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Reconstruction of Green's function for multiplicative Sturm-Liouville problem

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Abstract. We construct multiplicative Green's (or *Green's) function for *Sturm-Liouville (*SL) equation. The basic properties of *Green's function are given. Then, *SL equation is evaluated by using *Green's function. Effectiveness of Green's function in *case will thus be seen by some examples.

1. Introduction

In mathematical physics, Green's function creates a highly effective method to solve initial or boundary value problems (IVB's-BVP's). Green's functions are devices to solve crucial ordinary and partial differential equations which may be unsolvable by other techniques. To obtain a solution for IVP or BVP, we need a mapping whose kernel is Green's function. This function is an impulse reaction to a nonhomogeneous linear differential operator described on a domain with customized initial, boundary conditions. It is also used in quantum field theory, aerodynamics, electrodynamics, aeroacoustics, seismology. There are many studies in literature on Green's function in classical case [1, 2, 4, 10, 12, 21, 23, 24, 27]. It is important to generate Green's function for such problems. However, it was seen that Green's function was not defined and its properties were not given in *calculus. Before examining the Green's function and SL problem in *case, let's express some basic concepts and theorems for *calculus. First, let's express development process of *calculus.

*calculus was introduced by *Grossman and Katz* [16], [17] in 1967 as an alternative to Newtonian calculus. This type of calculus is also known as non-Newtonian because of its difference from classical calculus of Newton and Leibniz. *calculus is a beneficial supplement to classical calculus in that it is adapted to situations including exponential functions in same case that usual calculus is adapted to situations including linear functions. *calculus moves roles of substraction, addition to division, multiplication. There are actually many reasons to study *calculus. It improves the work of additive calculations indirectly. Problems that are difficult to solve in classical case can be solved with incredible ease in here. Every property in Newtonian case can be defined in *calculus within certain rules. The importance of this theory should be emphasized in terms of applications in physics and engineering.

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Many events in nature change exponentially. For example: populations of countries, magnitude of an earthquake [8] are events that behave in this manner. For this reason, using *case instead of classical calculus allows a better physical evaluation of these type events. It also gives better results than usual case in many fields such as finance, economics, biology and demography. A very limited number of studies have been conducted on this calculus until the beginning of the 2000s. Recently, various studies have been carried out on it and quality and effective results have been obtained (see [6, 7, 9, 11, 13, 18, 25, 26, 28]).

Definition 1.1. *[5] Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be differentiable in usual case where $f(t) > 0$ for all t. If the below limit exists and positive*

$$f^*(t) = \lim_{h\to 0}\left[\frac{f(t+h)}{f(t)}\right]^{\frac{1}{h}}, \tag{1.1}$$

*$f^*(t)$ is denominated as *derivative of f at t.*

Lemma 1.2. *[5] Let $f : A \to \mathbb{R}$ be positive and usual differentiable at t. Then, there is a relation between classical and *derivatives as follows:*

$$f^*(t) = e^{\left(\ln o f\right)'(t)}.$$

*Repeating this procedure n times, it can be obtained the relation between the $n-$th order classical derivative and $n-$th *derivative as*

$$f^{*(n)}(t) = e^{\left(\ln o f\right)^{(n)}(t)}.$$

Theorem 1.3. *[5] Let f, h be *differentiable and p be usual differentiable at t. The below expressions are provided for *derivative.*

i. $(cf)^*(t) = f^*(t), c \in \mathbb{R}^+$,
ii. $(fh)^*(t) = f^*(t)h^*(t)$,
iii. $(f/h)^*(t) = f^*(t)/h^*(t)$
iv. $(f^p)^*(t) = f^*(t)^{p(t)} f(t)^{p'(t)}$,
v. $(fop)^*(t) = f^*(p(t))^{p'(t)}$,
vi. $(f+h)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+h(t)}} h^*(t)^{\frac{h(t)}{f(t)+h(t)}}$.

Definition 1.4. *[5] Let f be positive, bounded on $[a, b]$ for $-\infty < a < b < \infty$. Then, $\int_a^b f(x)^{dx}$ is *integral of f on $[a, b]$. If f is positive, Riemann integrable on $[a, b]$, it is *integrable on $[a, b]$ where*

$$\int_a^b f(x)^{dx} = e^{\int_a^b \left(\ln o f\right)(x)dx}.$$

On the contrary, one can prove that

$$\int_a^b f(x)dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx},$$

if f is Riemann integrable on $[a, b]$.

Theorem 1.5. *[5] Let f, h be positive, bounded functions on $[a, b]$ where $-\infty < a < b < \infty$. If f, h are *integrable on $[a, b]$, the below expressions hold:*

i. $\int\limits_a^b [f(x)^p]^{dx} = \left[\int\limits_a^b f(x)^{dx}\right]^p, p \in \mathbb{R}$

ii. $\int\limits_a^b [f(x)h(x)]^{dx} = \int\limits_a^b f(x)^{dx} \int\limits_a^b h(x)^{dx},$

iii. $\int\limits_a^b \left[\frac{f(x)}{h(x)}\right]^{dx} = \frac{\int\limits_a^b f(x)^{dx}}{\int\limits_a^b h(x)^{dx}}$

iv. $\int\limits_a^b f(x)^{dx} = \int\limits_a^c f(x)^{dx} \int\limits_c^b f(x)^{dx}, a \leq c \leq b.$

Now let's explain how classical integration by parts method is expressed in *calculus.

Theorem 1.6. *[5] Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^+$ be *integrable, $h : [a, b] \subset \mathbb{R} \to \mathbb{R}^+$ be usual integrable. Then,*

$$\int\limits_a^b \left[f^*(x)^{h(x)}\right]^{dx} = \frac{f(b)^{h(b)}}{f(a)^{h(a)}} \frac{1}{\int\limits_a^b \left[f(x)^{h'(x)}\right]^{dx}}.$$

*This equality is known as *integration by parts method. We will use this formula frequently throughout our study.*

The main focus of this study is to construct usual Green's function in *case for given problem. After we have firmly established this function in *case and examined its properties, we will apply it to SL problem in *case to test its effectiveness. Solving a *problem includes SL equation, which has a very important place in mathematical physics, with *Green's function will bring a new perspective to spectral theory. As it is known, SL equation is very important for mathematical physics in the classical case and many studies have been done on this subject from many angles [3], [14], [19], [20], [22], [29].

This study is arranged as follows: we consider a *SL problem in Section 2. Then, we prove again some important theorems related to its spectral properties in *calculus since we will look at the proofs from a different perspective. Using constructed problem, we generate *Green's function in Section 3. Eventually, we give fundamental features of this valuble function. In Section 4, we give a conclusion to sum up our study.

2. Some Spectral Properties of *Sturm-Liouville Problem

In this section, SL problem will be introduced in *calculus and some its spectral properties will be examined. This is necessary so that Green's function can be set up in *calculus for this problem. Let us consider below *homogeneous SL problem

$$\mathcal{L}(y) = \left(e^{-1} \odot y^{**}\right) \oplus \left(e^{q(x)} \odot y\right) = e^{\lambda} \odot y, \quad t \in [a, b], \tag{2.1}$$

with conditions

$$L_1(y) = (e^{\cos\alpha} \odot y(a)) \oplus \left(e^{\sin\alpha} \odot y^*(a)\right) = 1, \tag{2.2}$$

$$L_2(y) = \left(e^{\cos\beta} \odot y(b)\right) \oplus \left(e^{\sin\beta} \odot y^*(b)\right) = 1, \tag{2.3}$$

where q is real valued function on $[a, b]$; α, β are arbitrary real constants [15]. In classical case, equation (2.1) is transformed into the following nonlinear equation. Actually, all results we will get are valid for this nonlinear equation. This shows influence of *calculus.

$$y''y - (y')^2 + [(\lambda - q(t)) \ln y]\, y^2 = 0. \tag{2.4}$$

The eigenfunctions of this equation coincide with the solutions of equation (2.1). Some spectral properties of the above problem will be made using the inner product on $L_2^*[a,b]$. For $\mathcal{L}(y) = e^{\lambda} \odot y$, $y \neq 1$ is called *eigenfunction of $\mathcal{L}$ and λ is *eigenvalue of given problem. The non-homogeneous version of Eq. (2.1) is as follows.

$$\left(e^{-1} \odot y^{**}\right) \oplus \left(e^{q(x)} \ominus e^{\lambda}\right) \odot y = e^{f(x)}. \tag{2.5}$$

For a better understanding of *theory, let's express *algebraic structures that we will encounter while constructing Green's function for *SL equation. Arithmetic operations created with exponential functions are called *algebraic operations. Let's show some properties of these operations with *arithmetic table for $f,h \in \mathbb{R}^+$.

$$f \ominus h = \frac{f}{h}, \quad f \oplus h = fh, \quad f \odot h = f^{\ln h} = h^{\ln f}.$$

These operations create some algebraic structures. If $\oplus : A \times A \to A$ is an operation where $A \neq \phi$ and $A \subset \mathbb{R}^+$, the algebraic structure $(A,\oplus)$ is called *group. Similarly, $(A,\oplus,\odot)$ is a *ring. This situation gives us the opportunity to use these processes easily and define different structures.

Lemma 2.1. *[15]* $^*L_2[a,b] = \left\{ f : \int_a^b [f(x) \odot f(x)]^{dx} < \infty \right\}$ *is *inner product space with*

$$<,>_*: {}^*L_2[a,b] \times {}^*L_2[a,b] \to \mathbb{R}^+, \qquad < f,h >_*= \int_a^b [f(x) \odot h(x)]^{dx},$$

*where $f,h \in {}^*L_2[a,b]$ are positive. The proof can be easily demonstrated using the notion of *inner product.*

Here, the theory will be built on this space.

Theorem 2.2. **SL operator $\mathcal{L}$ is formally self-adjoint on $^*L_2[a,b]$.*

Proof. By *inner product on $^*L_2[a,b]$, we get

$$< L(y), z >_*= \int_a^b [L(y) \odot z]^{dx} = \int_a^b \left[(y^{**})^{-\ln z}\right]^{dx} \int_a^b \left[\left(y^{q(x)}\right)^{\ln z}\right]^{dx}. \tag{2.6}$$

Using *integration by parts method to first *integral on right side yields

$$\int_a^b \left[(y^{**})^{-\ln z}\right]^{dx} = \frac{y^*(b)^{-\ln z(b)}}{y^*(a)^{-\ln z(a)}} \frac{1}{\int_a^b \left[y^*(x)^{\frac{-z'(x)}{z(x)}}\right]^{dx}}.$$

If *integration by parts method is used once again,

$$\int_a^b \left[(y^{**})^{-\ln z}\right]^{dx} = \frac{y^*(b)^{-\ln z(b)}}{y^*(a)^{-\ln z(a)}} \frac{y(a)^{-\ln z^*(a)}}{y(b)^{-\ln z^*(b)}} \int_a^b \left[y(x)^{-\ln z^{**}(x)}\right]^{dx}. \tag{2.7}$$

If the expression (2.7) is substituted in (2.6), we get

$$< L(y), z >_*= \frac{y(b)^{\ln z^*(b)}}{y^*(b)^{\ln z(b)}} \frac{y^*(a)^{\ln z(a)}}{y(a)^{\ln z^*(a)}} \int_a^b \left[(z^{**})^{-\ln y} \left(z^{q(x)}\right)^{\ln y}\right]^{dx}. \tag{2.8}$$

By the conditions (2.2),(2.3),

$$< L(y), z >_* = < y, L(z) >_* .$$

□

Theorem 2.3. *Eigenfunctions $y(x,\lambda_1)$, $z(x,\lambda_2)$ of (2.1)-(2.3) related to distinct eigenvalues λ_1 and λ_2 are orthogonal.*

Proof. Let $[y,z]_x = [y(x)\odot z^*(x)] \ominus [y^*(x)\odot z(x)]$. Thus, by definition of *derivative, we get

$$\left([y,z]_x\right)^* = \frac{(y^{**})^{-\ln z}\left(y^{q(x)}\right)^{\ln z}}{(z^{**})^{-\ln y}\left(z^{q(x)}\right)^{\ln y}}. \tag{2.9}$$

After some calculations

$$\left([y,z]_x\right)^* = (y\odot z)^{\lambda_1-\lambda_2}.$$

If *integral is taken for both sides of the last equation and the conditions are taken into account, we get

$$1 = \int_a^b \left[(y\odot z)^{\lambda_1-\lambda_2}\right]^{dx} = \left[\int_a^b (y\odot z)^{dx}\right]^{\lambda_1-\lambda_2}.$$

Since $\lambda_1 \neq \lambda_2$, it yields

$$\int_a^b (y\odot z)^{dx} = 1.$$

□

Theorem 2.4. *Multiplicative Wronskian of any two solutions to the problem (2.1)-(2.3) is independent of x.*

Proof. Let $y(x)$ and $z(x)$ be two solutions of (2.1)-(2.3). Since

$$\frac{[y,z]_x(x)}{[y,z]_x(a)} = \frac{< L(y), z >_*}{< y, L(z) >_*},$$

we get

$$\frac{[y,z]_x(x)}{[y,z]_x(a)} = \frac{< y^\lambda, z >_*}{< y, z^\lambda >_*} = 1 \Rightarrow [y,z]_x(x) = [y,z]_x(a). \tag{2.10}$$

Moreover, multiplicative Wronskian (*Wronskian) yields

$$W_m(y,z)(x) = \begin{vmatrix} \ln y & \ln z \\ \ln y^* & \ln z^* \end{vmatrix} = \ln [y,z]_x .$$

If this result is taken together with (2.10),

$$W_m(y,z)(x) = e^{W_m(y,z)(a)},$$

ensures that result is independent of x. □

Theorem 2.5. *Any two solutions of equation (2.1) are linearly dependent iff $W_m = 0$.*

Proof. Let $y(x)$, $z(x)$ be two linearly dependent solutions of (2.1). There is at least a constant $c \neq 1$ such $y(x) = e^c \odot z(x)$. Based on this information, *Wronskian is

$$W_m(y,z)(x) = 0.$$

Conversely, let's admit $W_m(y,z)(x) = 0$. In this case, $y(x) = e^c \odot z(x)$ can be seen with an easy calculation. It completes the proof. □

Lemma 2.6. *All eigenvalues of (2.1)-(2.2) are geometrically simple.*

Proof. Let $y(x)$, $z(x)$ be eigenfunctions of (2.1)-(2.2) related to λ. By the condition (2.2), we get

$$W_m(y,z)(a) = \ln y(a) \ln z^*(a) - \ln y^*(a) \ln z(a) = 0.$$

It shows the linearly dependence of $y(x)$ and $z(x)$. It completes the proof. □

Remark 2.7. *Now, we need to give some explanation about obtaining eigenvalues and eigenfunctions of given problem. Let $\phi_1(\cdot,\lambda)$ and $\phi_2(\cdot,\lambda)$ be linearly independent solutions of (2.1) which satisfy the condition*

$$\phi_i^{*(j-1)}(a,\lambda) = e^{\delta_{ij}}, i, j = 1, 2.,$$

where δ_{ij} is Kronecker delta in classical case. Thus, each solution of Eq. (2.1) will take the form below.

$$y(x,\lambda) = \phi_1(x,\lambda)^{A_1} \phi_2(x,\lambda)^{A_2},$$

where A_1, A_2 are constants independent of x. Here, if the solution of Eq. (2.1) satisfies conditions (2.2) and (2.3), it will be eigenfunction of the related problem. In other words, if one can find a non-trivial solution of

$$\begin{aligned} A_1 \ln L_1\left(\phi_1\right) + A_2 \ln L_1\left(\phi_2\right) &= 0, \\ A_1 \ln L_2\left(\phi_1\right) + A_2 \ln L_2\left(\phi_2\right) &= 0, \end{aligned}$$

then, it will be eigenfunction. Thus, λ is an eigenvalue of given problem iff

$$\Delta_m(\lambda) = \begin{vmatrix} \ln L_1\left(\phi_1\right) & \ln L_1\left(\phi_2\right) \\ \ln L_2\left(\phi_1\right) & \ln L_2\left(\phi_2\right) \end{vmatrix} = 0.$$

Here, $\Delta_m(\lambda)$ is characteristic equation related to (2.1)-(2.2) and zeros of $\Delta_m(\lambda)$ are eigenvalues of (2.1)-(2.2).

Theorem 2.8. *All eigenvalues of (2.1)-(2.2) are simple zeros of $\Delta_m(\lambda)$.*

Proof. Assume that $\theta_1(\cdot,\lambda)$, $\theta_2(\cdot,\lambda)$ are given by following equalities

$$\theta_1(x,\lambda) = \left[L_1\left(\phi_2\right) \odot \phi_1(x,\lambda)\right] \ominus \left[L_1\left(\phi_1\right) \odot \phi_2(x,\lambda)\right], \tag{2.11}$$

$$\theta_2(x,\lambda) = \left[L_2\left(\phi_2\right) \odot \phi_1(x,\lambda)\right] \ominus \left[L_2\left(\phi_1\right) \odot \phi_2(x,\lambda)\right]. \tag{2.12}$$

According to this definition, it can be written as

$$\theta_1(x,\lambda) = \frac{\left[\phi_2(a)^{\cos\alpha} \phi_2^*(a)^{\sin\alpha}\right]^{\ln\phi_1(x,\lambda)}}{\left[\phi_1(a)^{\cos\alpha} \phi_1^*(a)^{\sin\alpha}\right]^{\ln\phi_2(x,\lambda)}}, \quad \text{and} \quad \theta_2(x,\lambda) = \frac{\left[\phi_2(b)^{\cos\beta} \phi_2^*(b)^{\sin\beta}\right]^{\ln\phi_1(x,\lambda)}}{\left[\phi_1(b)^{\cos\beta} \phi_1^*(b)^{\sin\beta}\right]^{\ln\phi_2(x,\lambda)}}.$$

Consequently, below conditions are satisfied.

$$\begin{aligned} &\theta_1(a,\lambda) = e^{\sin\alpha},\ \theta_1^*(a,\lambda) = e^{-\cos\alpha}, \\ &\theta_2(b,\lambda) = e^{\sin\beta},\ \theta_2^*(b,\lambda) = e^{-\cos\beta}. \end{aligned} \tag{2.13}$$

On the other hand, if *Wronskian definition is used, we get

$$W_m(\theta_1(x,\lambda),\theta_2(x,\lambda)) = W_m(\phi_1(x,\lambda),\phi_2(x,\lambda))\Delta_m(\lambda) = \Delta_m(\lambda). \tag{2.14}$$

Now, let λ_0 be an eigenvalue of the problem (2.1)-(2.3). By (2.11) and (2.13), we get

$$\theta_1(x,\lambda_0) = \theta_2(x,\lambda_0)^{c_1}, \tag{2.15}$$

and

$$\theta_1(b,\lambda_0) = \theta_2(b,\lambda)^{c_1}, \quad \theta_1^*(b,\lambda_0) = \theta_2^*(b,\lambda)^{c_1}.$$

By setting $y(x) = \theta_1(x,\lambda)$, $z(x) = \theta_1(x,\lambda_0)$, we get

$$(\lambda-\lambda_0)\int_a^b \ln\theta_1(x,\lambda)^{\ln\theta_1(x,\lambda_0)}\,dx = c_1\Delta_m(\lambda).$$

Since $\Delta_m(\lambda)$ is *entire function of λ,

$$\Delta_m^*(\lambda_0) = \left(e^{\lim\limits_{\lambda\to\lambda_0}\frac{\Delta_m(\lambda)}{\lambda-\lambda_0}}\right)^{\frac{1}{\Delta_m(\lambda_0)}} = \left(e^{\frac{1}{c_1}\int_a^b \ln^2\theta_1(x,\lambda_0)dx}\right)^{\frac{1}{\Delta_m(\lambda_0)}} \neq 1. \tag{2.16}$$

Hence, λ_0 is simple zero of $\Delta_m(\lambda)$. □

3. Reconstruction of Green's Function for *SL Problem

In this section, *Green's function will be constructed for *SL problem which is not homogeneous and some of its properties will be given.

Theorem 3.1. *Let's admit that λ is not an eigenvalue of (2.1)- (2.3) and $\phi(\cdot,\lambda)$ satisfies the equation (2.5) and the conditions (2.2), (2.3). Then,*

$$\phi(x,\lambda) = \int_a^b \left(G_m(x,t,\lambda)\odot e^{f(t)}\right)^{dt}, t\in[a,b] \tag{3.1}$$

*where $e^{f(t)} \in {}^*L_2[a,b]$ and $G_m(x,t,\lambda)$ is *Green function for (2.1)-(2.3) defined by*

$$G_m(x,t,\lambda) = e^{\frac{-1}{\Delta_m(\lambda)}}\odot\begin{cases}\theta_2(x)\odot\theta_1(t), & a\le t\le x\\ \theta_1(x)\odot\theta_2(t), & x<t\le b\end{cases}. \tag{3.2}$$

Conversely, $\phi(\cdot,\lambda)$ defined by (3.1) satisfies equation (2.5) and conditions (2.2), (2.3). Furthermore, $G_m(x,t,\lambda)$ is unique. Here, θ_1 and θ_2 are linearly independent solutions of (2.1) which satisfy (2.2) and (2.3).

Proof. By (3.1),

$$\phi(x,\lambda) = \int_a^b \left(G_m(x,t,\lambda)\odot e^{f(t)}\right)^{dt} = e^{\int_a^x f(t)\ln G_m(x,t,\lambda)dt}\, e^{\int_x^b f(t)\ln G_m(x,t,\lambda)dt}. \tag{3.3}$$

By definition of *Green's function,

$$G_m(x,t,\lambda) = \begin{cases} \left(\theta_2(x)^{\ln\theta_1(t)}\right)^{\frac{-1}{\Delta_m(\lambda)}}, & a \le t \le x \\ \left(\theta_1(x)^{\ln\theta_2(t)}\right)^{\frac{-1}{\Delta_m(\lambda)}}, & x < t \le b \end{cases}. \tag{3.4}$$

Considering (3.3) and (3.4) together, we get

$$\phi(x,\lambda) = \theta_2(x)^{-\int\limits_a^x \frac{\ln\theta_1(t)f(t)}{\Delta_m(\lambda)}dt}\,\theta_1(x)^{-\int\limits_x^b \frac{\ln\theta_2(t)f(t)}{\Delta_m(\lambda)}dt}. \tag{3.5}$$

Indeed, the particular solution of Eq. (2.5) by change of variables method in *case is as follows

$$\phi(x,\lambda) = \theta_1(x,\lambda)^{c_1(x)}\theta_2(x,\lambda)^{c_2(x)}, \tag{3.6}$$

where $c_1(x)$ and $c_2(x)$ are functions of x. If method is applied in *case, we get

$$\theta_1(x,\lambda)^{c_1'(x)}\theta_2(x,\lambda)^{c_2'(x)} = 1, \tag{3.7}$$
$$\theta_1^*(x,\lambda)^{c_1'(x)}\theta_2^*(x,\lambda)^{c_2'(x)} = e^{-f(x)}.$$

If $c_1'(x)$ and $c_2'(x)$ are left alone here, it yields,

$$c_1'(x) = \frac{f(x)\ln\theta_2}{\Delta_m(\lambda)}, \quad c_2'(x) = -\frac{f(x)\ln\theta_1}{\Delta_m(\lambda)}. \tag{3.8}$$

By some calculations

$$c_1(x) = c_1(b) - \int\limits_x^b \frac{f(t)\ln\theta_2(t)}{\Delta_m(\lambda)}dt, \tag{3.9}$$

and

$$c_2(x) = c_2(a) - \int\limits_a^x \frac{f(t)\ln\theta_1(t)}{\Delta_m(\lambda)}dt. \tag{3.10}$$

Thus, general solution of Eq. (2.5) is

$$\phi(x,\lambda) = \theta_1(x,\lambda)^{c_1}\theta_2(x,\lambda)^{c_2}\theta_1(x,\lambda)^{-\int\limits_x^b \frac{f(t)\ln\theta_2(t)}{\Delta_m(\lambda)}dt}\,\theta_2(x,\lambda)^{-\int\limits_a^x \frac{f(t)\ln\theta_1(t)}{\Delta_m(\lambda)}dt}, \tag{3.11}$$

by (3.6) where c_1, c_2 are arbitrary constants. Now, let's determine these constants so that $\phi(x,\lambda)$ satisfies conditions (2.2), (2.3). By (2.2) and (3.11), we get

$$\phi(a,\lambda) = e^{-\sin\alpha}, \qquad \phi^*(a,\lambda) = e^{\cos\alpha}. \tag{3.12}$$

If conditions are handled carefully here, we get

$$c_1 = \frac{-1}{\Delta_m(\lambda)}\int\limits_a^b f(t)\ln\theta_2(t)dt, \quad \text{and} \quad c_2 = \frac{1}{\Delta_m(\lambda)}\int\limits_a^b f(t)\ln\theta_1(t)dt.$$

Thus, (3.4) and (3.5) are provided. Now, on the contrary, let's show that when $\phi(x,\lambda)$ is given by (3.1), it satisfies conditions (2.5) and (2.2), (2.3). By (3.5), we get

$$\phi^*(x,\lambda) = \theta_2^*(x)^{-\int\limits_a^x \frac{\ln\theta_1(t)f(t)}{\Delta_m(\lambda)}dt}\,\theta_2(x)^{-\frac{\ln\theta_1(x)f(x)}{\Delta_m(\lambda)}}\,\theta_1^*(x)^{-\int\limits_x^b \frac{\ln\theta_2(t)f(t)}{\Delta_m(\lambda)}dt}\,\theta_1(x)^{\frac{\ln\theta_2(x)f(x)}{\Delta_m(\lambda)}}$$

If we take *derivative of above equation once again

$$\phi^{**}(x,\lambda) = \theta_2^{**}(x)^{-\int_a^x \frac{\ln\theta_1(t)f(t)}{\Delta_m(\lambda)}dt} \left(\theta_2^{*}(x)^{-\frac{\ln\theta_1(x)f(x)}{\Delta_m(\lambda)}}\right)^2 \theta_2(x)^{\left(-\frac{\ln\theta_1(x)f(x)}{\Delta_m(\lambda)}\right)'}$$
$$\theta_1^{**}(x)^{-\int_x^b \frac{\ln\theta_2(t)f(t)}{\Delta_m(\lambda)}dt} \left(\theta_1^{*}(x)^{\frac{\ln\theta_2(x)f(x)}{\Delta_m(\lambda)}}\right)^2 \theta_1(x)^{\left(\frac{\ln\theta_2(x)f(x)}{\Delta_m(\lambda)}\right)'}. \tag{3.13}$$

Since θ_1 and θ_2 are solutions of (2.1), we have

$$\theta_1^{**}(x) = \theta_1(x)^{q(x)-\lambda}, \tag{3.14}$$

$$\theta_2^{**}(x) = \theta_2(x)^{q(x)-\lambda}. \tag{3.15}$$

Furthermore,

$$\left(\theta_2^{*}(x)^{-\frac{\ln\theta_1(x)f(x)}{\Delta_m(\lambda)}}\right)^2 \left(\theta_1^{*}(x)^{\frac{\ln\theta_2(x)f(x)}{\Delta_m(\lambda)}}\right)^2 = e^{-2f}, \tag{3.16}$$

and

$$\theta_2(x)^{\left(-\frac{\ln\theta_1(x)f(x)}{\Delta_m(\lambda)}\right)'} \theta_1(x)^{\left(\frac{\ln\theta_2(x)f(x)}{\Delta_m(\lambda)}\right)'} = e^{f}. \tag{3.17}$$

If the expressions (3.14), (3.15), (3.16) and (3.17) are also taken into account in (3.13), we get

$$\phi^{**}(x,\lambda) = \phi(x,\lambda)^{q(x)-\lambda}\, e^{-f(x)}.$$

This shows that Eq. (2.5) is satisfied. Finally, let's show that conditions are met.

$$\phi(a,\lambda)^{\cos\alpha}\, \phi^{*}(a,\lambda)^{\sin\alpha} = \left(\theta_1(a)^{\cos\alpha}\, \theta_1^{*}(a)^{\sin\alpha}\right)^{-\int_a^b \frac{\ln\theta_2 f}{\Delta_m(\lambda)}dt} = 1,$$

and

$$\phi(b,\lambda)^{\cos\beta}\, \phi^{*}(b,\lambda)^{\sin\beta} = \left(\theta_2(b)^{\cos\beta}\, \theta_2^{*}(b)^{\sin\beta}\right)^{-\int_a^b \frac{\ln\theta_1 f}{\Delta_m(\lambda)}dt} = 1.$$

Now, let us prove uniqueness of *Green's function for given problem. Let's face it, there is another *Green's function $\widetilde{G_m}(x,t,\lambda)$ for same problem. Then, we get

$$\phi(x,\lambda) = \int_a^b \left(G_m(x,t,\lambda)\odot e^{f(t)}\right)^{dt},$$

and

$$\phi(x,\lambda) = \int_a^b \left(\widetilde{G_m}(x,t,\lambda)\odot e^{f(t)}\right)^{dt}.$$

Thence, we get

$$\int_a^b f(t)\left[\ln G_m(x,t,\lambda) - \ln\widetilde{G_m}(x,t,\lambda)\right]dt = 0. \tag{3.18}$$

By setting $f(t) = \overline{\ln G_m(x,t,\lambda) - \ln \widetilde{G_m}(x,t,\lambda)}$, we get

$$\int_a^b \left| \ln G_m(x,t,\lambda) - \ln \widetilde{G_m}(x,t,\lambda) \right|^2 dt = 0 \Rightarrow G_m(x,t,\lambda) = \widetilde{G_m}(x,t,\lambda).$$

It completes the proof. □

Theorem 3.2. *Let $G_m(x,t,\lambda)$ be *Green's function of (2.1)-(2.3) which has below properties.*

i. $G_m(x,t,\lambda)$ is continuous at (a,a).
ii. $G_m(x,t,\lambda) = G_m(t,x,\lambda)$.
iii. $G_m(x,t,\lambda)$ satisfies Eq. (2.1) and conditions (2.2)-(2.3) for all $t \in \mathbb{R}$ as a function of x.
iv. Let λ_0 be an eigenvalue of $\Delta_m(\lambda)$. Then, λ_0 is simple pole point of $G_m(x,t,\lambda)$ and

$$G_m(x,t,\lambda) = \left[\psi_0(x)^{-\psi_0(t)}\right]^{\frac{1}{\lambda-\lambda_0}} \widetilde{G}_m(x,t,\lambda).$$

Here, $\widetilde{G}_m(x,t,\lambda)$ is *type holomorfic function of λ in the neighbourhood of λ_0. ψ is normalized eigenfunction related to λ_0.

Proof. *i.* Proof is obtained by continuity of $\theta_1(.,\lambda)$ and $\theta_2(.,\lambda)$ for all $\lambda \in \mathbb{R}$.
ii. It will be proved by using notion of *−Green's function. Indeed, the definition yields

$$G_m(x,t,\lambda) = e^{\frac{-1}{\Delta_m(\lambda)}} \odot \begin{cases} \theta_2(x) \odot \theta_1(t), & a \le t \le x, \\ \theta_1(x) \odot \theta_2(t), & x < t \le b, \end{cases}$$

and

$$G_m(t,x,\lambda) = e^{\frac{-1}{\Delta_m(\lambda)}} \odot \begin{cases} \theta_2(t) \odot \theta_1(x), & a \le x \le t, \\ \theta_1(t) \odot \theta_2(x), & t < x \le b \end{cases}.$$

If some basic features of multiplicative calculus are used, we conclude that

$$\ln G_m(x,t,\lambda) = \begin{cases} \frac{-\ln \theta_2(x)}{\Delta_m(\lambda)} \ln \theta_1(t), & \ln a \le \ln t \le \ln x \\ \frac{-\ln \theta_2(t)}{\Delta_m(\lambda)} \ln \theta_1(x), & \ln x < \ln t \le \ln b \end{cases}$$

and

$$\ln G_m(t,x,\lambda) = \begin{cases} \frac{-\ln \theta_1(x)}{\Delta_m(\lambda)} \ln \theta_2(t), & \ln a \le \ln x \le \ln t \\ \frac{-\ln \theta_2(x)}{\Delta_m(\lambda)} \ln \theta_1(t), & \ln t < \ln x \le \ln b \end{cases}.$$

This implies

$$G_m(x,t,\lambda) = G_m(t,x,\lambda).$$

iii. Let $x \in [a,t]$. Then,

$$G_m(x,t,\lambda) = \theta_2(x)^{\frac{-\ln \theta_1(t)}{\Delta_m(\lambda)}} \Rightarrow \mathcal{L}G_m(x,t,\lambda) = e^{\lambda} \odot G_m(x,t,\lambda).$$

Similarly, proof can be made for $x \in [t,b]$.

$$[e^{\cos\alpha} \odot G_m(a,t,\lambda)] \oplus \left[e^{\sin\alpha} \odot G_m^*(a,t,\lambda)\right] = \left[\theta_1(a)^{\cos\alpha} \theta_1^*(a)^{\sin\alpha}\right]^{\frac{-\ln\theta_2(t)}{\Delta_m(\lambda)}=} = 1$$

and

$$\left[e^{\cos\beta} \odot G_m(b,t,\lambda)\right] \oplus \left[e^{\sin\beta} \odot G_m^*(b,t,\lambda)\right] = \left[\theta_2(b)^{\cos\beta}\theta_2^*(b)^{\sin\beta}\right]^{\frac{-\ln\theta_1(t)}{\Delta m(\lambda)}=} = 1$$

iv. Let λ_0 be pole point of $G_m(x,t,\lambda)$ and $R_m(x,t)$ be *residue of $G_m(x,t,\lambda)$ at $\lambda = \lambda_0$. By (2.15), (2.16), we get

$$R_m(x,t) = \lim_{\lambda\to\lambda_0} [G_m(x,t,\lambda)]^{\lambda-\lambda_0} = \theta_1(x)^{\frac{-\ln\theta_1(t)}{c_1} \frac{1}{\frac{1}{c_1}\int\limits_a^b \ln^2\theta_1(x,\lambda_0)dx}} = \psi_0(x,\lambda_0)^{-\psi_0(t,\lambda_0)}.$$

It completes the proof. □

Example 3.3. *Consider *SL-BVP*

$$e^{-1} \odot y^{**}(x) = e^{\lambda} \odot y, \tag{3.19}$$

with Dirichlet conditions

$$L_1(y) = y(0) = 0,$$
$$L_2(y) = y(1) = 0.$$

A fundamental set of solutions for (3.19) is

$$\phi_1(x,\lambda) = e^{\cos\sqrt{\lambda}x}, \quad \phi_2(x,\lambda) = e^{\sin\sqrt{\lambda}x}.$$

As known, eigenvalues of (3.19) are zeros of

$$\Delta_m(\lambda) = \begin{vmatrix} \ln L_1(\phi_1) & \ln L_1(\phi_2) \\ \ln L_2(\phi_1) & \ln L_2(\phi_2) \end{vmatrix} = \sin\sqrt{\lambda} = 0.$$

Therefore, eigenvalues are zeros of $\sin\sqrt{\lambda}$ *as* $\lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. *Then, we have*

$$\theta_1(x,\lambda) = e^{-\sin\sqrt{\lambda}x},$$
$$\theta_2(x,\lambda) = e^{\cos\sqrt{\lambda}x\sin\sqrt{\lambda}-\sin\sqrt{\lambda}x\cos\sqrt{\lambda}}.$$

So, *Green's function is

$$G_m(x,t,\lambda) = e^{\frac{-1}{\sin\lambda}} \odot \begin{cases} e^{\cos\sqrt{\lambda}x\sin\sqrt{\lambda}-\sin\sqrt{\lambda}x\cos\sqrt{\lambda}} \odot \sin\sqrt{\lambda}t & 0 \le t \le x \\ \sin\sqrt{\lambda}x \odot e^{\cos\sqrt{\lambda}t\sin\sqrt{\lambda}-\sin\sqrt{\lambda}t\cos\sqrt{\lambda}} & x < t \le 1 \end{cases}.$$

Example 3.4. *Now let's examine Eq. (3.19) with *Neumann conditions*

$$L_1(y) = y^*(0) = 0,$$
$$L_2(y) = y^*(1) = 0.$$

A fundamental set of solutions for (3.19) are

$$\phi_1(x,\lambda) = e^{\cos\sqrt{\lambda}x}, \quad \phi_2(x,\lambda) = e^{\sin\sqrt{\lambda}x}.$$

Eigenvalues of (3.19) are zeros of

$$\Delta_m(\lambda) = \begin{vmatrix} \ln(1) & \ln\left(e^{\sqrt{\lambda}}\right) \\ \ln\left(e^{-\sqrt{\lambda}\sin\sqrt{\lambda}}\right) & \ln\left(e^{\sqrt{\lambda}\cos\sqrt{\lambda}}\right) \end{vmatrix} = \sqrt{\lambda}\sin\sqrt{\lambda},$$

as $\lambda_n = n^2\pi^2,\ n \in \mathbb{N}$. *Then, *Green's function is*

$$G_m(x,t,\lambda) = \begin{cases} \theta_2(x)^{-\frac{\ln\theta_1(t)}{\Delta_m(\lambda)}}, & 0 \le t \le x \\ \theta_1(x)^{-\frac{\ln\theta_2(t)}{\Delta_m(\lambda)}}, & x < t \le 1 \end{cases}$$

where

$$\theta_1(x,\lambda) = e^{\sqrt{\lambda}\cos\sqrt{\lambda}x - \sin\sqrt{\lambda}x},$$
$$\theta_2(x,\lambda) = e^{\sqrt{\lambda}\left(\cos\sqrt{\lambda}\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x\sin\sqrt{\lambda}\right)}.$$

Example 3.5. *Let's examine Eq. (3.19) with *Robin conditions*

$$L_1(y) = y(0) = 0,$$
$$L_2(y) = y(1)y^*(1) = 0.$$

For this problem, basic set of solutions are

$$\phi_1(x,\lambda) = e^{\cos\sqrt{\lambda}x}, \quad \phi_2(x,\lambda) = e^{\sin\sqrt{\lambda}x}.$$

Then, eigenvalues of (3.19) are the zeros of

$$\Delta_m(\lambda) = \begin{vmatrix} 1 & 0 \\ \cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} & \cos\sqrt{\lambda} + \sqrt{\lambda}\sin\sqrt{\lambda} \end{vmatrix} = \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}.$$

*Then, *Green's function is*

$$G_m(x,t,\lambda) = \begin{cases} \theta_2(x)^{\frac{\sin\sqrt{\lambda}t}{\sin\sqrt{\lambda}+\sqrt{\lambda}\cos\sqrt{\lambda}}}, & 0 \le t \le x \\ e^{\frac{\sin\sqrt{\lambda}x\ln\theta_2(t)}{\sin\sqrt{\lambda}+\sqrt{\lambda}\cos\sqrt{\lambda}}}, & x < t \le 1 \end{cases}$$

where

$$\theta_1(x,\lambda) = e^{-\sin\sqrt{\lambda}x},$$
$$\theta_2(x,\lambda) = e^{\left(\sin\sqrt{\lambda}\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}\sin\sqrt{\lambda}x\right) + \sqrt{\lambda}\left(\cos\sqrt{\lambda}x\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x\sin\sqrt{\lambda}\right)}.$$

4. Conclusion

In this study, nonhomogeneous SL problem is defined in *case. Then, important spectral properties of homogeneous *problem were examined. In last part, *Green's function is created for nonhomogeneous problem. Some indispensable properties of this function have been given and proven. The relevant nonhomogeneous problem was examined in *case with help of this function. Green's function in *spectral theory is defined for the first time with this study. It will provide an important convenience to mathematicians who will work in this field and will accelerate the work in spectral theory in *case. In classical case, provision of many properties of this function in *calculus shows that many concepts and theorems in spectral theory can be defined in *calculus and very important results can be obtained. This shows that *calculus will be effective in many areas of mathematical calculus as in many fields and will lead to important results from different angles.

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