsilon}|y_\lambda^\varepsilon(T) - \psi(x_\lambda^\varepsilon(T))|^2\Big]^2 - \Big[E^{\nu_\lambda}|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big]^2 \\ &\quad = 2\Big[J(y_{\lambda 0}, u_\lambda(.), \nu_\lambda(.)) - J(y_0^*, u^*(.), \nu^*(.)) + \lambda\Big] \\ &\quad \times \Bigg\{E\int_0^T \Bigg[(Z_\lambda^1(t) + Z_\lambda^2(t))l(t) + Z_\lambda(t)\Big[l_x(t)(x_\lambda^1(t) + x_\lambda^2(t)) \\ &\quad + l_{x'}(t)(x_\lambda^1(t-\tau) + x_\lambda^2(t-\tau)) + l_y(t)(y_\lambda^1(t) + y_\lambda^2(t)) + \frac{1}{2}l_{xx}(t)(x_\lambda^1(t))^2 \\ &\quad + \frac{1}{2}l_{x'x'}(t)(x_\lambda^1(t-\tau))^2 + l_{xx'}(t)(x_\lambda^1(t-\tau)x_\lambda^1(t)) \\ &\quad + \frac{1}{2}l_{yy}(t)(y_\lambda^1(t))^2 + l_{x'y}(t)(x_\lambda^1(t-\tau)y_\lambda^1(t)) + \Delta l(t)\Big]\Bigg]dt \\ &\quad + E\Big[(Z_\lambda^1(T) + Z_\lambda^2(T))\phi(x_\lambda(T)) + Z_\lambda(T)\Big[\phi_x(x_\lambda(T)(x_\lambda^1(T) + x_\lambda^2(T)) \\ &\quad + \frac{1}{2}\phi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2\Big]\Big] + \gamma_y(y_{\lambda 0})\varepsilon y_0\Bigg\} + 2E\Big[Z_\lambda(T)|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big] \\ &\quad \times \Bigg\{2Z_\lambda(T)E\Big[(y_\lambda(T) - \psi(x_\lambda(T)))(y_\lambda^1(T) + y_\lambda^2(T))\Big] \\ &\quad + E((y_\lambda^1(T))^2) + 2E\Big[(y_\lambda(T) - \psi(x_\lambda(T))) \\ &\quad \times (-\psi_x(x_\lambda(T))(x_\lambda^1(T) + x_\lambda^2(T))\Big] \\ &\quad + E\Big[(y_\lambda(T) - \psi(x_\lambda(T)))(-\psi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2)\Big] \\ &\quad + E\Big[\psi_x(x_\lambda(T))^2(x_\lambda^1(T))^2\Big] + E\Big[(Z_\lambda^1(T) + Z_\lambda^2(T))|y_\lambda(T) - \psi(x_\lambda(T))|^2\Big]^2\Bigg\} + o(\varepsilon). \end{aligned}$$

So, from the above relation and by (24), we have the following variational inequality

$$\beta_\lambda^\varepsilon\bigg\{E\int_0^T\bigg[(Z_\lambda^1(t)+Z_\lambda^2(t))l(t)+Z_\lambda(t)\Big[l_x(t)(x_\lambda^1(t)+x_\lambda^2(t))$$
$$+l_{x'}(t)(x_\lambda^1(t-\tau)+x_\lambda^2(t-\tau))+l_y(t)(y_\lambda^1(t)+y_\lambda^2(t))+\frac{1}{2}l_{xx}(t)(x_\lambda^1(t))^2$$
$$+\frac{1}{2}l_{x'x'}(t)(x_\lambda^1(t-\tau))^2+l_{xx'}(t)(x_\lambda^1(t-\tau)x_\lambda^1(t))$$
$$+\frac{1}{2}l_{yy}(t)(y_\lambda^1(t))^2+l_{x'y}(t)(x_\lambda^1(t-\tau)y_\lambda^1(t))+\Delta l(t)\Big]\bigg]dt$$
$$+E\Big[(Z_\lambda^1(T)+Z_\lambda^2(T))\phi(x_\lambda(T))+Z_\lambda(T)\Big[\phi_x(x_\lambda(T)(x_\lambda^1(T)+x_\lambda^2(T))$$
$$+\frac{1}{2}\phi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2\Big]\Big]+\gamma_y(y_{\lambda 0})\varepsilon y_0\bigg\}$$
$$+\mu_\lambda^\varepsilon\bigg\{2Z_\lambda(T)E\Big[(y_\lambda(T)-\psi(x_\lambda(T)))(y_\lambda^1(T)+y_\lambda^2(T))\Big]+E((y_\lambda^1(T))^2) \tag{25}$$
$$+2E\Big[(y_\lambda(T)-\psi(x_\lambda(T)))\times(-\psi_x(x_\lambda(T))(x_\lambda^1(T)+x_\lambda^2(T))\Big]$$
$$+E\Big[(y_\lambda(T)-\psi(x_\lambda(T)))(-\psi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2)\Big]$$
$$+E\Big[\psi_x(x_\lambda(T))^2(x_\lambda^1(T))^2\Big]+E\Big[(Z_\lambda^1(T)+Z_\lambda^2(T))|y_\lambda(T)-\psi(x_\lambda(T))|^2\Big]^2\bigg\}$$
$$+o(\varepsilon)+\varepsilon\sqrt{\lambda}\sqrt{2+|y_0|^2}\geq 0, \tag{26}$$

where

$$\beta_\lambda^\varepsilon=\frac{2\Big[J(y_{\lambda 0},u_\lambda(.),\nu_\lambda(.))-J(y^*,u^*(.),\nu^*(.))+\lambda\Big]}{J_\lambda(y_{\lambda 0}^\varepsilon,u_\lambda^\varepsilon(.),\nu_\lambda^\varepsilon(.))+J_\lambda(y_{\lambda 0},u_\lambda(.),\nu_\lambda(.))},$$
$$\mu_\lambda^\varepsilon=\frac{2E^{\nu_\lambda}|y_\lambda(T)-\phi(x_\lambda(T))|^2}{J_\lambda(y_{\lambda 0}^\varepsilon,u_\lambda^\varepsilon(.),\nu_\lambda^\varepsilon(.))+J_\lambda(y_{\lambda 0},u_\lambda(.),\nu_\lambda(.))}.$$

Let $\Gamma_1(t)=Z^1(t).Z^{-1}(t)$, $\Gamma_2(t)=Z^2(t).Z^{-1}(t)$, using Itô's formula, we get

$$d\Gamma^1(t)=\Big[h_x(t,x(t),x(t-\tau),\nu(t))x^1(t)+h_{x'}(t,x(t),x(t-\tau),\nu(t))x^1(t-\tau)$$
$$+h(t,x(t),x(t-\tau),\nu^\varepsilon(t))-h(t,x(t),x(t-\tau),\nu(t))\Big]d\tilde{W}(t), \tag{27}$$

$$d\Gamma^2(t)=\Big[h_x(t,x(t),x(t-\tau),\nu(t))x^2(t)+h_{x'}(t,x(t),x(t-\tau),\nu(t))x^2(t-\tau)$$
$$+\frac{1}{2}h_{xx}(t,x(t),x(t-\tau),\nu(t))(x^1(t))^2$$
$$+\frac{1}{2}h_{x'x'}(t,x(t),x(t-\tau),\nu(t))(x^1(t-\tau))^2$$
$$+h_{xx'}(t,x(t),x(t-\tau),\nu(t))(x^1(t)x^1(t-\tau))$$
$$+(h_x(t,x(t),x(t-\tau),\nu^\varepsilon(t))-h_x(t,x(t),x(t-\tau),\nu(t)))x^1(t)$$
$$+(h_{x'}(t,x(t),x(t-\tau),\nu^\varepsilon(t))-h_{x'}(t,x(t),x(t-\tau),\nu(t)))x^1(t-\tau)$$
$$+\Gamma_1(t)(h(t,x(t),x(t-\tau),\nu^\varepsilon(t))-h(t,x(t),x(t-\tau),\nu(t)))\Big]d\tilde{W}(t). \tag{28}$$

We can rewrite inequality (25) as

$$\begin{aligned}
&\beta_\lambda^\varepsilon\Big\{E^{\nu_\lambda}\int_0^T\Big[(\Gamma_\lambda^1(t)+\Gamma_\lambda^2(t))l(t)+l_x(t)(x_\lambda^1(t)+x_\lambda^2(t))\\
&+l_{x'}(t)(x_\lambda^1(t-\tau)+x_\lambda^2(t-\tau))+l_y(t)(y_\lambda^1(t)+y_\lambda^2(t))+\frac{1}{2}l_{xx}(t)(x_\lambda^1(t))^2\\
&+\frac{1}{2}l_{x'x'}(t)(x_\lambda^1(t-\tau))^2+l_{xx'}(t)(x_\lambda^1(t-\tau)x_\lambda^1(t))+\frac{1}{2}l_{yy}(t)(y_\lambda^1(t))^2\\
&+l_{x'y}(t)(x_\lambda^1(t-\tau)y_\lambda^1(t))+\Delta l(t)\Big]dt+E^{\nu_\lambda}\Big[(\Gamma_\lambda^1(T)+\Gamma_\lambda^2(T))\phi(x_\lambda(T))\\
&+\phi_x(x_\lambda(T))(x_\lambda^1(T)+x_\lambda^2(T))+\frac{1}{2}\phi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2\Big]+\gamma_y(y_{\lambda 0})\varepsilon y_0\Big\}\\
&+\mu_\lambda^\varepsilon\Big\{2E^{\nu_\lambda}\Big[(y_\lambda(T)-\psi(x_\lambda(T)))(y_\lambda^1(T)+y_\lambda^2(T))\Big]+E^{\nu_\lambda}((y_\lambda^1(T))^2)\\
&+2E^{\nu_\lambda}\Big[y_\lambda(T)-\psi(x_\lambda(T))\times(-\psi_x(x_\lambda(T))(x_\lambda^1(T)+x_\lambda^2(T))\Big]\\
&+E^{\nu_\lambda}\Big[y_\lambda(T)-\psi(x_\lambda(T)))(-\psi_{xx}(x_\lambda(T)(x_\lambda^1(T))^2\Big]\\
&+E^{\nu_\lambda}\Big[(\psi_x(x_\lambda(T))^2(x_\lambda^1(T))^2\Big]+E^{\nu_\lambda}\Big[(\Gamma_\lambda^1(T)+\Gamma_\lambda^2(T))|y_\lambda(T)-\psi(x_\lambda(T))|^2\Big]^2\Big\}\\
&+o(\varepsilon)+\varepsilon\sqrt{\lambda}\sqrt{2+|y_0|^2}\geq 0. \qquad (29)
\end{aligned}$$

As before, we introduce the following adjoint equations:

$$\begin{cases}
dm_\lambda^\varepsilon(t)=-\Big(N_\lambda^\varepsilon(t)h(t)+\beta_\lambda^\varepsilon l(t)\Big)dt+N_\lambda^\varepsilon(t)dY(t),\\
dp_\lambda^\varepsilon(t)=-\Big[p_\lambda^\varepsilon(t)b_x(t)+q_\lambda^\varepsilon(t)a_x(t)+k_\lambda^\varepsilon(t)\sigma_x(t)+N_\lambda^\varepsilon(t)h_x(t)+\beta_\lambda^\varepsilon l_x(t)\\
\qquad +E^{\mathcal{F}_t}(p_\lambda^\varepsilon(t+\tau)b_{x'}(t+\tau))+E^{\mathcal{F}_t}(q_\lambda^\varepsilon(t+\tau)a_{x'}(t+\tau))+E^{\mathcal{F}_t}(k_\lambda^\varepsilon(t+\tau)\sigma_{x'}(t+\tau))\\
\qquad +E^{\mathcal{F}_t}(N_\lambda^\varepsilon(t+\tau)h_{x'}(t+\tau))+\beta E^{\mathcal{F}_t}({}_\lambda^\varepsilon l_{x'}(t+\tau))\Big]dt+k_\lambda^\varepsilon(t)dW(t),\\
dq_\lambda^\varepsilon(t)=-\Big(q_\lambda^\varepsilon(t)a_y(t)+\beta_\lambda^\varepsilon l_y(t)\Big)dt+r_\lambda^\varepsilon(t)dW(t),\\
m_\lambda^\varepsilon(T)=\beta_\lambda^\varepsilon\phi(x_\lambda(T))+\mu_\lambda^\varepsilon(y_\lambda(T)-\psi(x_\lambda(T))^2,\quad q_\lambda^\varepsilon(T)=2\mu_\lambda^\varepsilon(y_\lambda(T)-\psi(x_\lambda(T)),\\
p_\lambda^\varepsilon(T)=\beta_\lambda^\varepsilon\phi_x(x_\lambda(T))-2\mu_\lambda^\varepsilon\psi_x(x_\lambda(T)(y_\lambda(T)-\psi(x_\lambda(T)),\quad p_\lambda^\varepsilon(t)=0,\quad t\in(T,T+\tau].
\end{cases} \qquad (30)$$

This is a time-advanced BSDE. Then by Theorem 2.1 in [20], we know that (30) admits unique solutions under hypothesis 1.

Define the Hamiltonian function H as (16). Applying Itô's formula to $p_\lambda^\varepsilon(t)(x_\lambda^1(t)+x_\lambda^2(t))+q_\lambda^\varepsilon(t)(y_\lambda^1(t)+y_\lambda^2(t))+m_\lambda^\varepsilon(t)(\Gamma_\lambda^1(t)+\Gamma_\lambda^2(t))$ and by the variational inequality (29) we get

$$\begin{aligned}
&E^{\nu_\lambda}\int_0^T\Delta H(t)dt+\frac{1}{2}E^{\nu_\lambda}\int_0^T\Big[H_{xx}(t)(x_\lambda^1(t))^2+H_{yy}(t)(y_\lambda^1(t))^2+2H_{xy}(t)(x_\lambda^1(t)y_\lambda^1(t))\\
&+H_{x'x'}(t)(x_\lambda^1(t-\tau))^2+2H_{xx'}(t)(x_\lambda^1(t)x_\lambda^1(t-\tau))+2H_{yx'}(t)(y_\lambda^1(t)x_\lambda^1(t-\tau))\Big]dt\\
&+\frac{1}{2}E^{\nu_\lambda}\beta_\lambda^\varepsilon\Big[\phi_{xx}(x_\lambda(T))(x_\lambda^1(T))^2\Big]+\beta_\lambda^\varepsilon\gamma_y(y_{\lambda 0})\varepsilon y_0+q_\lambda^\varepsilon(0)\varepsilon y_0\\
&-E\mu_\lambda^\varepsilon\Big[y_\lambda(T)-\psi(x_\lambda(T)))(-\psi_{xx}(x_\lambda(T)(x_\lambda^1(T))^2\Big]\\
&+E^{\nu_\lambda}\Big[(\psi_x(x_\lambda(T))^2(x_\lambda^1(T))^2\Big]+E^{\nu_\lambda}\mu_\lambda^\varepsilon\Big[y_\lambda^1(T)\Big]^2+o(\varepsilon)+\varepsilon\sqrt{\lambda}\sqrt{2+|y_0|^2}\geq 0, \qquad (31)
\end{aligned}$$

where

$$L(t)=L(t,x_\lambda(t),x_\lambda(t-\tau),y_\lambda(t),u_\lambda(t),\nu_\lambda(t),p_\lambda^\varepsilon(t),q_\lambda^\varepsilon(t),k_\lambda^\varepsilon(t),r_\lambda^\varepsilon(t),N_\lambda^\varepsilon(t),\beta_\lambda^\varepsilon(t)),$$
$$L=H,H_{xx},H_{yy},H_{xy},$$
$$\Delta H(t)=H(t,x_\lambda(t),x_\lambda(t-\tau),y_\lambda(t),u_\lambda^\varepsilon(t),\nu_\lambda^\varepsilon(t),p_\lambda^\varepsilon(t),q_\lambda^\varepsilon(t),k_\lambda^\varepsilon(t),r_\lambda^\varepsilon(t),N_\lambda^\varepsilon(t),\beta_\lambda^\varepsilon(t))-H(t).$$

Now, we formulate the following adjoint equations:

$$\begin{cases} dP^{\varepsilon}_{\lambda 1}(t) = -\Big[2P^{\varepsilon}_{\lambda 1}(t)b_x(t) + \sigma_x^2(t)P^{\varepsilon}_{\lambda 1}(t) + 2\sigma_x(t)Q^{\varepsilon}_{\lambda 1}(t) + P^{\varepsilon}_{\lambda 3}(t)a_x(t) + H_{xx}(t) \\ \qquad +E^{\mathcal{F}_t}(P'^{\varepsilon}_{\lambda 1}(t+\tau)b_{x'}(t+\tau)) + E^{\mathcal{F}_t}(P^{\varepsilon}_{\lambda 1}(t+\tau)\sigma^2_{x'}(t+\tau)) \\ \qquad +E^{\mathcal{F}_t}(P'^{\varepsilon}_{\lambda 1}(t+\tau)\sigma_x(t)\sigma_{x'}(t+\tau)) + E^{\mathcal{F}_t}(Q'^{\varepsilon}_{\lambda 1}(t+\tau)\sigma_{x'}(t+\tau)) \\ \qquad +E^{\mathcal{F}_t}(P'^{\varepsilon}_{\lambda 1}(t+\tau)a_{x'}(t+\tau)) + H_{x'x'}(t+\tau)\Big]dt + Q^{\varepsilon}_{\lambda 1}(t)dW(t), \\ P^{\varepsilon}_{\lambda 1}(T) = \beta^{\varepsilon}_{\lambda}\phi_{xx}(x_\lambda(T)) + 2\mu^{\varepsilon}_{\lambda}\Big[(y_\lambda(T) - \psi(x_\lambda(T))(-\psi_{xx}(x_\lambda(T)) + (\psi_x(x_\lambda(T))^2\Big], \\ P^{\varepsilon}_{\lambda 1}(t) = 0, \quad t \in (T, T+\tau] \end{cases} \tag{32}$$

$$\begin{cases} dP^{\varepsilon}_{\lambda 2}(t) = -\Big[2P^{\varepsilon}_{\lambda 2}(t)a_y(t) + H_{yy}(t)\Big]dt + Q^{\varepsilon}_{\lambda 2}(t)dW(t), \\ P^{\varepsilon}_{\lambda 2}(T) = 2\mu^{\varepsilon}_{\lambda}, \end{cases} \tag{33}$$

$$\begin{cases} dP^{\varepsilon}_{\lambda 3}(t) = -\Big[P^{\varepsilon}_{\lambda 3}(t)a_y(t) + P^{\varepsilon}_{\lambda 3}(t)b_x(t) + 2a_x(t)P^{\varepsilon}_{\lambda 2}(t) + \sigma_x(t)Q^{\varepsilon}_{\lambda 3}(t) + H_{xy}(t) \\ \qquad +E^{\mathcal{F}_t}(P^{\varepsilon}_{\lambda 3}(t+\tau)b_{x'}(t+\tau)) + E^{\mathcal{F}_t}(Q^{\varepsilon}_{\lambda 3}(t+\tau)\sigma_{x'}(t+\tau)) \\ \qquad +2E^{\mathcal{F}_t}(P^{\varepsilon}_{\lambda 2}(t+\tau)a_{x'}(t+\tau))\Big]dt + Q^{\varepsilon}_{\lambda 3}(t)dW(t), \\ P^{\varepsilon}_{\lambda 3}(T) = 0, \ P^{\varepsilon}_{\lambda 3}(t) = 0, \ t \in (T, T+\tau], \end{cases} \tag{34}$$

$$\begin{cases} dP'_1(t) = -\Big[P'^{\varepsilon}_{\lambda 1}(t)b_x(t) + 2P^{\varepsilon}_{\lambda 1}(t)b_{x'}(t) + \sigma_x(t)Q'^{\varepsilon}_{\lambda 1}(t) + P'^{\varepsilon}_{\lambda 1}(t)b_x(t-\tau) \\ \qquad +2Q^{\varepsilon}_{\lambda 1}(t)\sigma_{x'}(t) + Q'^{\varepsilon}_{\lambda 1}(t)\sigma_x(t-\tau) + P'^{\varepsilon}_{\lambda 1}(t)\sigma_x(t)\sigma_x(t-\tau) + P^{\varepsilon}_{\lambda 3}(t)a_{x'}(t) \\ \qquad +P'^{\varepsilon}_{\lambda 3}(t)a_x(t) + 2P^{\varepsilon}_{\lambda 1}(t)\sigma_{x'}(t)\sigma_x(t) + H_{xx'}(t)\Big]dt + Q'^{\varepsilon}_{\lambda 1}(t)dW(t), \\ P'^{\varepsilon}_{\lambda 1}(T) = 0, \ t \in [T, T+\tau], \end{cases} \tag{35}$$

$$\begin{cases} dP'^{\varepsilon}_{\lambda 3}(t) = -\Big[P'^{\varepsilon}_{\lambda 3}(t)a_y(t) + P'^{\varepsilon}_{\lambda 3}(t)b_x(t-\tau) + Q'^{\varepsilon}_{\lambda 3}(t)\sigma_x(t-\tau) + H_{x'y}(t)\Big]dt + Q'^{\varepsilon}_{\lambda 3}(t)dW(t), \\ P'^{\varepsilon}_{\lambda 3}(t) = 0, \quad t \in [T, T+\tau]. \end{cases} \tag{36}$$

Clearly

$$\lim_{\varepsilon\to 0}(|\beta^{\varepsilon}_{\lambda}|^2 + |\mu^{\varepsilon}_{\lambda}|^2) = 1,$$

so there exists a subsequence still denoted by $(\beta^{\varepsilon}_{\lambda}, \mu^{\varepsilon}_{\lambda})$ that converges to $(\beta_\lambda, \mu_\lambda)$, which $|\beta_\lambda|^2 + |\mu_\lambda|^2 = 1$. Consequently, we can verify the following limits as $\varepsilon \to 0$ in $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R})$

$$m^{\varepsilon}_{\lambda}(.) \to m_\lambda(.), \quad ,N^{\varepsilon}_{\lambda}(.) \to N_\lambda(.), \quad p^{\varepsilon}_{\lambda}(.) \to p_\lambda(.), \quad q^{\varepsilon}_{\lambda}(.) \to q_\lambda(.), \quad k^{\varepsilon}_{\lambda}(.) \to k_\lambda(.),$$
$$r^{\varepsilon}_{\lambda}(.) \to r_\lambda(.), \quad P^{\varepsilon}_{\lambda 1}(t) \to P_{\lambda 1}(t), \quad P^{\varepsilon}_{\lambda 2}(t) \to P_{\lambda 2}(t), \quad P^{\varepsilon}_{\lambda 3}(t) \to P_{\lambda 3}(t),$$

$$Q^{\varepsilon}_{\lambda 1}(t) \to Q_{\lambda 1}(t), \quad Q^{\varepsilon}_{\lambda 2}(t) \to Q_{\lambda 2}(t), \quad Q^{\varepsilon}_{\lambda 3}(t) \to Q_{\lambda 3}(t),$$
$$P'^{\varepsilon}_{\lambda 1}(t) \to P'_{\lambda 1}(t), \quad P'^{\varepsilon}_{\lambda 3}(t) \to P'_{\lambda 3}(t), \quad Q'^{\varepsilon}_{\lambda 1}(t) \to Q'_{\lambda 1}(t), \quad Q'^{\varepsilon}_{\lambda 3}(t) \to Q'_{\lambda 3}(t),$$

where $(m_\lambda(.), N_\lambda(.))$, $(p_\lambda(.), k_\lambda(.))$, $(q_\lambda(.), r_\lambda(.))$, $(P_{\lambda 1}(.), Q_{\lambda 1}(.))$, $(P_{\lambda 2}(.), Q_{\lambda 2}(.))$ and $(P_{\lambda 3}(.), Q_{\lambda 3}(.))$, $(P'_{\lambda 1}(.), Q'_{\lambda 1}(.))$, $(P'_{\lambda 3}(.), Q'_{\lambda 3}(.))$ satisfy in (30)-(34) with $(\beta^{\varepsilon}_{\lambda}, \mu^{\varepsilon}_{\lambda})$ replaced by $(\beta_\lambda, \mu_\lambda)$ and
$L(t) = L(t, x_\lambda(t), x_\lambda(t-\tau), y_\lambda(t), u_\lambda(t), \nu_\lambda(t))$ for $L = H_{xx}, H_{yy}, H_{xy}, H_{x'x'}, H_{xx'}, H_{x'y}$.
Applying Itô's formula to $P_{\lambda 1}(t)(x^1_\lambda(t))^2$, $P'_{\lambda 1}(t)(x^1_\lambda(t)x^1_\lambda(t-\tau))$, $P_{\lambda 2}(t)(y^1_\lambda(t))^2$, $P_{\lambda 3}(t)x^1_\lambda(t)y^1_\lambda(t)$,
$P'_{\lambda 3}(t)x^1_\lambda(t-\tau)y^1_\lambda(t)$ and letting (31)$\times\frac{1}{\varepsilon}$ and $\varepsilon \to 0$, also, we note that $x^1(t-2\tau) = 0$, because we use one

pointwise delay, so we have

$$
\begin{aligned}
E^{\nu}\int_0^T &\Big[H(t,x_\lambda(t),x_\lambda(t-\tau),y_\lambda(t),u(t),\nu(t),p_\lambda(t),q_\lambda(t),k_\lambda(t),r_\lambda(t),N_\lambda(t),\beta_\lambda(t))\\
&-H(t,x_\lambda(t),x_\lambda(t-\tau),y_\lambda(t),u_\lambda(t),\nu_\lambda(t),p_\lambda(t),q_\lambda(t),k_\lambda(t),r_\lambda(t),N_\lambda(t),\beta_\lambda(t))\Big]dt\\
&+\frac{1}{2}\Big[\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu(t))-(\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu_\lambda(t)))\Big]^2P_{\lambda1}(t)\\
&+\Big[(\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu(t))-\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu_\lambda(t)))\\
&\times(\sigma(t-\tau,x_\lambda(t-\tau),x_\lambda(t-2\tau),\nu(t))-\sigma(t-\tau,x_\lambda(t-\tau),x_\lambda(t-2\tau),\nu_\lambda(t)))\Big]P'_{\lambda1}(t)\\
&+\Big[(\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu(t))-\sigma(t,x_\lambda(t),x_\lambda(t-\tau),\nu_\lambda(t)))(u(t)-u_\lambda(t))\Big]P_{\lambda3}(t)\\
&+\Big[(\sigma(t-\tau,x_\lambda(t-\tau),x_\lambda(t-2\tau),\nu(t))-\sigma(t-\tau,x_\lambda(t-\tau),x_\lambda(t-2\tau),\nu_\lambda(t)))\\
&\times(u(t)-u_\lambda(t))\Big]P'_{\lambda3}(t)\\
&+\frac{1}{2}\Big[(u(t)-u_\lambda(t))^2\Big]P_{\lambda2}(t)+\beta_\lambda\gamma_y(y_{\lambda0})\varepsilon y_0+q_\lambda(0)y_0+\sqrt{\lambda}\sqrt{2+|y_0|^2}\geq 0.
\end{aligned}
\tag{37}
$$

Similarly, there exists a subsequence of $(\beta_\lambda,\mu_\lambda)$ that converges to (β,μ), with $|\beta|^2+|\mu|^2=1$. Since $y_{\lambda0}\to y_0^*$, $u_\lambda(.)\to u^*(.)$, $\nu_\lambda(.)\to\nu^*(.)$ as $\lambda\to 0$, we have the following limits as $\lambda\to 0$

$$
x_\lambda(.)\to x^*(.),\quad y_\lambda(.)\to y^*(.),\quad Z_\lambda(.)\to Z^*(.),\quad ,y^*(T)=\psi(x^*(T)),
$$

also, in $\mathcal{L}^2_{\mathcal{F}}([0,T],\mathbb{R})$, we have the following limits

$$
\begin{aligned}
&m_\lambda(.)\to m(.),\quad ,N_\lambda(.)\to N(.),\quad p_\lambda(.)\to p(.),\quad q_\lambda(.)\to q(.),\quad k_\lambda(.)\to k(.),\\
&r_\lambda(.)\to r(.),\quad P_{\lambda1}(t)\to P_1(t),\quad P_{\lambda2}(t)\to P_2(t),\quad P_{\lambda3}(t)\to P_3(t),\\
&Q_{\lambda1}(t)\to Q_1(t),\quad Q_{\lambda2}(t)\to Q_2(t),\quad Q_{\lambda3}(t)\to Q_3(t),\\
&P'_{\lambda1}(t)\to P'_1(t),\quad P'_{\lambda3}(t)\to P'_3(t),\quad Q'_{\lambda1}(t)\to Q'_1(t),\quad Q'_{\lambda3}(t)\to Q'_3(t),
\end{aligned}
$$

sending $\lambda\to 0$ in (37), we derive (23). □

Step 2. For the general case of control domains, let

$$
\begin{aligned}
&\mathcal{M}^n=\{y_0\in\mathbb{R};|y_0|\leq|y_0^*|+n\},\quad \mathcal{N}^n=\{u(t)\in\mathbb{R};|u(t)|\leq|u^*(t)|+n\},\\
&\mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{N}^n)=\{u(.)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R});u(t)\in\mathcal{N}^n\},\\
&\mathcal{U}^n_{ad}=\{\nu(.)\in\mathcal{U}_{ad}|\sup_{0\leq t\leq T}E|\nu(t)|\leq\sup_{0\leq t\leq T}E|\nu^*(t)|+n\},\quad n=1,2,....
\end{aligned}
$$

It is easy to see that $\mathcal{M}^n$ is convex and $\mathcal{M}^n$, $L^2_{\mathcal{F}}(0,T;\mathcal{N}^n)$, $\mathcal{U}^n_{ad}$ for n=1,2,... are all bounded. Then by the proof in Step 1 and in a similar way as Step 2, in [13] we have the general maximum principle (23). Since $(y_0,u(.),\nu(.))$ is arbitrary we have the following theorem:

Theorem 4.2. *Assume hypothesis 1 and hypothesis 2 be satisfied, $(x^*(.),y^*(.),z^*(.),\nu^*(.))$ and $Z^*(.)$ is a solution of the optimal control problem (4), (6) and (7). Then there exist β and μ with $\beta^2+\mu^2=1$, such that for any $y_0\in\mathbb{R}$, $u\in\mathbb{R}$ and $\nu\in\mathcal{U}$ the condition (23) holds, where H is defined by (16) and $(m(.),N(.))$, $(p(.),k(.))$, $(q(.),r(.))$, $(P_1(.),Q_1(.))$, $(P'_1(.),Q'_1(.))$, $(P_2(.),Q_2(.))$, $(P_3(.),Q_3(.))$, $(P'_3(.),Q'_3(.))$ are the solutions of the equations (30)-(36) with $u^*(.)$ replaced by $z^*(.)$, respectively.*

5. Example

In this section, we will study a linear-quadratic partially observed problem with delay. We consider the following control system:

$$\begin{cases} dx(t) = (ax(t) + a^{'}x(t-\tau) + bv(t))dt + (cx(t) + c^{'}x(t-\tau) + ev(t))dW(t), \\ dy(t) = (fx(t) + gy(t) + mz(t) + nv(t))dt + z(t)dW(t), \quad t \in [0, T], \\ x(t) = x_0(t), \quad t \in [-\tau, 0], \quad y(T) = kx(T), \end{cases} \tag{38}$$

and the observation process

$$\begin{cases} dY(t) = hdt + d\tilde{W}(t), \\ Y(0) = 0. \end{cases} \tag{39}$$

Our objective is to minimize the following cost functional over $\mathcal{U}$

$$J(v(.)) = \frac{1}{2}E^{v}\Big[\int_0^T (lv^2(t) + sz^2(t))dt + \alpha x(T)^2 + wy^2(0)\Big], \tag{40}$$

subject to (38) and (39), where $a, a^{'}, b, c, c^{'}, e, f, g, m, n, h, l, \alpha, s$ and w are constants satisfying $l > 0,\ \alpha \geq 0$ and $w > 0$.
The Hamiltonian function is

$$\begin{aligned} H(t, x, x', y, u, v, p, q, k, r, N, \beta) =& p(ax(t) + a^{'}x(t-\tau) + bv(t)) + q(fx(t) + gy(t) + mu(t) + nv(t)) \\ &+ k(cx(t) + c^{'}x(t-\tau) + ev(t)) + ru(t) + Nh \\ &+ \frac{1}{2}\beta(lv^2(t) + sz^2(t)). \end{aligned} \tag{41}$$

The adjoint processes take the form:

$$\begin{cases} dm(t) = -\Big(N(t)h + \frac{1}{2}\beta(lv^2(t) + sz^2(t))\Big)dt + N(t)dY(t), \\ dp(t) = -\Big[p(t)a + q(t)f + k(t)c + E^{\mathcal{F}_t}(p(t+\tau)a^{'}) + E^{\mathcal{F}_t}(k(t+\tau)c^{'})\Big]dt + k(t)dW(t), \\ dq(t) = -q(t)gdt + r(t)dW(t), \\ m(T) = \frac{1}{2}\beta\alpha x^{*2}(T), \quad p(T) = \alpha\beta x^*(T), \quad p(t) = 0, \quad t \in (T, T+\tau], \quad q(T) = 0, \end{cases} \tag{42}$$

$$\begin{cases} dP_1(t) = -\Big[2P_1(t)a + c^2P_1(t) + 2cQ_1(t) + P_3(t)f \\ \qquad + E^{\mathcal{F}_t}(P_1'(t+\tau)a^{'}) + E^{\mathcal{F}_t}(P_1(t+\tau)c^{'2}) \\ \qquad + E^{\mathcal{F}_t}(P_1'(t+\tau)cc^{'}) + E^{\mathcal{F}_t}(Q_1'(t+\tau)c^{'})\Big]dt + Q_1(t)dW(t), \\ P_1(T) = \beta\alpha + 2\mu k^2, \quad P_1(t) = 0, \quad t \in (T, T+\tau], \end{cases} \tag{43}$$

$$\begin{cases} dP_2(t) = -\Big[2P_2(t)g\Big]dt + Q_2(t)dW(t), \\ P_2(T) = 2\mu, \end{cases} \tag{44}$$

$$\begin{cases} dP_3(t) = -\Big[P_3(t)g + P_3(t)a + 2fP_2(t) + Q_3(t)c \\ \qquad + E^{\mathcal{F}_t}(P_3(t+\tau)a^{'}) + E^{\mathcal{F}_t}(Q_3(t+\tau)c^{'})\Big]dt + Q_3(t)dW(t), \\ P_3(t) = 0, \quad t \in [T, T+\tau], \end{cases} \tag{45}$$

$$\begin{cases} dP_1'(t) = -\Big[P_1'(t)a + 2P_1(t)a' + cQ_1'(t) + P_1'(t)a \\ \qquad + 2Q_1(t)c' + Q_1'(t)c + P_1'(t)c^2 \\ \qquad + P_3'(t)f + 2P_1(t)c'c\Big]dt + Q_1'(t)dW(t), \\ P_1'(T) = 0 \ \ t \in [T, T+\tau], \end{cases} \tag{46}$$

$$\begin{cases} dP'_3(t) = -\Big[P'_3(t)g + P'_3(t)a + Q'_3(t)c\Big]dt + Q'_3(t)dW(t), \\ P'_3(t) = 0, \quad t \in [T, T+\tau], \end{cases} \tag{47}$$

where μ and β are constants satisfying $\mu^2 + \beta^2 = 1$, $\mu \geq 0, \beta \geq 0$.
Because the case of $\beta = 0$ is trivial, so we only study the case $\beta > 0$. Let $u = z^*$ and $\nu = \nu^*$ in the maximum condition (23), so we have $\beta w y_0^* + q(0) = 0$, i.e.,

$$y_0^* = -\frac{q(0)}{\beta w}. \tag{48}$$

Let $\nu(t) = \nu^*(t)$ and $y_0 = y_0^*$ in the condition (23) , so we have $q(t)m + r(t) + \beta s z^*(t) = 0$, therefore

$$z^*(t) = -\frac{mq(t) + r(t)}{\beta s}. \tag{49}$$

Letting $u(t) = z^*(t)$ and $y_0 = y_0^*$ in the inequality (23), the optimal control $\nu^*(t)$ should satisfy

$$(p(t)b + q(t)n + k(t)e)(\nu(t) - \nu^*(t)) + \frac{1}{2}\beta l(\nu^2(t) - (\nu^*(t))^2)$$
$$+ \frac{1}{2}P_1(t)e^2(\nu(t) - \nu^{*2}(t)) + e^2(\nu(t) - \nu^*(t))^2 P'_1(t) \geq 0.$$

Then we have

$$\nu^*(t) = -\frac{p(t)b + q(t)a + k(t)e}{\beta l}. \tag{50}$$

Proposition 5.1. *If $y^*(0)$ and $z^*(.)$ satisfy in (48) and (49),then for given $\beta > 0$, $\nu^*(.)$ given in (50) is an optimal control for LQ optimal problem (38), (39) and (40).*

Proof. Using Itô formula to $p(t)(x(t) - x^*(t)) + q(t)(y(t) - y^*(t))$ from 0 to T, and according to (48), we derive

$$E^\nu\Big[\alpha\beta x^*(T)(x(T) - x^*(T)) + \beta w y^*(0)(y(0) - y^*(0))\Big]$$
$$= E^\nu \int_0^T (p(t)b + k(t)l + q(t)n)(\nu(t) - \nu^*(t)) + (m\,q(t) + r(t))(z(t) - z^*(t)). \tag{51}$$

Noting $\alpha > 0$,$w \geq 0$, we have

$$\begin{aligned} &J(\nu(.)) - J(\nu^*(.)) \\ &\quad= \frac{1}{2}E^\nu \int_0^T \Big[l(\nu^2(t) - \nu^{*2}(t)) + s(z^2(t) - z^{*2}(t)) \\ &\qquad+ \alpha(x^2(T) - x^{*2}(T)) + w(y^2(0) - y^{*2}(0))\Big]dt \\ &\quad\geq E^\nu \int_0^T \Big[\frac{1}{2}l(\nu^2(t) - \nu^{*2}(t)) + \frac{1}{2}s(z^2(t) - z^{*2}(t)) \\ &\qquad+ \alpha\, x^*(T)(x(T) - x^*(T)) + w\, y^*(0)(y(0) - y^*(0))\Big]dt. \end{aligned}$$

So, using (51), we have

$$J(\nu(.)) - J(\nu^*(.)) \tag{52}$$
$$\geq E^\nu \int_0^T \Big[\frac{1}{2}l(\nu^2(t) - \nu^{*2}(t)) + \frac{1}{2}s(z^2(t) - z^{*2}(t))$$
$$+ \frac{1}{\beta}\Big(p(t)b + k(t)l + q(t)n)(\nu(t) - \nu^*(t)) + (m\,q(t) + r(t))(z(t) - z^*(t))\Big)\Big]dt. \tag{53}$$

From (49) and $s > 0$, we have

$$\begin{aligned} &E^{\nu}\int_0^T \Big[\frac{1}{2}s(z^2(t)-z^{*2}(t))+\frac{1}{\beta}(m\,q(t)+r(t))(z(t)-z^*(t))\Big]dt \\ &\quad = E^{\nu}\int_0^T \Big[\frac{1}{2}s(z(t)-z^*(t))^2+(sz^*(t)+\frac{1}{\beta}(m\,q(t)+r(t)))(z(t)-z^*(t))\Big]dt \\ &\quad \geq 0. \end{aligned} \tag{54}$$

From $l > 0$ and (50), we get

$$\Big[\frac{1}{2}l(\nu^2(t)-\nu^{*2}(t))+\frac{1}{\beta}\Big(p(t)b+k(t)l+q(t)n)(\nu(t)-\nu^*(t))\Big)\geq 0. \tag{55}$$

Then from (52)-(55), we have

$$J(\nu(.)-J(\nu^*(.))\geq 0,\ \forall \nu(.)\in \mathcal{U}_{ad}.$$

Thus, $\nu^*(.)$ is the optimal control. □

6. Conclusions

In this paper, we have studied the optimal control of forward-backward stochastic delay systems with a partial observation noise. The maximum principle is obtained under the assumption that the control region is not necessarily convex and the forward diffusion coefficient contains control variables and the control enters into the observation. We have regarded the martingale term in the backward equation as the control and have employed Ekeland's variational principle to obtain the maximum principle. Also, we have derived the first and second-order adjoint processes.

In the future, we would like to obtain the maximum principle for forward-backward stochastic control systems involving both delays in the state variable and the control variable.

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Appendix

Proof of Lemma 3.1. Because the higher-order moments imply lower-order ones, thus we only prove estimations 3, 6, 8.

From hypothesis 1 and Doob's martingale inequality, we get

$$\begin{aligned} \sup_{-\tau\leq t\leq T} E|x^1_\lambda(t)|^{16} &\leq C\Big[\sup_{-\tau\leq t\leq T} E\int_0^t |x^1_\lambda(s)|^{16}ds+\sup_{-\tau\leq t\leq T} E\int_0^t |x^1_\lambda(s-\tau)|^{16}ds \\ &\quad +\varepsilon^{15}E\int_d^{d+\varepsilon}|b(\nu^\varepsilon_\lambda(s))-b(s)|^{16}ds+\varepsilon^7 E\int_d^{d+\varepsilon}|\sigma(\nu^\varepsilon_\lambda(s))-\sigma(s)|^{16}ds\Big] \\ &\leq C(\sup_{-\tau\leq t\leq T} E\int_0^t |x^1_\lambda(s)|^{16}ds+\varepsilon^8). \end{aligned}$$

Thus with Gronwall's inequality, the proof of estimation 3 is completed.
Similar to the above analysis,

$$\begin{aligned}\sup_{-\tau\le t\le T} E|x_\lambda^2(t)|^8 \le & C\Big[\sup_{-\tau\le t\le T} E\int_0^t |x_\lambda^2(s)|^8 ds + \sup_{-\tau\le t\le T} E\int_0^t |x_\lambda^2(s-\tau)|^8 ds \\ & + \sup_{-\tau\le t\le T} E|x_\lambda^1(t)|^{16} + \sup_{-\tau\le t\le T} E|x_\lambda^1(t-\tau)|^{16} + \sup_{-\tau\le t\le T} E|x_\lambda^1(t)|^8|x_\lambda^1(t-\tau)|^8 \\ & + \varepsilon^7 E\int_d^{d+\varepsilon} |b_x(\nu_\lambda^\varepsilon(s)) - b_x(s)|^8 |x_\lambda^1(s)|^8 ds \\ & + \varepsilon^3 E\int_d^{d+\varepsilon} |\sigma_x(\nu_\lambda^\varepsilon(s)) - \sigma_x(s)|^8 |x_\lambda^1(s)|^8 ds \\ & + \varepsilon^7 E\int_d^{d+\varepsilon} |(b_{x'}(\nu_\lambda^\varepsilon(s)) - b_{x'}(s))|^8 |x_\lambda^1(s-\tau)|^8 ds\Big]\end{aligned}$$

$$\begin{aligned}&\le C(\sup_{-\tau\le t\le T} E\int_0^t |x_\lambda^2(s)|^8 ds + \sup_{-\tau\le t\le T} E|x_\lambda^1(t)|^{16} + \varepsilon^8 \sup_{-\tau\le t\le T} E|x_\lambda^1(t)|^8) \\ &\le C(\sup_{-\tau\le t\le T} E\int_0^t |x_\lambda^2(s)|^8 ds + \varepsilon^8).\end{aligned}$$

From estimations 2, 3, and Gronwall's inequality, we derive estimation 6.
By Taylor expansion we have

$$\begin{aligned}&\int_0^t b(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s), x_\lambda(s-\tau) + x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau), \nu_\lambda^\varepsilon(s))ds \\ &+ \int_0^t \sigma(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s), x_\lambda(s-\tau) + x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau), \nu_\lambda^\varepsilon(s))dW(s) \\ &= x_\lambda + x_\lambda^1 + x_\lambda^2 - x_0(t) + \int_0^t [A_1^\varepsilon(s) + A_2^\varepsilon(s) + A_3^\varepsilon(s)]ds \\ &+ \int_0^t [B_1^\varepsilon(s) + B_2^\varepsilon(s) + B_3^\varepsilon(s)]dW(s) \\ &+ \int_0^t (b_x(\nu_\lambda^\varepsilon) - b_x(s))x_\lambda^2(s)ds + \int_0^t (b_{x'}(\nu_\lambda^\varepsilon) - b_{x'}(s))x_\lambda^2(s-\tau)ds \\ &+ \int_0^t (\sigma_x(\nu_\lambda^\varepsilon) - \sigma_x(s))x_\lambda^2(s)dW(s) + \int_0^t (\sigma_{x'}(\nu_\lambda^\varepsilon) - \sigma_{x'}(s))x_\lambda^2(s-\tau)dW(s),\end{aligned}$$

where (using (14) and (15)),

$$\begin{aligned}A_1^\varepsilon(s) =& \frac{1}{2} b_{xx}(s)((x_\lambda^2(s))^2 + 2x_\lambda^1(s)x_\lambda^2(s)) \\ &+ \int_0^1\int_0^1 \lambda\Big[b_{xx}(s, x_\lambda(s) + \lambda\mu(x_\lambda^1(s) + x_\lambda^2(s)) \\ &, x_\lambda(s-\tau) + \lambda\mu(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau)), \nu_\lambda^\varepsilon(s)) - b_{xx}(s)\Big] d\lambda d\mu (x_\lambda^1(s) + x_\lambda^2(s))^2,\end{aligned}$$

$$\begin{aligned}A_2^\varepsilon(s) =& \frac{1}{2} b_{x'x'}(s)((x_\lambda^2(s-\tau))^2 + 2x_\lambda^1(s-\tau)x_\lambda^2(s-\tau)) \\ &+ \int_0^1\int_0^1 \lambda\Big[b_{x'x'}(s, x_\lambda(s) + \lambda\mu(x_\lambda^1(s) + x_\lambda^2(s)) \\ &, x_\lambda(s-\tau) + \lambda\mu(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau)), \nu_\lambda^\varepsilon(s)) - b_{xx}(s)\Big] d\lambda d\mu (x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau))^2,\end{aligned}$$

$$\begin{aligned}A_3^\varepsilon(s) =& b_{xx'}(s)(x_\lambda^1(s)x_\lambda^2(s-\tau)+x_\lambda^2(s)x_\lambda^1(s-\tau)+x_\lambda^1(s-\tau)x_\lambda^2(s-\tau))\\&+2\int_0^1\int_0^1\lambda\Big[b_{xx'}(s,x_\lambda(s)+\lambda\mu(x_\lambda^1(s)+x_\lambda^2(s))\\&,x_\lambda(s-\tau)+\lambda\mu(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\nu_\lambda^\varepsilon(s))-b_{xx}(s)\Big]d\lambda d\mu\\&\times(x_\lambda^1(s)+x_\lambda^2(s))(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\end{aligned}$$

$$\begin{aligned}B_1^\varepsilon(s) =& \frac{1}{2}\sigma_{xx}(s)((x_\lambda^2(s))^2+2x_\lambda^1(s)x_\lambda^2(s))\\&+\int_0^1\int_0^1\lambda\Big[\sigma_{xx}(s,x_\lambda(s)+\lambda\mu(x_\lambda^1(s)+x_\lambda^2(s))\\&,x_\lambda(s-\tau)+\lambda\mu(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\nu_\lambda^\varepsilon(s))-\sigma_{xx}(s)\Big]d\lambda d\mu(x_\lambda^1(s)+x_\lambda^2(s))^2,\end{aligned}$$

$$\begin{aligned}B_2^\varepsilon(s) =& \frac{1}{2}\sigma_{x'x'}(s)((x_\lambda^2(s-\tau))^2+2x_\lambda^1(s-\tau)x_\lambda^2(s-\tau))\\&+\int_0^1\int_0^1\lambda\Big[\sigma_{x'x'}(s,x_\lambda(s)+\lambda\mu(x_\lambda^1(s)+x_\lambda^2(s))\\&,x_\lambda(s-\tau)+\lambda\mu(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\nu_\lambda^\varepsilon(s))-\sigma_{xx}(s)\Big]d\lambda d\mu(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau))^2,\end{aligned}$$

$$\begin{aligned}B_3^\varepsilon(s) =& \sigma_{xx'}(s)(x_\lambda^1(s)x_\lambda^2(s-\tau)+x_\lambda^2(s)x_\lambda^1(s-\tau)+x_\lambda^1(s-\tau)x_\lambda^2(s-\tau))\\&+2\int_0^1\int_0^1\lambda\Big[\sigma_{xx'}(s,x_\lambda(s)+\lambda\mu(x_\lambda^1(s)+x_\lambda^2(s))\\&,x_\lambda(s-\tau)+\lambda\mu(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\nu_\lambda^\varepsilon(s))-\sigma_{xx}(s)\Big]d\lambda d\mu\\&\times(x_\lambda^1(s)+x_\lambda^2(s))(x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\end{aligned}$$

which using estimation 3, we can verify

$$\begin{aligned}&\sup_{-\tau\le t\le T}E\left(\left|\int_0^t A_1^\varepsilon(s)ds\right|^4\right)+\sup_{-\tau\le t\le T}E\left(\left|\int_0^t A_2^\varepsilon(s)dW(s)\right|^4\right)\\&+\sup_{-\tau\le t\le T}E\left(\left|\int_0^t A_3^\varepsilon(s)dY(s)\right|^4\right)+\sup_{-\tau\le t\le T}E\left(\left|\int_0^t B_1^\varepsilon(s)ds\right|^4\right)\\&+\sup_{-\tau\le t\le T}E\left(\left|\int_0^t B_2^\varepsilon(s)dW(s)\right|^4\right)+\sup_{-\tau\le t\le T}E\left(\left|\int_0^t B_3^\varepsilon(s)dY(s)\right|^4\right)=o(\varepsilon^4).\end{aligned}$$

Thus we can derive

$$\begin{aligned}x_\lambda^\varepsilon(t)-x_\lambda(t)-x_\lambda^1(t)-x_\lambda^2(t) =& \int_0^t\Big[b(s,x_\lambda^\varepsilon(s),x_\lambda^\varepsilon(s-\tau),\nu_\lambda^\varepsilon(s))\\&-\Big(b(s,x_\lambda(s)+x_\lambda^1(s)+x_\lambda^2,x_\lambda(s-\tau)+x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau)),\nu_\lambda^\varepsilon)\Big)\Big]dt\\&+\int_0^t\Big[\sigma(s,x_\lambda^\varepsilon(s),x_\lambda^\varepsilon(s-\tau),\nu_\lambda^\varepsilon(s))\\&-\sigma(s,x_\lambda(s)+x_\lambda^1(s)+x_\lambda^2(s),x_\lambda(s-\tau)+x_\lambda^1(s-\tau)+x_\lambda^2(s-\tau),\nu_\lambda^\varepsilon(s))\Big]dW(s)\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t [A_1^\varepsilon(s) + A_2^\varepsilon(s) + A_3^\varepsilon(s)]ds - \int_0^t [B_1^\varepsilon(s) + B_2^\varepsilon(s) + B_3^\varepsilon(s)]dW(s)\\
&+ \int_0^t (b_x(\nu_\lambda^\varepsilon) - b_x(s))x_\lambda^2(s)ds + \int_0^t (b_{x'}(\nu_\lambda^\varepsilon) - b_{x'}(s))x_\lambda^2(s-\tau)ds\\
&+ \int_0^t (\sigma_x(\nu_\lambda^\varepsilon) - \sigma_x(s))x_\lambda^2(s)dW(s) + \int_0^t (\sigma_{x'}(\nu_\lambda^\varepsilon) - \sigma_{x'}(s))x_\lambda^2(s-\tau)dW(s).
\end{aligned}
$$

Thus, the estimation 8 can be obtained from Itô's formula and Gronwall's inequality. The proof is completed.
Proof of Lemma 3.3. Since the proof of the estimations 1, 2 are similar to Lemma 3.3 in [18], we omit it for simplicity.
Because the higher-order moments imply lower-order ones, thus we only prove estimations 2, 4 and 5.
From hypothesis 1 and Doob's martingale inequality, we get

$$
\begin{aligned}
\sup_{0\le t\le T} E|Z_\lambda^2(t)|^4 \le{}& C\Bigg[\sup_{0\le t\le T} E\int_0^t |Z_\lambda^2(s)|^4 ds + \sup_{0\le t\le T} E\int_0^t |Z_\lambda(s)x_\lambda^2(s)|^4 ds\\
&+ \sup_{0\le t\le T} E\int_0^t |Z_\lambda(s)x_\lambda^2(s-\tau)|^4 ds + \sup_{0\le t\le T} E\int_0^t |Z_\lambda(s)(x_\lambda^1(s))^2|^4 ds\\
&+ \sup_{0\le t\le T} E\int_0^t |Z_\lambda(s)(x_\lambda^1(s-\tau))^2|^4 ds + \sup_{0\le t\le T} E\int_0^t |Z_\lambda(s)(x_\lambda^1(s)x_\lambda^1(s-\tau))|^4 ds\\
&+\varepsilon E\int_d^{d+\varepsilon} |Z_\lambda(s)(h_x(s,x(s),x(s-\tau),\nu(s)) - h_x(s))x_\lambda^1(s)|^4 ds\\
&+ \varepsilon E\int_d^{d+\varepsilon} |Z_\lambda(s)(h_{x'}(s,x(s),x(s-\tau),\nu(s)) - h_{x'}(s))x_\lambda^1(s-\tau)|^4\\
&+\varepsilon E\int_d^{d+\varepsilon} |Z_\lambda^1(s)(h(s,x(s),x(s-\tau),\nu(s)) - h(s))|^4 ds\Bigg]\\
\le{}& C\Bigg(\sqrt{E\int_0^t |x_\lambda^2(s)|^8} + \sup_{0\le t\le T} E\int_0^t |Z_\lambda^2(s)|^4 ds\\
&+ \sqrt{E\int_0^t |x_\lambda^1(s)|^{16}} + \varepsilon E\int_d^{d+\varepsilon} |Z_\lambda(s)x_\lambda^1(s)|^4 ds + \varepsilon E\int_d^{d+\varepsilon} |Z_\lambda^1(s)|^4 ds\Bigg)\\
\le{}& C\Bigg(\sqrt{T\sup_{0\le t\le T} E|x_\lambda^2(t)|^8 ds} + \sup_{0\le t\le T} E\int_0^t |Z_\lambda^2(s)|^4 ds\\
&+ \sqrt{T\sup_{0\le t\le T} E|x_\lambda^1(t)|^{16} ds} + \varepsilon^2\sqrt{\sup_{0\le t\le T} E|x_\lambda^1(t)|^8} + \varepsilon^2 \sup_{0\le t\le T} E|Z_\lambda^1(t)|^4 ds\Bigg).
\end{aligned}
$$

Thus, with Gronwall's inequality and by Lemma 3.1 and estimation 2, the proof of estimation 4 is completed.
We have

$$
\begin{aligned}
&\int_0^t (Z_\lambda^1(s) + Z_\lambda^2(s))h(s)dY(s) + \int_0^t Z_\lambda^1(s)\Delta h(s)ds\\
&+ \int_0^t Z_\lambda(s)h(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s), x_\lambda(s-\tau) + x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau), \nu_\lambda^\varepsilon(s))dY(s)\\
&= Z_\lambda(t) + Z_\lambda^1(t) + Z_\lambda^2(t) - 1 + \int_0^t Z_\lambda(s)\Lambda^\varepsilon(s)dY(s),
\end{aligned}
$$

where

$$\begin{aligned}
\Lambda^{\varepsilon}(s) &= \frac{1}{2}h_{xx}(s)((x_{\lambda}^{2}(s))^{2} + 2x_{\lambda}^{1}(s)x_{\lambda}^{2}(s)) + \Delta h_{x}(s)x_{\lambda}^{2}(s) \\
&+ \frac{1}{2}h_{x'x'}(s)((x_{\lambda}^{2}(s-\tau))^{2} + 2x_{\lambda}^{1}(s-\tau)x_{\lambda}^{2}(s-\tau)) + \Delta h_{x'}(s)x_{\lambda}^{2}(s-\tau) \\
&+ h_{xx'}(s)((x_{\lambda}^{1}(s-\tau)x_{\lambda}^{2}(s-\tau) + x_{\lambda}^{1}(s)x_{\lambda}^{2}(s-\tau) + x_{\lambda}^{1}(s-\tau)x_{\lambda}^{2}(s)) \\
&+ \int_{0}^{1}\int_{0}^{1} \lambda h_{xx}(s, x_{\lambda}(s) + \lambda\mu(x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s)), x_{\lambda}(s-\tau) + \lambda\mu(x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau)), \nu_{\lambda}^{\varepsilon}(s)) \\
&\times d\lambda d\mu(x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s))^{2} \\
&+ \int_{0}^{1}\int_{0}^{1} \lambda h_{x'x'}(s, x_{\lambda}(s) + \lambda\mu(x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s)), x_{\lambda}(s-\tau) + \lambda\mu(x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau)), \nu_{\lambda}^{\varepsilon}(s)) \\
&\times d\lambda d\mu(x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau))^{2} \\
&+ 2\int_{0}^{1}\int_{0}^{1} \lambda h_{xx'}(s, x_{\lambda}(s) + \lambda\mu(x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s)), x_{\lambda}(s-\tau) + \lambda\mu(x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau)), \nu_{\lambda}^{\varepsilon}(s)) \\
&\times d\lambda d\mu(x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s))(x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau))
\end{aligned}$$

From hypothesis 1 and Lemma 3.1, we can verify

$$\sup_{0\le t\le T} E\left(\int_{0}^{t} Z_{\lambda}(s)\Lambda^{\varepsilon}(s)dY(s)\right)^{2} = o(\varepsilon^{2}).$$

We have

$$\begin{aligned}
Z_{\lambda}^{\varepsilon}(t) &- Z_{\lambda}(t) - Z_{\lambda}^{1}(t) - Z_{\lambda}^{2}(t) \\
&= \int_{0}^{t} Z_{\lambda}^{\varepsilon}(s)h(s, x_{\lambda}^{\varepsilon}(s), x_{\lambda}^{\varepsilon}(s-\tau), \nu_{\lambda}^{\varepsilon}(s)) - \int_{0}^{t}(Z_{\lambda}^{1}(s) + Z_{\lambda}^{2}(s))h(s)dY(s) \\
&- \int_{0}^{t} Z_{\lambda}^{1}(s)\Delta h(s)ds + \int_{0}^{t} Z_{\lambda}(s)\Lambda^{\varepsilon}(s)dY(s) \\
&- \int_{0}^{t} Z_{\lambda}(s)h(s, x_{\lambda}(s) + x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s), x_{\lambda}(s-\tau) + x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau), \nu_{\lambda}^{\varepsilon}(s))dY(s) \\
&= \int_{0}^{t}(Z_{\lambda}^{\varepsilon}(s) - Z_{\lambda}(s) - Z_{\lambda}^{1}(s) - Z_{\lambda}^{2}(s))h(s, x_{\lambda}^{\varepsilon}(s), x_{\lambda}^{\varepsilon}(s-\tau), \nu_{\lambda}^{\varepsilon}(s))dY(s) \\
&+ \int_{0}^{t}(Z_{\lambda}(s) + Z_{\lambda}^{1}(s) + Z_{\lambda}^{2}(s))\Big[h(s, x_{\lambda}^{\varepsilon}(s), x_{\lambda}^{\varepsilon}(s-\tau), \nu_{\lambda}^{\varepsilon}(s)) \\
&- h(s, x_{\lambda}(s) + x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s), x_{\lambda}(s-\tau) + x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau), \nu_{\lambda}^{\varepsilon}(s))\Big]dY(s) \\
&+ \int_{0}^{t}(Z_{\lambda}^{1}(s) + Z_{\lambda}^{2}(s))\Big[h(s, x_{\lambda}(s) + x_{\lambda}^{1}(s) + x_{\lambda}^{2}(s), x_{\lambda}(s-\tau) + x_{\lambda}^{1}(s-\tau) + x_{\lambda}^{2}(s-\tau), \nu_{\lambda}^{\varepsilon}(s)) \\
&- h(\nu_{\lambda}^{\varepsilon}(s))\Big]dY(s) + \int_{0}^{t}(Z_{\lambda}^{2}(s))\Delta h(s)dY(s) + \int_{0}^{t} Z_{\lambda}(s)\Lambda^{\varepsilon}(s)dY(s) \\
&= \int_{0}^{t}(Z_{\lambda}^{\varepsilon}(s) - Z_{\lambda}(s) - Z_{\lambda}^{1}(s) - Z_{\lambda}^{2}(s))h(s, x_{\lambda}^{\varepsilon}(s), x_{\lambda}^{\varepsilon}(s-\tau), \nu_{\lambda}^{\varepsilon}(s))dY(s) \\
&+ \int_{0}^{t}((Z_{\lambda}(s) + Z_{\lambda}^{1}(s) + Z_{\lambda}^{2}(s))\Xi^{\varepsilon}(s)dY(s) + \int_{0}^{t}(Z_{\lambda}^{1}(s) + Z_{\lambda}^{2}(s))\Gamma^{\varepsilon}(s)dY(s) \\
&+ \int_{0}^{t}(Z_{\lambda}^{2}(s))\Delta h(s)dY(s) + \int_{0}^{t} Z_{\lambda}(s)\Lambda^{\varepsilon}(s)dY(s),
\end{aligned}$$

where

$$
\begin{aligned}
\Xi^{\varepsilon}(s) = &\int_0^1 h_x(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s) + \lambda(x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)) \\
&, x_\lambda(s-\tau) + x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau) + \lambda(x_\lambda^{\varepsilon}(s-\tau) - x_\lambda(s-\tau) \\
&- x_\lambda^1(s-\tau) - x_\lambda^2(s-\tau)), \nu_\lambda^{\varepsilon}(s)) d\lambda (x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)) \\
&+ \int_0^1 h_{x'}(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s) + \lambda(x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)) \\
&, x_\lambda(s-\tau) + x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau) + \lambda(x_\lambda^{\varepsilon}(s-\tau) - x_\lambda(s-\tau) \\
&- x_\lambda^1(s-\tau) - x_\lambda^2(s-\tau)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda \times (x_\lambda^{\varepsilon}(s-\tau) - x_\lambda(s-\tau) - x_\lambda^1(s-\tau) - x_\lambda^2(s-\tau)) \\
&+ \int_0^1 \int_0^1 \lambda h_{xx}(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s) + \lambda\mu(x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s))^2 \\
&+ \int_0^1 \int_0^1 \lambda h_{x'x'}(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s) + \lambda\mu(x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^{\varepsilon}(s-\tau) - x_\lambda(s-\tau) - x_\lambda^1(s-\tau) - x_\lambda^2(s-\tau))^2 \\
&+ 2\int_0^1 \int_0^1 \lambda h_{xx'}(s, x_\lambda(s) + x_\lambda^1(s) + x_\lambda^2(s) + \lambda\mu(x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^{\varepsilon}(s) - x_\lambda(s) - x_\lambda^1(s) - x_\lambda^2(s)) \\
&(x_\lambda^{\varepsilon}(s-\tau) - x_\lambda(s-\tau) - x_\lambda^1(s-\tau) - x_\lambda^2(s-\tau)), \\
\Gamma^{\varepsilon}(s) = & h_x(s)(x_\lambda^1(s) + x_\lambda^2(s)) + \int_0^1 \int_0^1 \lambda \\
&\times h_{xx}(s, x_\lambda(s) + \lambda\mu(x_\lambda^1(s) + x_\lambda^2(s)), x_\lambda(s-\tau) + \lambda\mu(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^1(s) + x_\lambda^2(s))^2 + \int_0^1 \int_0^1 \lambda \\
&\times h_{x'x'}(s, x_\lambda(s) + \lambda\mu(x_\lambda^1(s) + x_\lambda^2(s)), x_\lambda(s-\tau) + \lambda\mu(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau))^2 + \int_0^1 \int_0^1 2\lambda \\
&\times h_{xx'}(s, x_\lambda(s) + \lambda\mu(x_\lambda^1(s) + x_\lambda^2(s)), x_\lambda(s-\tau) + \lambda\mu(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau)), \nu_\lambda^{\varepsilon}(s)) \\
&d\lambda d\mu \times (x_\lambda^1(s) + x_\lambda^2(s))(x_\lambda^1(s-\tau) + x_\lambda^2(s-\tau))
\end{aligned}
$$

From this relation and by hypothesis 1 and Itô's isometry, we have

$$
\begin{aligned}
E|Z_\lambda^{\varepsilon}(t) - Z_\lambda(t) - Z_\lambda^1(t) - Z_\lambda^2(t)|^2 \leq C\Bigg[& E\int_0^t |Z_\lambda^{\varepsilon}(s) - Z_\lambda(s) - Z_\lambda^1(s) - Z_\lambda^2(s)|^2 ds \\
&+ E\int_0^t (|Z_\lambda(t)|^2 + |Z_\lambda^1(s)|^2 + |Z_\lambda^2(s)|^2)|\Xi^{\varepsilon}(s)|^2 ds + E\int_0^t (|Z_\lambda^1(s)|^2 + |Z_\lambda^2(s)|^2)(|\Gamma^{\varepsilon}(s)|^2) ds \\
&+ \varepsilon E\int_0^t |Z_\lambda^2(s)|^2 ds + \sup_{0\leq t\leq T} E\left(\int_0^t Z_\lambda(s)\Lambda^{\varepsilon}(s) dY(s)\right)^2 \Bigg].
\end{aligned}
$$

Using Gronwall's inequality and Lemma 3.1, we can obtain the estimation 5.

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