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A note on an integral operator induced by Zygmund function

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Abstract. In this note, by means of a kernel function induced by a continuous function f on the unit circle, we show that corresponding integral operator on Banach space $\mathcal{A}^p$ is bounded or compact precisely when f belongs to the big Zygmund class Λ_* or the little Zygmund class λ_*, where $\mathcal{A}^p$ consists of all holomorphic functions ϕ on $\overline{\mathbb{C}}\backslash S^1$ with the finite corresponding norm. This generalizes the result in Hu, Song, Wei and Shen (2013) [5] and meanwhile may be considered as the infinitesimal version of main result obtained in Tang and Wu (2019) [8].

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane, $\overline{\mathbb{C}} = \{\mathbb{C}\} \cup \{\infty\}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $\Delta^* = \overline{\mathbb{C}}\backslash\overline{\Delta}$ and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Set f be a continuous function on the unit circle S^1. Y.Hu, J.R.Song, H.Y.Wei and Y.L.Shen [5] introduced a integral operator T_f induced by kernel function

$$K_f(\zeta, z) = (\chi_\Delta(\zeta)\chi_\Delta(z) - \chi_{\Delta^*}(\zeta)\chi_{\Delta^*}(z))\phi_f(\zeta, z), \tag{1.1}$$

for $(\zeta, z) \in (\Delta \cup \Delta^*) \times (\Delta \cup \Delta^*)$, where χ stands for the characteristic function of a set and

$$\phi_f(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{f(w)}{(1-\zeta w)^2(1-zw)^2}dw, \tag{1.2}$$

for $(\zeta, z) \in (\Delta \cup \Delta) \times (\Delta^* \cup \Delta^*)$. In particular, when $\zeta = z$, we *have*

$$\phi_f(z) = \phi_f(z, z) = \frac{1}{2\pi i}\int_{S^1} \frac{f(w)}{(1-zw)^4}dw. \tag{1.3}$$

The integral operator T_f is defined as for any holomorphic function ψ in $\Delta \cup \Delta^*$,

$$T_f\psi(\zeta) = \frac{1}{\pi}\iint_{\Delta\cup\Delta^*} K_f(\zeta, z)\psi(\overline{z})dxdy. \tag{1.4}$$

That is to say, when $\zeta \in \Delta$,

$$T_f\psi(\zeta) = \frac{1}{\pi}\iint_{\Delta} K_f(\zeta, z)\psi(\overline{z})dxdy, \tag{1.5}$$

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while $\zeta \in \Delta^*$,

$$T_f\psi(\zeta) = -\frac{1}{\pi}\iint_{\Delta^*} K_f(\zeta, z)\psi(\overline{z})dxdy. \tag{1.6}$$

Let $p > 1$. The Banach space $\mathcal{A}^p$ consists of all holomorphic functions ϕ in the unit disk Δ with finite norm

$$\|\phi\|^p_{\mathcal{A}^p} = \frac{1}{\pi}\iint_{\Delta\cup\Delta^*} |\phi(z)|^p|1-|z|^2|^{p-2}dxdy < \infty. \tag{1.7}$$

When $p = 2$, $\mathcal{A}^2$ is a Hilbert space with inner product and norm

$$<\phi, \psi>= \frac{1}{\pi}\iint_{\Delta\cup\Delta^*} \phi(z)\overline{\psi(z)}dxdy, \qquad \|\phi\| =<\phi, \phi>^{\frac{1}{2}}. \tag{1.8}$$

Next, we recall the definitions of the two class Λ_*, λ_* and give some background on the paper.

A continuous function f on S^1 is said to belong to the big Zygmund class Λ_* if there exists some constant $M > 0$ such that

$$|f(e^{i(\theta+t)}) - 2f(e^{i\theta}) + f(e^{i(\theta-t)})| \leq Mt \tag{1.9}$$

for all $\theta \in [0, 2\pi)$ and $t \in (0, \pi)$. A Zygmund function f is said to belong to the little Zygmund class λ_* if

$$\lim_{t\to 0+} \frac{f(e^{i(\theta+t)}) - 2f(e^{i\theta}) + f(e^{i(\theta-t)})}{t} = 0 \tag{1.10}$$

uniformly for $\theta \in [0, 2\pi)$. Zygmund [10] firstly introduced these functions. Zygmund functions have been found many applications in the study of Teichmüller space (see [2], [3], [7]). On the other hand, some researchers want to give some new characterizations for Λ_* and λ_* in view of methods in the Teichmüller space. In 2012, Y.Hu and Y.L.Shen [4] introduced a kernel function ϕ_h induced by a quasisymmetric homeomorphism h and discussed the compactness of the corresponding operator T_h^- on $\mathcal{A}^2$. Meanwhile, Y.Hu, J.R.Song, H.Y. Wei and Y.L.Shen [5] gave the infinitesimal version of the main result in [4]. Subsequently, on the base of [4], S.A.Tang and P.C.Wu [8] studied the compactness of the corresponding operator T_h^- on $\mathcal{A}^p$. In this paper, we show the infinitesimal version of the main result in [8]. Indeed, we will prove the following theorems.

Theorem 1.1. *Let f be a continuous function on the unit circle and $p > 1$. Then the following statements are equivalent:*

(i) $f \in \Lambda_*$;

(ii) The integral operator $T_f : \mathcal{A}^p \to \mathcal{A}^p$ is bounded;

(iii)For each fixed $z \in \Delta \cup \Delta^*$, $K_f(\cdot, z) \in \mathcal{A}^p$, and $\|K_f(\cdot, z)\|_{\mathcal{A}^p} = O(|1-|z|^2|^{-1})$;

(iv) $\phi_f(z) = O((1-|z|^2)^{-2})$.

Theorem 1.2. *Let f be a continuous function on the unit circle and $p > 1$. Then the following statements are equivalent:*

(i) $f \in \lambda_*$;

(ii) The integral operator $T_f : \mathcal{A}^p \to \mathcal{A}^p$ is compact;

(iii)For each fixed $z \in \Delta \cup \Delta^*$, $K_f(\cdot, z) \in \mathcal{A}^p$, and $\|K_f(\cdot, z)\|_{\mathcal{A}^p} = o(|1-|z|^2|^{-1})$;

(iv) $\phi_f(z) = o((1-|z|^2)^{-2})$.

When $p = 2$, Thorem 1.1 or Theorem 1.2 coincides with Theorem 3.4 or Theorem 3.5 in [5], respectively.

We end this introduction section with the organization of the paper. In section 2, we give some lemmas which will be used to prove main theorems. In section 3, we will give a proof of Theorem 1.1. The last section we will prove Theorem 1.2.

2. Some lemmas

According to Ahlfors [1], a complex-valued function F defined in a domain Ω is called a quasiconformal deformation if it has the generalized derivative $\bar{\partial}F$ such that $\bar{\partial}F \in L^{\infty}(\Omega)$. The following lemma on the quasiconformal deformation will be used later.

Lemma 2.1. *(see Theorem 2.1 in [5] or [4]) Let f be a continuous function on the unit circle. Then $f \in \Lambda_*$ if and only if f can be extended to a quasiconformal deformation $\widetilde{f}$ of the whole plane $\mathbb{C}$ so that $\widetilde{f} = O(z^2)$ as $z \to \infty$; Furthermore, $f \in \lambda_*$ if and only if the quasiconformal deformation extension $\widetilde{f}$ can be so chosen that $\bar{\partial}\widetilde{f}(z) \to 0$ as $|z| \to 1$.*

The following lemma gives the relationship between $\phi_f(z)$ defined in (1.3) and the quasiconformal deformation.

Lemma 2.2. *(see Proposition 2.3 in [5]) Let f be a continuous function on the unit circle S^1. Then the following statements holds:*

(i) $\phi_f(z) = O((1-|z|^2)^{-2})$ if and only if f can be extended to a quasiconformal deformation $\widetilde{f}$ of the whole plane $\mathbb{C}$ so that $\widetilde{f}(z) = O(z^2)$ as $z \to \infty$;

(ii) $\phi_f(z) = o((1-|z|^2)^{-2})$ as $|z| \to 1$ if and only if f can be extended to a quasiconformal deformation $\widetilde{f}$ of the whole plane $\mathbb{C}$ so that $\bar{\partial}\widetilde{f}(z) \to 0$ as $z \to 1$.

Furthermore, in each case,

$$E(f)(z) = \frac{|1-|z|^2|^3}{2\pi i}\int_{S^1}\frac{f(w)}{(1-\bar{z}w)^3(w-z)}dw, \qquad z \in \Delta \cup \Delta^*, \tag{2.1}$$

provides the required extension.

To prove the boundedness and compactness of the integral operator T_f, we need to introduce the following lemma.

Lemma 2.3. *(see Theorem 3.6 of [9]) Suppose that (X, μ) is a measure space and $K(x, y)$ is a nonnegative measurable function on $X \times X$, K is the integral operator with kernel $K(x, y)$, that is*

$$K_{\psi}(x) = \iint_X K(x, y)\psi(y)d\mu(y).$$

Let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exist positive constants C_1 and C_2 and a positive measurable function h on X such that

$$\iint_X K(x, y)h^q(y)d\mu(y) \le C_1 h^q(x), \tag{2.2}$$

for almost every $x \in X$ and

$$\iint_X K(x, y)h^p(x)d\mu(x) \le C_2 h^p(y), \tag{2.3}$$

for almost every $y \in X$, then K is a bounded operator on $L^p(X, d\mu)$ with norm less then or equal to $C_1^{\frac{1}{q}}C_2^{\frac{1}{p}}$.

We also need the following integral estimates (see [9]).

Lemma 2.4. *([9]) Suppose that $z \in \Delta$, $s > 0$ and $t > -1$. Then there exists constant $C > 0$ so that*

$$\frac{1}{C}\frac{1}{(1-|z|^2)^s} \le \iint_{\Delta}\frac{(1-|w|^2)^t}{|1-z\overline{w}|^{2+t+s}}dudv \le C\frac{1}{(1-|z|^2)^s}. \tag{2.4}$$

Remark 2.5. *When $z \in \Delta^*$,we can obtain*

$$\frac{1}{C}\frac{1}{(|z|^2-1)^s} \le \iint_{\Delta^*} \frac{(|w|^2-1)^t}{|1-z\overline{w}|^{2+t+s}} dudv \le C\frac{1}{(|z|^2-1)^s}. \tag{2.5}$$

We also consider a special subset of $\mathcal{A}^p$. For fixed $z \in \Delta \cup \Delta^*$, we set $\psi_z \in \mathcal{A}^p$ by

$$\psi_z(\zeta) = \frac{1-|z|^2}{(1-z\zeta)^2}\chi_\Delta(\zeta), \qquad z \in \Delta, \tag{2.6}$$

and

$$\psi_z(\zeta) = \frac{1-|z|^2}{(1-z\zeta)^2}\chi_{\Delta^*}(\zeta), \qquad z \in \Delta^*. \tag{2.7}$$

It follows from Lemma 2.4 and Remark 1 that $\psi_z \in \mathcal{A}^p$.

To prove our main theorems, we need the following two lemmas in [5].

Lemma 2.6. *(see Lemma 3.2 in [5]) Let f be continuous function on the unit circle. Then it holds that*

$$T_f\psi_z(\zeta) = (1-|z|^2)K_f(\zeta, z), \qquad (\zeta, z) \in (\Delta \cup \Delta^*) \times (\Delta \cup \Delta^*). \tag{2.8}$$

Lemma 2.7. *(see Lemma 3.3 in [5]) Let f be continuous function on the unit circle. If for each fixed $z \in \Delta \cup \Delta^*$, $K_f(\cdot, z) \in \mathcal{A}^2$, then for all $z \in \Delta \cup \Delta^*$, it holds that*

$$\langle K_f(\cdot, z), \psi_{\overline{z}}\rangle = |1-|z|^2|\phi_f(z). \tag{2.9}$$

3. Proof of Theorem 1.1

Proof. "(i)⇒(ii)" It is known that the following two equations ((3.1) and (3.2)) hold (see [5]). when $(\zeta, z) \in \Delta \times \Delta$,

$$\phi_f(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{f(w)}{(1-\zeta w)^2(1-zw)^2}dw = \frac{1}{\pi}\iint_\Delta \frac{\overline{\partial}F(w)}{(1-\zeta w)^2(1-zw)^2}dudv, \tag{3.1}$$

when $(\zeta, z) \in \Delta^* \times \Delta^*$,

$$\phi_f(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{f(w)}{(1-\zeta w)^2(1-zw)^2}dw = -\frac{1}{\pi}\iint_{\Delta^*} \frac{\overline{\partial}F(w)}{(1-\zeta w)^2(1-zw)^2}dudv, \tag{3.2}$$

where $F(w)$ is the quasiconformal deformation of f. Meanwhile, for any holomorphic function $\psi \in \mathcal{A}^p$, by a result of [5], we know that

$$\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \frac{\psi(\overline{z})}{(1-z\zeta)^2}dxdy, \quad \zeta \in \Delta, \tag{3.3}$$

and

$$\psi(\zeta) = \frac{1}{\pi}\iint_{\Delta^*} \frac{\psi(\overline{z})}{(1-z\zeta)^2}dxdy, \quad \zeta \in \Delta^*. \tag{3.4}$$

Following from (3.1)-(3.4), Y.Hu, J.R.Song, H.Y. Wei and Y.L.Shen [5] got that

$$T_f\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv, \quad \texttt{for} \quad \zeta \in \Delta, \tag{3.5}$$

and

$$T_f\psi(\zeta) = \frac{1}{\pi}\iint_{\Delta^*} \frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv, \quad \texttt{for} \quad \zeta \in \Delta^*. \tag{3.6}$$

Then one side is for $\zeta \in \Delta$,

$$\begin{aligned}
&\frac{1}{\pi}\iint_{\Delta}|T_f\psi(\zeta)|^p(1-|\zeta|^2)^{p-2}d\xi d\eta\\
=&\frac{1}{\pi}\iint_{\Delta}\left|\frac{1}{\pi}\iint_{\Delta}\frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta\\
\leq&\frac{1}{\pi^{p+1}}\iint_{\Delta}\left|\iint_{\Delta}\frac{|\overline{\partial}F(w)\psi(w)|}{|1-\zeta w|^2}dudv\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta\\
=&\frac{1}{\pi^{p+1}}\iint_{\Delta}\left|\iint_{\Delta}\frac{(1-|w|^2)^{2-p}|\overline{\partial}F(w)\psi(w)|}{|1-\zeta w|^2}(1-|w|^2)^{p-2}dudv\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta.
\end{aligned}$$

Let $d\mu(w)=(1-|w|^2)^{p-2}dudv$ and $K(\zeta,w)=\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}$, $(\zeta,w)\in\Delta\times\Delta$. Consider the function $h(w)=(1-|w|^2)^{\alpha}$, where $-1+\frac{1}{p}<\alpha<min\{0,-1+\frac{2}{p}\}$. By Lemma 2.4, there exists a constant $C_3>0$ such that

$$\begin{aligned}
&\iint_{\Delta}\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}h^q(w)d\mu(w)\\
=&\iint_{\Delta}\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}(1-|w|^2)^{\alpha q}(1-|w|^2)^{p-2}dudv\\
=&\iint_{\Delta}\frac{(1-|w|^2)^{\alpha q}}{|1-\zeta w|^2}dudv\\
\leq&\ C_3(1-|\zeta|^2)^{\alpha q}=C_3h^q(\zeta).
\end{aligned}\tag{3.8}$$

Similarly, there exists a constant $C_4>0$ such that

$$\begin{aligned}
&\iint_{\Delta}\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}h^p(\zeta)d\mu(\zeta)\\
=&\iint_{\Delta}\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}(1-|\zeta|^2)^{\alpha p}(1-|\zeta|^2)^{p-2}d\xi d\eta\\
=&\iint_{\Delta}\frac{(1-|w|^2)^{2-p}}{|1-\zeta w|^2}(1-|\zeta|^2)^{\alpha p+p-2}d\xi d\eta\\
=&\ (1-|w|^2)^{2-p}\iint_{\Delta}\frac{(1-|\zeta|^2)^{\alpha p+p-2}}{|1-\zeta w|^2}d\xi d\eta\\
\leq&\ C_4(1-|w|^2)^{\alpha p}\\
=&\ C_4h^p(w).
\end{aligned}\tag{3.9}$$

Combining (3.8) and (3.9), by Lemma 2.3 we deduce that

$$K_{\psi}(\zeta)=\iint_{\Delta}K(\zeta,w)\psi(w)d\mu(w)$$

is bounded on $L^p(\Delta, d\mu)$. Thus there exists a constant C_5 such that

$$\begin{aligned}
&\frac{1}{\pi}\iint_{\Delta} |T_f\psi(\zeta)|^p(1-|\zeta|^2)^{p-2}d\xi d\eta \\
=\ & \frac{1}{\pi}\iint_{\Delta}\left|\frac{1}{\pi}\iint_{\Delta}\frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta \\
\leq\ & \frac{C_5}{\pi^{p+1}}\iint_{\Delta}\left|\overline{\partial}F(w)\psi(w)\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta \\
\leq\ & \frac{C_5}{\pi^{p+1}}\parallel\overline{\partial}F\parallel_{\infty}^p\iint_{\Delta}|\psi(\zeta)|^p(1-|\zeta|^2)^{p-2}d\xi d\eta \\
\leq\ & \frac{C_5}{\pi^{p+1}}\parallel\overline{\partial}F\parallel_{\infty}^p\parallel\psi\parallel_{\mathcal{A}^p}^p.
\end{aligned} \tag{3.10}$$

And the other side is for $\zeta \in \Delta^*$. By (3.6) we get

$$\begin{aligned}
T_f\psi(\zeta) &= \frac{1}{\pi}\iint_{\Delta^*}\frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv \\
&= \frac{1}{\pi}\iint_{\Delta}\frac{\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})}{(1-\frac{\zeta}{w})^2}\frac{1}{|w|^2}dudv,
\end{aligned}$$

then

$$\begin{aligned}
\left|T_f\psi(\zeta)\right|^p &= \frac{1}{\pi^p}\left|\iint_{\Delta^*}\frac{\overline{\partial}F(w)\psi(w)}{(1-\zeta w)^2}dudv\right|^p \\
&\leq \frac{1}{\pi^p}\left|\iint_{\Delta}\frac{|\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})|}{|1-\frac{\zeta}{w}|^2}\frac{1}{|w|^2}dudv\right|^p \\
&= \frac{1}{\pi^p}\left|\iint_{\Delta}\frac{|\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})|}{|w-\zeta|^2}dudv\right|^p.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{\pi}\iint_{\Delta^*}|T_f\psi(\zeta)|^p(|\zeta|^2-1)^{p-2}d\xi d\eta \\
\leq\ & \frac{1}{\pi^{p+1}}\iint_{\Delta^*}\left|\iint_{\Delta}\frac{|\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})|}{|w-\zeta|^2}dudv\right|^p(|\zeta|^2-1)^{p-2}d\xi d\eta \\
=\ & \frac{1}{\pi^{p+1}}\iint_{\Delta}\left|\iint_{\Delta}\frac{|\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})|}{|1-w\zeta|^2}dudv\right|^p(1-|\zeta|^2)^{p-2}|\zeta|^2 d\xi d\eta \\
\leq\ & \frac{1}{\pi^{p+1}}\iint_{\Delta}\left|\iint_{\Delta}\frac{|\overline{\partial}F(\frac{1}{w})\psi(\frac{1}{w})|}{|1-w\zeta|^2}dudv\right|^p(1-|\zeta|^2)^{p-2}d\xi d\eta.
\end{aligned} \tag{3.11}$$

Similar to the proof of (3.10), it follows from (3.11) that there exists a constant $C_6 > 0$ such that

$$\begin{aligned}
&\frac{1}{\pi}\iint_{\Delta^*}|T_f\psi(\zeta)|^p(1-|\zeta|^2)^{p-2}d\xi d\eta \\
\leq\ & \frac{C_6}{\pi^{p+1}}\parallel\overline{\partial}F\parallel_{\infty}^p\parallel\psi\parallel_{\mathcal{A}^p}^p.
\end{aligned} \tag{3.12}$$

In view of (3.10) and (3.12), there exists a constant $C_7 > 0$ such that

$$\| T_f\psi \|^p_{\mathcal{A}^p} \leq C_7 \| \bar{\partial} F \|^p_{\infty} \| \psi \|^p_{\mathcal{A}^p} . \tag{3.13}$$

This shows that $T_f : \mathcal{A}^p \to \mathcal{A}^p$ is bounded.

"(ii)⇒(iii)" By Lemma 2.5, we know

$$T_f\psi_z(\zeta) = (1 - |z|^2)K_f(\zeta, z). \tag{3.14}$$

Since

$$\begin{aligned}
& \iint_{\Delta\cup\Delta^*} |T_f\psi_z(\zeta)|^p |1 - |\zeta|^2|^{p-2} d\xi d\eta \\
= & \iint_{\Delta\cup\Delta^*} |1 - |z|^2|^p K^p_f(\zeta, z) |1 - |\zeta|^2|^{p-2} d\xi d\eta \\
= & |1 - |z|^2|^p \iint_{\Delta\cup\Delta^*} K^p_f(\zeta, z) |1 - |\zeta||^{p-2} d\xi d\eta,
\end{aligned} \tag{3.15}$$

we get

$$\|T_f\psi_z\|_{\mathcal{A}^p} = |1 - |z|^2| \|K_f(\cdot, z)\|_{\mathcal{A}^p}, \tag{3.16}$$

that is to say $\|K_f(\cdot, z)\|_{\mathcal{A}^p} = O(|1 - |z|^2|^{-1})$.

"(iii)⇒(iv)" By Lemma 2.6, we know

$$\langle K_f(\cdot, z), \psi_{\bar{z}}\rangle = |1 - |z|^2|\phi_f(z),$$

namely,

$$\frac{1}{\pi} \iint_{\Delta\cup\Delta^*} K_f(\zeta, z)\overline{\psi_{\bar{z}}} d\xi d\eta = |1 - |z|^2|\phi_f(z). \tag{3.17}$$

In view of Hölder's inequality, we obtain

$$\begin{aligned}
& \left| \frac{1}{\pi} \iint_{\Delta\cup\Delta^*} K_f(\zeta, z)\overline{\psi_{\bar{z}}} d\xi d\eta \right| \\
= & \left| \frac{1}{\pi} \iint_{\Delta\cup\Delta^*} K_f(\zeta, z)\overline{\psi_{\bar{z}}} |1 - |\zeta|^2|^{\frac{p-2}{p}} |1 - |\zeta|^2|^{\frac{2-p}{p}}) d\xi d\eta \right| \\
\leq & \frac{1}{\pi} \left(\iint_{\Delta\cup\Delta^*} \left|K_f(\zeta, z)\right|^p |1 - |\zeta|^2|^{p-2} d\xi d\eta \right)^{\frac{1}{p}} \left(\iint_{\Delta\cup\Delta^*} |\psi_{\bar{z}}(\zeta)|^q |1 - |\zeta|^2|^{q-2} d\xi d\eta \right)^{\frac{1}{q}} .
\end{aligned} \tag{3.18}$$

Because $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $1 < q < \infty$. From Lemma 2.4, we can assert that $\psi_z \in \mathcal{A}^q$. Therefore, we conclude that

$$\phi_f(z) = O((1 - |z|^2)^{-2}). \tag{3.19}$$

Similarly, when $z \in \Delta^*$, by Hölder inequality and Remark 1, we obtain

$$\phi_f(z) = O((1 - |z|^2)^{-2}), z \in \Delta^*. \tag{3.20}$$

"(iv)⇒(i)" This follows from Lemma 2.1 and Lemma 2.2.

□

4. proof of Theorem 1.2

Proof. "(i)⇒(ii)" It is sufficient to show that $T_f(\psi_n) \to 0$ for each sequence $\{\psi_n\}$ converges to zero locally uniformly in $\Delta \cup \Delta^*$.

Since $f \in \lambda_*$, from Lemma 2.1, we get that for any $\epsilon > 0$, there exists some $r < 1$ such that as $r < |w| < \frac{1}{r}$, $|\bar{\partial}F(w)| < \varepsilon$. Therefore, by the (3.13) we obtain

$$\begin{aligned}
& ||T_f\psi||^p_{\mathcal{A}^p} \\
\leq\ & C_7||\bar{\partial}F||^p_\infty ||\psi||^p_{\mathcal{A}^p} \\
=\ & C_7||\bar{\partial}F||^p_\infty \iint_{\Delta\cup\Delta^*} |\psi|^p(1-|\zeta|)^{p-2}d\zeta d\eta \\
\leq\ & C_7\left[||\bar{\partial}F||^p_\infty \iint_{\{|\zeta|<r\}\cup\{|\zeta|>\frac{1}{r}\}} |\psi|^p(1-|\zeta|)^{p-2}d\zeta d\eta + \varepsilon^p||\psi||^p_{\mathcal{A}^p}\right].
\end{aligned} \tag{4.1}$$

Thus, $T_f \to 0$ for each sequence $\{\psi_n\}$ which converges to zero weakly.

Proofs of (ii)⇒(iii), (iii)⇒(iv) and (iv)⇒(i) are same as Theorem 3.5 in [5]. □

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