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Simulations and bisimulations for fuzzy multimodal logics over Heyting algebras

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Abstract. In the present paper, we study fuzzy multimodal logics over complete Heyting algebras and Kripke models for these logics. We introduce two types of simulations (forward and backward) and five types of bisimulations (forward, backward, forward-backward, backward-forward and regular) between Kripke models, as well as the corresponding presimulations and prebisimulations, which are simulations and bisimulations with relaxed conditions. For each type of simulations and bisimulations an efficient algorithm has been provided that works as follows: it computes the greatest presimulation/prebisimulation of that type, and then checks whether it meets the additional condition: if it does, then it is also the greatest simulation/bisimulation of that type, otherwise, there is not any simulation/bisimulation of that type. The algorithms are inspired by algorithms for checking the existence and computing the greatest simulations and bisimulations between fuzzy automata. We also demonstrate the application of these algorithms in the state reduction of Kripke models. We show that forward bisimulation fuzzy equivalences on the Kripke model provide reduced models equivalent to the original model concerning plus-formulas, backward bisimulation fuzzy equivalences provide reduced models equivalent concerning minus-formulas, while regular bisimulation fuzzy equivalences provide reduced models equivalent concerning all modal formulas.

1. Introduction

Bisimulations have significantly contributed to the application of concurrency theory in computer sciences. They were introduced by Milner [35] and Park [45] with the original purpose of modeling behavioural equivalence among processes and reducing the state-space of processes. Later, the field of their use expanded to many other areas of computer science, and today they are employed in areas such as functional languages, object-oriented languages, types, data types, domains, databases, compiler optimizations, program analysis, verification tools, etc. Even a little earlier, bisimulations were discovered in mathematics, that is, in modal logic, by van Benthem [57]. He came up with the result known today as the *van Benthem's theorem*, which states that propositional modal logic is the bisimulation-invariant fragment of first-order logic. Another famous result that emphasizes the importance of bisimulations in modal logic is the *Hennessy-Milner*

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theorem which states that two worlds, from image-finite Kripke models, are bisimulation-equivalent if and only if they satisfy the same set of modal formulas (cf. [25, 26] as well as [5, Theorems 2.20 and 2.24]). For more information on origins of bisimulations and their applications we refer to [52, 53].

Fuzzy modal logic is a kind of many-valued modal logic obtained by applying the fuzzy approach to classical modal logic, which allows not only to reason about modalities but also to cope with uncertainty. After an early attempt to combine fuzzy logic and modal logic [54], fuzzy modal logic has flourished over the last few decades (cf., e.g., [7, 9, 10, 24, 59, 60, 64]). Connections between bisimulations and fuzzy modal logic have been unexplored until recently, but have been intensively studied in recent years, especially for a special type of fuzzy modal logic – fuzzy description logics (cf. [19, 20, 23, 32, 38–43, 61]). Note also that the logical characterizations of the Hennessy-Milner type for bisimulations between fuzzy social networks and fuzzy labelled transition systems have been provided in [21, 22, 62, 63].

There have been different approaches to bisimulations for fuzzy modal logics in the literature dealing with this subject. In the first paper on this subject [19], bisimulations were defined as ordinary crisp relations that satisfy some additional "fuzzy conditions". A similar approach has been used in the recent paper [32]. However, in most of recent papers, a complete fuzzy approach has been used and bisimulations have been defined as fuzzy relations. Fan [20] studied two types of fuzzy bisimulations, which are called here forward and regular bisimulations. In a series of papers, Nguyen and others [23, 38–43] have dealt with a special type of fuzzy forward bisimulations for fuzzy description logics. In the case when the considered Kripke models are image-finite and the underlying structure of truth values is linearly ordered, this type of bisimulations coincides with forward bisimulations. The third approach to bisimulations for fuzzy modal logics, based on bisimulation games and pseudometrics, was used in [61].

The purpose of this paper is to conduct a comprehensive study that will include more different types of simulations and bisimulations for fuzzy multimodal logics. The motivation for this came from papers [13, 14] where two types of simulations and four types of bisimulations for fuzzy finite automata were introduced. These are forward and backward simulations, and forward, backward, forward-backward, and backward-forward bisimulations. We also introduce the fifth type of bisimulations, regular bisimulations, which originate from research on fuzzy social networks [30]. Research on bisimulations (and simulations) for various types of relational systems was mainly focused on forward bisimulations. Backward bisimulations are much less common in the literature, probably due to their duality with the forward ones, as a result of which many results on forward bisimulations can be easily transformed into corresponding results concerning the backward ones. However, there are situations when backward and forward bisimulations behave completely differently. For instance, it has been shown in [14] that there are situations in which for any of the four types of bisimulations for fuzzy automata, there is only a bisimulation of that type, and there are no bisimulations of the other three types. Also, it has been shown in [55] that backward bisimulations can be very successfully used in the fuzzy discrete event systems theory, in the conflict analysis, while forward bisimulations cannot. Interesting differences between forward and backward bisimulations will be demonstrated in this paper as well (cf. Theorems 7.7–7.9). All this suggests that all types of bisimulations and simulations should be equally in the focus of research.

The main results of the paper are as follows. First, for each of the two types of simulations and the five types of bisimulations we have introduced here, we create an algorithm that tests the existence of a simulation or bisimulation of that type between the given Kripke models. In the case where such simulation or bisimulation exists, the same algorithm computes the greatest one. As these algorithms do not always finish in a finite number of steps, we also provide their modifications which determine whether there are crisp simulations or bisimulations of a given type, and compute the greatest ones when they exist. Such algorithms always finish in finitely many steps. Second, we provide an application of bisimulations in the state reduction of fuzzy Kripke models, while preserving their semantic properties. Using an arbitrary fuzzy quasi-order on a given fuzzy Kripke model, we construct a new model, called the afterset fuzzy Kripke model, and in the case when this fuzzy quasi-order is regular, forward or backward bisimulation fuzzy equivalence, we show that the corresponding afterset model is equivalent to the original one with respect to all modal formulas, to all formulas valid in future worlds or to all formulas valid in past worlds (viewed in the temporal interpretation of Kripke models).

The paper is organized as follows. After this introductory section, in Section 2 we give basic definitions and notation concerning Heyting algebra, fuzzy sets and fuzzy relations, and in Section 3, we introduce syntax and semantics for fuzzy multimodal logics over a Heyting algebra. In Section 4, we introduce two types of simulations and five types of bisimulations for fuzzy multimodal logics and describe their basic properties. These simulations and bisimulations are defined by means of particular systems of fuzzy relation inequations, and we transform these systems into equivalent systems consisting only of two inequations. The main results of the paper are presented in Section 5, where we provide algorithms for testing the existence of simulations or bisimulations of a given type, and computing the greatest ones, in the cases when they exist, and in Section 7, where we provide a method for the state reduction of fuzzy Kripke models, while preserving their semantic properties. In Sections 6 and 8, we present computational examples which demonstrate applications of the results from Sections 5 and 7.

2. Preliminaries

A Dutch mathematician Luitzen Brouwer had founded the mathematical philosophy of intuitionism in the early 20th century. His student Arend Heyting developed formal systems in order to provide a formal basis for Brouwer's programme in 1930 (cf. [27]). The algebras thus obtained are called *Heyting algebras*. Instead of Heyting algebra, certain authors used a term *pseudo-Boolean algebra* or *relatively pseudocomplemented distributive lattice with* 0 (for example, see [46]), and *Brouwerian algebras* for algebraic duals of Heyting algebras (see [44]). For more information about Heyting algebras see [1, 3, 4, 6].

Now, we give a definition of a Heyting algebra.

Definition 2.1. An algebra $\mathscr{H} = (H, \wedge, \vee, \to, 0, 1)$ with three binary and two nullary operations is a *Heyting algebra* if it satisfies:

(H1) $(H, \wedge, \vee)$ is a distributive lattice;
(H2) $x \wedge 0 = 0, \quad x \vee 1 = 1$;
(H3) $x \to x = 1$;
(H4) $(x \to y) \wedge y = y, \quad x \wedge (x \to y) = x \wedge y$;
(H5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z), \quad (x \vee y) \to z = (x \to z) \wedge (y \to z)$.

A binary operation $\to$ is called *relative pseudocomplementation*, or *residuum*, in many sources. The *relative pseudocomplement* $x \to z$ *of* x *with respect to* z can be characterized as follows:

$$x \to z = \bigvee \left\{ y \in H \mid x \wedge y \leqslant z \right\}. \tag{1}$$

Equivalently, we say that operations $\wedge$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property* or *residuation property*: for all $x, y, z \in H$,

$$x \wedge y \leqslant z \qquad \Leftrightarrow \qquad x \leqslant y \to z. \tag{2}$$

If, in addition, $(H, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{H}$ is called a *complete Heyting algebra*. In the rest of the paper, unless otherwise stated, $\mathscr{H} = (H, \wedge, \vee, \to, 0, 1)$ will be a complete Heyting algebra.

The operations $\wedge$ and $\to$ are intended for modeling the conjunction and implication of the corresponding logical calculus, respectively. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), \tag{3}$$

called *bi-implication*, is used for modeling the equivalence of truth values.

A complete Heyting algebra $\mathscr{H} = (H, \wedge, \vee, \to, 0, 1)$ satisfies the following infinite distributive law:

$$x \wedge \left(\bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \wedge y_i), \tag{4}$$

as well as

$$x \wedge \left(\bigwedge_{i\in I} y_i \right) = \bigwedge_{i\in I} (x \wedge y_i), \tag{5}$$

for all $x \in H$, $\{y_i\}_{i\in I} \subseteq H$ and for every index set I. Also, if a complete Heyting algebra satisfies the following condition:

$$x \vee \left(\bigwedge_{i\in I} y_i \right) = \bigwedge_{i\in I} (x \vee y_i), \tag{6}$$

then for all non-increasing sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq H$ we have

$$\bigwedge_{k\in\mathbb{N}} (x_k \vee y_k) = \left(\bigwedge_{k\in\mathbb{N}} x_k \right) \vee \left(\bigwedge_{k\in\mathbb{N}} y_k \right), \tag{7}$$

as shown in [16] for more general context of residuated lattices. Further, this can be generalized for all non-increasing sequences $\{x_k^j\}_{k\in\mathbb{N}} \subseteq H$, $j \in J$, in the following way:

$$\bigwedge_{k\in\mathbb{N}} \bigvee_{j\in J} x_k^j = \bigvee_{j\in J} \bigwedge_{k\in\mathbb{N}} x_k^j, \tag{8}$$

where J is a finite set of indices.

We also point out the well-known equation that holds in Heyting algebras:

$$x \wedge (y \to z) = x \wedge (x \wedge y \to z). \tag{9}$$

It is generally known that Heyting algebra $\mathscr{H} = (H, \wedge, \vee, \to, 0, 1)$ can be defined as a commutative residuated lattice $\mathscr{H} = (H, \wedge, \vee, \otimes, \to, 0, 1)$ in which operation $\otimes$ coincide with $\wedge$, i.e., $x \otimes y = x \wedge y$ for all $x, y \in H$. Therefore, the terminology and basic notions given in this section are according to [3, 4], but we set them up for a Heyting algebra.

A *fuzzy subset* of a set A *over* $\mathscr{H}$, or simply a *fuzzy subset* of A is a function from A to H. Ordinary crisp subsets of A are considered as fuzzy subsets of A taking membership values in the set $\{0, 1\} \subseteq H$. Let f and g be two fuzzy subsets of A. The *equality* of f and g is defined as the usual equality of functions, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set $\mathscr{F}(A)$ of all fuzzy subsets of A forms a complete Heyting algebra, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are functions from A to H defined by

$$\left(\bigwedge_{i\in I} f_i \right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i \right)(x) = \bigvee_{i\in I} f_i(x).$$

We can define the *product* $f \wedge g$ the same as binary meet: $f \wedge g(x) = f(x) \wedge g(x)$, for every $x \in A$, due to the relationship between Heyting algebra and residuated lattice.

The *crisp part* of fuzzy subset f of A is a crisp subset $\hat{f} = \{a \in A \mid f(a) = 1\}$ of A. We will also consider $\hat{f}$ as a function $\hat{f} : A \to H$ defined by $\hat{f}(a) = 1$, if $f(a) = 1$, and $\hat{f}(a) = 0$, if $f(a) < 1$.

Let A and B be non-empty sets. A *fuzzy relation between sets* A *and* B (in this order) is any function from $A \times B$ to H, i.e., any fuzzy subset of $A \times B$, and the equality, inclusion (ordering), joins and meets of fuzzy relations are defined as for fuzzy sets. In particular, a *fuzzy relation on a set* A is a function from $A \times A$ to H, i.e., any fuzzy subset of $A \times A$. The set of all fuzzy relations from A to B will be denoted by $\mathscr{R}(A, B)$, and the set of all fuzzy relations on a set A will be denoted by $\mathscr{R}(A)$. The *inverse* of a fuzzy relation $\varphi \in \mathscr{R}(A, B)$ is a fuzzy relation $\varphi^{-1} \in \mathscr{R}(B, A)$ defined by $\varphi^{-1}(b, a) = \varphi(a, b)$, for all $a \in A$ and $b \in B$. A *crisp relation* is a fuzzy relation which takes values only in the set $\{0,1\}$, and if φ is a crisp relation of A to B, then the expressions "$\varphi(a,b)=1$" and "$(a,b)\in\varphi$" will have the same meaning.

For non-empty sets A, B and C, and fuzzy relations $\varphi\in\mathscr{R}(A,B)$ and $\psi\in\mathscr{R}(B,C)$, their *composition* $\varphi\circ\psi$ is a fuzzy relation from $\mathscr{R}(A,C)$ defined by

$$(\varphi\circ\psi)(a,c)=\bigvee_{b\in B}\varphi(a,b)\wedge\psi(b,c), \tag{10}$$

for all $a\in A$ and $c\in C$. If φ and ψ are crisp relations, then $\varphi\circ\psi$ is the ordinary composition of relations, i.e.,

$$\varphi\circ\psi=\{(a,c)\in A\times C\mid(\exists b\in B)(a,b)\in\varphi\ \&\ (b,c)\in\psi\},$$

and if φ and ψ are functions, then $\varphi\circ\psi$ is an ordinary composition of functions, i.e., $(\varphi\circ\psi)(a)=\psi(\varphi(a))$, for every $a\in A$. Next, if $f\in\mathscr{F}(A)$, $\varphi\in\mathscr{R}(A,B)$ and $g\in\mathscr{F}(B)$, the compositions $f\circ\varphi$ and $\varphi\circ g$ are fuzzy subsets of B and A, respectively, which are defined by

$$(f\circ\varphi)(b)=\bigvee_{a\in A}f(a)\wedge\varphi(a,b),\qquad(\varphi\circ g)(a)=\bigvee_{b\in B}\varphi(a,b)\wedge g(b), \tag{11}$$

for every $a\in A$ and $b\in B$.

In particular, for $f,g\in\mathscr{F}(A)$ we write

$$f\circ g=\bigvee_{a\in A}f(a)\wedge g(a). \tag{12}$$

The value $f\circ g$ can be interpreted as the "degree of overlapping" of f and g. In particular, if f and g are crisp sets and φ is a crisp relation, then

$$f\circ\varphi=\{b\in B\mid(\exists a\in f)(a,b)\in\varphi\},\qquad\varphi\circ g=\{a\in A\mid(\exists b\in g)(a,b)\in\varphi\}.$$

Let A,B,C and D be non-empty sets. Then for any $\varphi_1\in\mathscr{R}(A,B)$, $\varphi_2\in\mathscr{R}(B,C)$ and $\varphi_3\in\mathscr{R}(C,D)$ we have

$$(\varphi_1\circ\varphi_2)\circ\varphi_3=\varphi_1\circ(\varphi_2\circ\varphi_3), \tag{13}$$

and for $\varphi_0\in\mathscr{R}(A,B)$, $\varphi_1,\varphi_2\in\mathscr{R}(B,C)$ and $\varphi_3\in\mathscr{R}(C,D)$ we have that

$$\varphi_1\leqslant\varphi_2\quad\text{implies}\quad\varphi_1^{-1}\leqslant\varphi_2^{-1},\qquad\varphi_0\circ\varphi_1\leqslant\varphi_0\circ\varphi_2\qquad\text{and}\qquad\varphi_1\circ\varphi_3\leqslant\varphi_2\circ\varphi_3. \tag{14}$$

Further, for any $\varphi\in\mathscr{R}(A,B)$, $\psi\in\mathscr{R}(B,C)$, $f\in\mathscr{F}(A)$, $g\in\mathscr{F}(B)$ and $h\in\mathscr{F}(C)$ the one can easily verify that

$$(f\circ\varphi)\circ\psi=f\circ(\phi\circ\psi),\qquad(f\circ\varphi)\circ g=f\circ(\varphi\circ g),\qquad(\varphi\circ\psi)\circ h=\varphi\circ(\psi\circ h) \tag{15}$$

and consequently, the parentheses in (15) can be omitted, as well as the parentheses in (13). Finally, for all $\varphi,\varphi_i\in\mathscr{R}(A,B)(i\in I)$ and $\psi,\psi_i\in\mathscr{R}(B,C)(i\in I)$ we have that

$$(\varphi\circ\psi)^{-1}=\psi^{-1}\circ\varphi^{-1}, \tag{16}$$

$$\varphi\circ\left(\bigvee_{i\in I}\psi_i\right)=\bigvee_{i\in I}(\varphi\circ\psi_i),\qquad\left(\bigvee_{i\in I}\varphi_i\right)\circ\psi=\bigvee_{i\in I}(\varphi_i\circ\psi), \tag{17}$$

$$\left(\bigvee_{i\in I}\varphi_i\right)^{-1}=\bigvee_{i\in I}\varphi_i^{-1}. \tag{18}$$

Let A and B be fuzzy sets. A fuzzy relation $\varphi\in\mathscr{R}(A,B)$ is called *image-finite* if for every $a\in A$ the set $\{b\in B\mid\varphi(a,b)>0\}$ is finite, it is called *domain-finite* if for every $b\in B$ the set $\{a\in A\mid\varphi(a,b)>0\}$ is finite, and it is called *degree-finite* if it is both image-finite and domain finite.

We note that if A, B and C are finite sets of cardinality $|A| = k$, $|B| = m$ and $|C| = n$, then $\varphi \in \mathscr{R}(A, B)$ and $\psi \in \mathscr{R}(B, C)$ can be treated as $k \times m$ and $m \times n$ fuzzy matrices over $\mathscr{H}$, and $\varphi \circ \psi$ is the matrix product. Analogously, for $f \in \mathscr{F}(A)$ and $g \in \mathscr{F}(B)$ we can treat $f \circ \varphi$ as the product of a $1 \times k$ matrix f and a $k \times m$ matrix φ, and $\varphi \circ g$ as the product of a $k \times m$ matrix R and an $m \times 1$ matrix g^t (the transpose of g).

A fuzzy relation $R \in \mathscr{R}(W)$ is *reflexive* if $R(u, u) = 1$, for each $u \in W$, it is *symmetric* if $R(u, v) = R(v, u)$, for all $u, v \in W$, and it is *transitive* if $R(u, v) \wedge R(v, w) \leqslant R(u, w)$, for all $u, v, w \in W$. A reflexive and transitive fuzzy relation is called a *fuzzy quasi-order*, and a reflexive, symmetric and transitive fuzzy relation is called a *fuzzy equivalence*. Moreover, a reflexive and transitive crisp relation is called a *quasi-order*, and a reflexive, symmetric and transitive crisp relation is called an *equivalence relation* or just an *equivalence*. If Q is a fuzzy quasi-order on a set W, then a fuzzy relation E_Q defined by $E_Q = Q \wedge Q^{-1}$ is a fuzzy equivalence on W, and is called a *natural fuzzy equivalence* of Q. A fuzzy quasi-order Q on W is called a *fuzzy order* if for all $u, v \in W$ we have that $Q(u, v) = Q(v, u) = 1$ implies $u = v$, i.e., if we have that $E_Q(u, v) = 1$ implies $u = v$.

Let Q be a fuzzy quasi-order on a set W. For each $w \in W$, the *Q-afterset* of w is the fuzzy set $Q_w \in H^W$ defined by $Q_w(u) = Q(w, u)$, for any $u \in W$, while the *Q-foreset* of w is the fuzzy set $Q^w \in H^W$ defined by $Q^w(u) = Q(u, w)$, for any $u \in W$ (see [2, 17, 18, 55]). The set of all Q-aftersets will be denoted by W/Q, and the set of all Q-foresets will be denoted by $W \backslash Q$. If E is a fuzzy equivalence on W, then for each $w \in W$ we have that $E_w = E^w$, and it is called an *equivalence class* of w (corresponding to the fuzzy equivalence E).

3. Fuzzy multimodal logics

There are plenty many-valued logics which differ in their syntax as well as in their semantics. There are also various studies of modal expansions of many-valued logics (cf. [7, 9, 10, 24, 59, 60]). In the listed papers, logic systems are interpreted in MTL-algebras or residuated lattices, and in [19] for a Heyting-valued modal language. Here, a fuzzy multimodal logic over a Heyting algebra will be defined in a similar fashion.

In the sequel, unless otherwise stated, $\mathscr{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ will be a complete Heyting algebra and I will be a non-empty set of indices. An alphabet of a many-valued multimodal logic $\mathscr{H}(\{\Box_i, \Diamond_i, \Box_i^-, \Diamond_i^-\}_{i \in I})$ consists of an enumerable set of *propositional symbols PV*, a set of *truth constants* $\overline{H} = \{\overline{t} \mid t \in H\}$, *logical connectives* $\wedge$ (*conjunction*) and $\rightarrow$ (*implication*), and four families of *modal operators*: $\{\Box_i\}_{i \in I}$ and $\{\Box_i^-\}_{i \in I}$ (*necessity operators*) and $\{\Diamond_i\}_{i \in I}$ and $\{\Diamond_i^-\}_{i \in I}$ (*possibility operators*).

The set of formulas $\Phi_{I,\mathscr{H}}$ of a many-valued modal logic is the smallest set containing propositional symbols and truth constants, and is closed under logical connectives and modal operators:

$$A ::= \overline{t} \mid p \mid A \wedge B \mid A \rightarrow B \mid \Box_i A \mid \Diamond_i A \mid \Box_i^- A \mid \Diamond_i^- A$$

where $t \in H$, $p \in PV$, $i \in I$, and A and B are formulas from $\Phi_{I,\mathscr{H}}$. In fact, symbols t, p, and A, B are meta-variables that range over H, PV and $\Phi_{I,\mathscr{H}}$, respectively. The following well-known abbreviations will be used:

(1) $\neg A \equiv A \rightarrow \overline{0}$ (*negation*),

(2) $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$ (*equivalence*),

(3) $A \vee B \equiv ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$ (*disjunction*).

Recall that 0 is the least element in $\mathscr{H}$ and $\overline{0}$ is the corresponding truth constant. Also, $\overline{0} \rightarrow \overline{0}$ gives the top element $\overline{1}$. The set of all formulas over the alphabet $\mathscr{H}(\{\Box_i, \Diamond_i\}_{i \in I})$, i.e., the set of those formulas from $\Phi_{I,\mathscr{H}}$ that do not contain any of the modal operators $\Box_i^-$ and $\Diamond_i^-$, $i \in I$, will be denoted by $\Phi^+_{I,\mathscr{H}}$. Similarly, the set of all formulas over the alphabet $\mathscr{H}(\{\Box_i^-, \Diamond_i^-\}_{i \in I})$, i.e., the set of those formulas from $\Phi_{I,\mathscr{H}}$ that do not contain any of the modal operators $\Box_i$ and $\Diamond_i$, $i \in I$, will be denoted by $\Phi^-_{I,\mathscr{H}}$. For the sake of simplicity, formulas from $\Phi^+_{I,\mathscr{H}}$ will be called *plus-formulas*, and formulas from $\Phi^-_{I,\mathscr{H}}$ will be called *minus-formulas*.

A *fuzzy Kripke frame* is a structure $\mathfrak{F} = (W, \{R_i\}_{i \in I})$ where W is a non-empty set of possible *worlds* (or *states* or *points*) and $R_i \in \mathscr{F}(W \times W)$ is a binary fuzzy relation on W, for every $i \in I$, called the *accessibility fuzzy relation* of the frame. It is usually assumed that I is a finite set with n elements, and then $\mathfrak{F}$ is called a *fuzzy Kripke n-frame*.

A *fuzzy Kripke model for* $\Phi_{I,\mathscr{H}}$ is a structure $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ such that $(W, \{R_i\}_{i\in I})$ is a fuzzy Kripke frame and $V : W \times (PV \cup \overline{H}) \to H$ is a truth assignment function, called the *evaluation of the model*, which assigns an H-truth value to propositional variables (and truth constants) in each world, such that $V(w, \bar{t}) = t$, for every $w \in W$ and $t \in H$. It is usually assumed that I is a finite set with n elements, and then $\mathfrak{M}$ is called a *fuzzy Kripke n-model*.

Note that the defined notion of a Kripke n-model for $\mathscr{H}$ should not be identified with the notion of an n-model defined in [34], i.e., models with the assignment function V restricted to the propositional variables $p_1, \dots, p_n$ and thereby to n-formulas, formulas formed from $p_1, \dots, p_n$.

The truth assignment function V can be inductively extended to a function $V : W \times \Phi_{I,\mathscr{H}} \to H$ by:

(V1) $V(w, A \wedge B) = V(w, A) \wedge V(w, B)$;

(V2) $V(w, A \to B) = V(w, A) \to V(w, B)$;

(V3) $V(w, \Box_i A) = \bigwedge_{u\in W} R_i(w, u) \to V(u, A)$, for every $i \in I$;

(V4) $V(w, \Diamond_i A) = \bigvee_{u\in W} R_i(w, u) \wedge V(u, A)$, for every $i \in I$;

(V5) $V(w, \Box_i^- A) = \bigwedge_{u\in W} R_i(u, w) \to V(u, A)$, for every $i \in I$;

(V6) $V(w, \Diamond_i^- A) = \bigvee_{u\in W} R_i(u, w) \wedge V(u, A)$, for every $i \in I$.

Note that the same symbols are used for $\wedge$ and $\to$ in both sides of formulas (V1)–(V6). The meaning is clear from the context, so we keep the notation simple. For each world $w \in W$ the truth assignment V determines a function $V_w : \Phi_{I,\mathscr{H}} \to H$ given by $V_w(A) = V(w, A)$, for every $A \in \Phi_{I,\mathscr{H}}$, and vice versa, for each $A \in \Phi_{I,\mathscr{H}}$ the truth assignment V determines a function $V_A : W \to H$ given by $V_A(w) = V(w, A)$, for every $w \in W$.

Usually, we will denote the models with $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{N}$, $\mathfrak{N}'$ etc., not emphasizing specifically the alphabet $\mathscr{H}(\{\Box_i, \Diamond_i, \Box_i^-, \Diamond_i^-\}_{i\in I})$, except when necessary. If W is a finite set, then we will say that $\mathfrak{M}$ is a fuzzy Kripke model with a *finite number of worlds (states)*.

Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R_i'\}_{i\in I}, V')$ be two fuzzy Kripke models, and let $\Phi \subseteq \Phi_{I,\mathscr{H}}$ be some set of formulas. Worlds $w \in W$ and $w' \in W'$ are said to be Φ-*equivalent* if $V(w, A) = V'(w', A)$, for all $A \in \Phi$. Moreover, $\mathfrak{M}$ and $\mathfrak{M}'$ are said to be Φ-*equivalent fuzzy Kripke models* if each $w \in W$ is Φ-equivalent to some $w' \in W'$, and vice versa, if each $w' \in W'$ is Φ-equivalent to some $w \in W$.

Two fuzzy Kripke models $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R_i'\}_{i\in I}, V')$ are said to be *isomorphic* if there exists a bijective function $\phi : W \to W'$ such that $R_i(u, v) = R_i'(\phi(u), \phi(v))$ and $V(w, p) = V'(\phi(w), p)$, for all $i \in I$, $p \in PV$ and $u, v, w \in W$.

4. Simulations and bisimulations

Two types of simulations and four types of bisimulations for fuzzy automata were introduced in [13]. In a similar fashion, we also define two types of simulations and four types of bisimulations between two fuzzy Kripke models. Additionally, we define a fifth type of bisimulation called *regular bisimulation*, as in the case of social networks (cf. [30]). Each of these types of simulations and bisimulations is defined using an appropriate system of fuzzy relation inequations, consisting of three types of inequations.

Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R_i'\}_{i\in I}, V')$ be two fuzzy Kripke models and let $\varphi \in \mathscr{R}(W, W')$ be a non-empty fuzzy relation. If φ satisfies

$$V_p \leqslant V_p' \circ \varphi^{-1}, \qquad \text{for every } p \in PV, \tag{fs-1}$$

$$\varphi^{-1} \circ R_i \leqslant R_i' \circ \varphi^{-1}, \qquad \text{for every } i \in I, \tag{fs-2}$$

$$\varphi^{-1} \circ V_p \leqslant V_p', \qquad \text{for every } p \in PV, \tag{fs-3}$$

then it is called a *forward simulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if it satisfies only (*fs*-2) and (*fs*-3), then it is called a *forward presimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$. On the other hand, if φ satisfies

$$V_p \leqslant \varphi \circ V'_p, \qquad \text{for every } p \in PV, \tag{bs-1}$$
$$R_i \circ \varphi \leqslant \varphi \circ R'_i, \qquad \text{for every } i \in I, \tag{bs-2}$$
$$V_p \circ \varphi \leqslant V'_p, \qquad \text{for every } p \in PV, \tag{bs-3}$$

then it is called a *backward simulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if it satisfies only (*bs-3*) and (*bs*-2), it is called a *backward presimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$.

Next, if both φ and φ^{-1} are forward simulations, i.e., if

$$V_p \leqslant V'_p \circ \varphi^{-1}, \quad V'_p \leqslant V_p \circ \varphi, \qquad \text{for every } p \in PV, \tag{fb-1}$$
$$\varphi^{-1} \circ R_i \leqslant R'_i \circ \varphi^{-1}, \quad \varphi \circ R'_i \leqslant R_i \circ \varphi, \qquad \text{for every } i \in I, \tag{fb-2}$$
$$\varphi^{-1} \circ V_p \leqslant V'_p, \quad \varphi \circ V'_p \leqslant V_p, \qquad \text{for every } p \in PV. \tag{fb-3}$$

then we call φ a *forward bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if φ satisfies only (*fb*-2) and (*fb*-3), then we call it a *forward prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$. Similarly, if both φ and φ^{-1} are backward simulation, i.e. if

$$V_p \leqslant \varphi \circ V'_p, \quad V'_p \leqslant \varphi^{-1} \circ V_p, \qquad \text{for every } p \in PV, \tag{bb-1}$$
$$R_i \circ \varphi \leqslant \varphi \circ R'_i, \quad R'_i \circ \varphi^{-1} \leqslant \varphi^{-1} \circ R_i, \qquad \text{for every } i \in I, \tag{bb-2}$$
$$V_p \circ \varphi \leqslant V'_p, \quad V'_p \circ \varphi^{-1} \leqslant V_p, \qquad \text{for every } p \in PV. \tag{bb-3}$$

then we call φ a *backward bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if φ satisfies only (*bb*-2) and (*bb*-3), then we call it a *backward prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$.

We also define two "mixed" types of bisimulations. Namely, if φ is a forward simulation and φ^{-1} is a backward simulation, i.e., if

$$V_p \leqslant V'_p \circ \varphi^{-1}, \quad V'_p \leqslant V_p \circ \varphi^{-1}, \qquad \text{for every } p \in PV, \tag{fbb-1}$$
$$\varphi^{-1} \circ R_i = R'_i \circ \varphi^{-1}, \quad \text{for every } i \in I, \tag{fbb-2}$$
$$\varphi^{-1} \circ V_p \leqslant V'_p, \quad V'_p \circ \varphi^{-1} \leqslant V_p, \qquad \text{for every } p \in PV, \tag{fbb-3}$$

then we say that φ is a *forward-backward bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if only (*fbb*-2) and (*fbb*-3) hold, we say that φ is a *forward-backward prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$.

Similarly, if φ is a backward simulation and φ^{-1} is a forward simulation, i.e., if

$$V'_p \leqslant V_p \circ \varphi, \quad V_p \leqslant \varphi \circ V'_p, \qquad \text{for every } p \in PV, \tag{bfb-1}$$
$$\varphi \circ R'_i = R_i \circ \varphi, \qquad \text{for every } i \in I, \tag{bfb-2}$$
$$\varphi \circ V'_p \leqslant V_p, \quad V_p \circ \varphi \leqslant V'_p, \qquad \text{for every } p \in PV, \tag{bfb-3}$$

then we say that φ is a *backward-forward bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if only (*bfb*-2) and (*bfb*-3) hold, then we say that φ is a *backward-forward prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$.

Finally, if φ is both a forward and backward bisimulation, i.e., if

$$V_p \leqslant V'_p \circ \varphi^{-1}, \quad V'_p \leqslant V_p \circ \varphi, \quad V_p \leqslant \varphi \circ V'_p, \quad V'_p \leqslant \varphi^{-1} \circ V'_p, \qquad \text{for every } p \in PV, \tag{rb-1}$$
$$\varphi^{-1} \circ R_i = R'_i \circ \varphi^{-1}, \quad \varphi \circ R'_i = R_i \circ \varphi, \qquad \text{for every } i \in I, \tag{rb-2}$$
$$\varphi^{-1} \circ V_p \leqslant V'_p, \quad \varphi \circ V'_p \leqslant V_p, \quad V_p \circ \varphi \leqslant V'_p, \quad V'_p \circ \varphi^{-1} \leqslant V_p, \qquad \text{for every } p \in PV, \tag{rb-3}$$

then we call φ a *regular bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and if φ satisfies only (*rb*-2) and (*rb*-3), then we call it a *regular prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$. Note that the prefix "regular" comes from the social network analysis (cf. [30, 56]).

For any $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, a fuzzy relation which satisfies (θ-1), (θ-2) and (θ-3) will be called a *simulation/bisimulation of type* θ or a θ*-simulation/bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$, and a fuzzy relation satisfying (θ-2) and (θ-3) will be called a *presimulation/prebisimulation of type* θ or a θ*-presimulation/prebisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$. In addition, if $\mathfrak{M}$ and $\mathfrak{M}'$ are the same fuzzy Kripke model, then we use the name *simulation/bisimulation of type* θ or θ*-simulation/bisimulation* on the fuzzy Kripke model $\mathfrak{M}$.

It can be easily verified that

$$\varphi^{-1} \circ V_p = V_p \circ \varphi, \qquad \text{for every } p \in PV, \tag{19}$$

$$\varphi \circ V'_p = V'_p \circ \varphi^{-1}, \qquad \text{for every } p \in PV. \tag{20}$$

It follows that the definitions of forward and backward simulations/presimulations differ only in the second conditions (*fs*-2) and (*bs*-2), which are mutually dual. Similarly, the definitions of all five types of bisimulations/prebisimulations differ only in the second conditions (θ-2), for $\theta \in \{fb, bb, fbb, bfb, rb\}$, and conjunctions of conditions (θ-1) and (θ-3) in these definitions can be replaced by

$$V'_p = V_p \circ \varphi, \qquad V_p = \varphi \circ V'_p, \qquad \text{for every } p \in PV. \tag{21}$$

However, although the definitions of bisimulations with condition (21) seem simpler, in the further text we will see that the original definitions are much more suitable for testing the existence of bisimulations and computing the greatest ones, in cases when they exist.

The meaning of simulations and bisimulations can best be explained in the case when $\mathfrak{M}$ and $\mathfrak{M}'$ are crisp (Boolean-valued) Kripke models and φ is an ordinary crisp (Boolean-valued) binary relation. The condition (*fs-1*) means that if the valuation V assigns the value "true" to the propositional variable p in some world w, then the valuation V' assigns to this variable the value "true" in some world w' which simulates w. On the other hand, the condition (*fs-3*) means that if w' simulates w and the valuation V assigns the value "true" to the propositional variable p in the world w, then the valuation V' also assigns to this variable the value "true" in the world w'. The meaning of the conditions (*fs*-2) and (*bs*-2) can be explained as follows: (*fs*-2) means that if u' simulates u and v is accessible from u, then there is v' accessible from u' which simulates v, and (*bs-2*) means that if u is accessible from v and u' simulates u, then u' is accessible from some v' which simulates v. This is explained in Figure 1. In both cases, accessibility is considered with respect to R_i, for each $i \in I$.

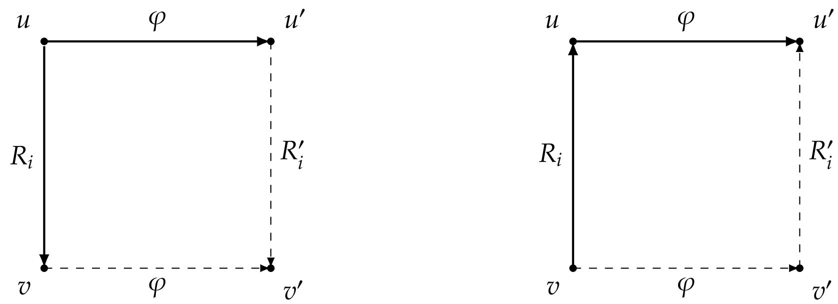

Figure 1: A forward simulation (the condition (*fs*-2), on the left) and backward simulation (the condition (*bs*-2), on the right).

Most researchers who have dealt with simulations and bisimulations in different contexts have considered only forward simulations and forward bisimulations, for which they have used the names strong simulations and strong bisimulations, or just simulations and bisimulations (cf., e.g., [20, 36, 37, 47]). The greatest bisimulation equivalence has usually been called a *bisimilarity*. However, our research is motivated by the study of different types of simulations and bisimulations between fuzzy automata, so here we also intend to study different types of simulations and bisimulations between Kripke models of fuzzy multimodal logics.

It has been noted in [13] that every forward simulation between two fuzzy automata is a backward simulation between the reverse fuzzy automata. This means that forward and backward simulations, forward and backward bisimulations, and backward-forward and forward-backward bisimulations, are mutually dual concepts. Here, we consider such duality for fuzzy Kripke models.

For a fuzzy Kripke model $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$, its *reverse fuzzy Kripke model* is the fuzzy Kripke model $\mathfrak{M}^{-1} = (W, \{R_i^{-1}\}_{i\in I}, V)$.

Let a mapping $\theta \mapsto \theta^d$ from the set $\{fs, bs, fb, bb, fbb, bfb, rb\}$ into itself be defined as follows:

$$\begin{pmatrix} fs & bs & fb & bb & fbb & bfb & rb \\ bs & fs & bb & fb & bfb & fbb & rb \end{pmatrix}$$

Now we can state and prove the following:

Theorem 4.1. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two fuzzy Kripke models, let $\mathfrak{M}^{-1} = (W, \{R_i^{-1}\}_{i\in I}, V)$ and $\mathfrak{M}'^{-1} = (W', \{R_i'^{-1}\}_{i\in I}, V')$ be the reverse fuzzy Kripke models for $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, let $\varphi \in \mathscr{R}(W, W')$ be a fuzzy relation, and let $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$.*

Then the following is true:

(a) *φ is a simulation/bisimulation of type θ between models $\mathfrak{M}$ and $\mathfrak{M}'$ if and only if φ is a simulation/bisimulation of type θ^d between the reverse fuzzy Kripke models $\mathfrak{M}^{-1}$ and $\mathfrak{M}'^{-1}$.*

(b) *The assertion* (a) *remains valid if the terms simulation and bisimulation are replaced with presimulation and prebisimulation, respectively.*

Proof. We will prove only the assertion in (a) concerning the case $\theta = fs$. The others can be proved similarly.

Let φ be forward simulation between $\mathfrak{M}$ and $\mathfrak{M}'$, i.e., let φ satisfy (*fs-1*), (*fs-2*) and (*fs-3*). As we know, conditions (*fs-1*) and (*fs-3*) can be easily transformed into (*bs-1*) and (*bs-3*), respectively, using (19) and (20).

Also, for each $i \in I$ we have that

$$\varphi^{-1} \circ R_i \leqslant R'_i \circ \varphi^{-1} \quad \Rightarrow \quad \left(\varphi^{-1} \circ R_i\right)^{-1} \leqslant \left(R'_i \circ \varphi^{-1}\right)^{-1} \quad \Rightarrow \quad R_i^{-1} \circ \varphi \leqslant \varphi \circ R_i'^{-1},$$

and it follows that φ satisfies (*bs*-2) for models $\mathfrak{M}^{-1}$ and $\mathfrak{M}'^{-1}$. □

We also state the following lemma that can be easily proved.

Lemma 4.2. *Let $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$.*

(a) *If $\{\varphi_\alpha\}_{\alpha\in Y}$ are simulations/bisimulations of type θ between models $\mathfrak{M}$ and $\mathfrak{M}'$, then $\bigvee_{\alpha\in Y} \varphi_\alpha$ is also a simulation/bisimulation of type θ between these models.*

(b) *If φ_1 is a simulation/bisimulation of type θ between models $\mathfrak{M}$ and $\mathfrak{M}'$ and φ_2 is a simulation/bisimulation of type θ between models $\mathfrak{M}'$ and $\mathfrak{M}''$, then $\varphi_1 \circ \varphi_2$ is a simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}''$.*

(c) *The assertions* (a) *and* (b) *remain valid if the terms simulation and bisimulation are replaced with presimulation and prebisimulation, respectively.*

Now, several useful notions and notation will be introduced in the same manner as in [14].

For non-empty sets of worlds W and W' and fuzzy subsets $\eta \in \mathscr{F}(W)$ and $\xi \in \mathscr{F}(W')$, fuzzy relations $\eta\backslash\xi \in \mathscr{R}(W, W')$ and $\eta/\xi \in \mathscr{R}(W, W')$ are defined as follows:

$$(\eta\backslash\xi)(w, w') = \eta(w) \to \xi(w'), \tag{22}$$

$$(\eta/\xi)(w, w') = \xi(w') \to \eta(w), \tag{23}$$

for arbitrary $w \in W$ and $w' \in W'$. Let us note that $\eta\backslash\xi = (\xi/\eta)^{-1}$ and $\eta/\xi = (\xi\backslash\eta)^{-1}$.

Next we state the well-know results by Sanchez (cf. [49–51]).

Lemma 4.3. *Let W and W' be non-empty sets of worlds and let $\eta \in \mathscr{F}(W)$ and $\xi \in \mathscr{F}(W')$.*

(a) The set of all solutions to the inequation $\eta \circ \chi \leqslant \xi$, where χ is an unknown fuzzy relation between W and W', is the principal ideal of $\mathscr{R}(W, W')$ generated by the fuzzy relation $\eta\backslash\xi$.

(b) *The set of all solutions to the inequation $\chi \circ \xi \leqslant \eta$, where χ is an unknown fuzzy relation between W and W', is the principal ideal of $\mathscr{R}(W, W')$ generated by the fuzzy relation η/ξ.*

In other words, the following residuation properties hold:

$$\eta \circ \chi \leqslant \xi \iff \chi \leqslant \eta\backslash\xi, \qquad \chi \circ \xi \leqslant \eta \iff \chi \leqslant \eta/\xi. \tag{24}$$

Note that $(\eta\backslash\xi) \wedge (\eta/\xi) = \eta \leftrightarrow \xi$, where $\eta \leftrightarrow \xi$ is a fuzzy relation between W and W' defined by

$$(\eta \leftrightarrow \xi)(w, w') = \eta(w) \leftrightarrow \xi(w'), \tag{25}$$

for arbitrary $w \in W$ and $w' \in W'$.

Next, let W and W' be non-empty sets of worlds and let $\alpha \in \mathscr{R}(W)$, $\beta \in \mathscr{R}(W')$ and $\gamma \in \mathscr{R}(W, W')$. The *right residual* of γ by α is a fuzzy relation $\alpha\backslash\gamma \in \mathscr{R}(W, W')$ defined by

$$(\alpha\backslash\gamma)(w, w') = \bigwedge_{u \in W} \alpha(u, w) \to \gamma(u, w'), \tag{26}$$

for all $w \in W$ and $w' \in W'$, and the *left residual* of γ by β is a fuzzy relation $\gamma/\beta \in \mathscr{R}(W, W')$ defined by

$$(\gamma/\beta)(w, w') = \bigwedge_{u' \in W'} \beta(w', u') \to \gamma(w, u'), \tag{27}$$

for all $w \in W$ and $w' \in W'$. We think of the right residual $\alpha\backslash\gamma$ as what remains of on the right after "dividing" γ on the left by α, and of the left residual γ/β as what remains of γ on the left after "dividing" γ on the right by β. In other words,

$$\alpha \circ \gamma' \leqslant \gamma \iff \gamma' \leqslant \alpha\backslash\gamma, \qquad \gamma' \circ \beta \leqslant \gamma \iff \gamma' \leqslant \gamma/\beta, \tag{28}$$

for all $\alpha \in \mathscr{R}(W)$, $\beta \in \mathscr{R}(W')$ and $\gamma', \gamma \in \mathscr{R}(W, W')$. In the case when $W = W'$, these two concepts become the well-known concepts of right and left residuals of fuzzy relations on a set (cf. [28]).

The following statements in the next lemma are also results by Sanchez (cf. [49–51]).

Lemma 4.4. *Let W and W' be non-empty sets of worlds and let $\alpha \in \mathscr{R}(W)$, $\beta \in \mathscr{R}(W')$ and $\gamma \in \mathscr{R}(W, W')$.*

(a) *The set of all solutions to the inequation $\alpha \circ \chi \leqslant \gamma$, where χ is an unknown fuzzy relation between W and W', is the principal ideal of $\mathscr{R}(W, W')$ generated by the right residual $\alpha\backslash\gamma$ of γ by α.*
(b) *The set of all solutions to the inequation $\chi \circ \beta \leqslant \gamma$, where χ is an unknown fuzzy relation between W and W', is the principal ideal of $\mathscr{R}(W, W')$ generated by the left residual γ/β of γ by β.*

As stated in the Introduction, now we will define isotone function ϕ on the lattice of fuzzy relations by which we will reduce problem of computation of the greatest (pre)simulation/(pre)bisimulation to the problem of computing the greatest post-fixed point, contained in a given fuzzy relation. Let's emphasize once again that greatest simulation/bisimulation do not always have to exist and in that case we just have decision-making procedure whether there is a simulation or bisimulation of a given type. First, we define initial fuzzy relations which are obtained from residuals and propositional variables in the model.

Let $\mathfrak{M} = (W, \{R_i\}_{i \in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i \in I}, V')$ be two fuzzy Kripke models. We define fuzzy relations $\pi^\theta \in \mathscr{R}(W, W')$, for $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, in the following way:

$$\pi^{fs} = \pi^{bs} = \bigwedge_{p \in PV} (V_p\backslash V'_p), \tag{29}$$

$$\pi^{fb} = \pi^{bb} = \pi^{fbb} = \pi^{bfb} = \pi^{rb} = \bigwedge_{p \in PV} [(V_p\backslash V'_p) \wedge (V_p/V'_p)] = \bigwedge_{p \in PV} (V_p \leftrightarrow V'_p). \tag{30}$$

Moreover, we define functions $\phi^\theta : \mathscr{R}(W,W') \to \mathscr{R}(W,W')$, for $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, as follows:

$$\phi^{fs}(\varphi) = \bigwedge_{i\in I}[(R'_i \circ \varphi^{-1})/R_i]^{-1}, \tag{31}$$

$$\phi^{bs}(\varphi) = \bigwedge_{i\in I} R_i\backslash(\varphi \circ R'_i), \tag{32}$$

$$\phi^{fb}(\varphi) = \bigwedge_{i\in I}[(R'_i \circ \varphi^{-1})/R_i]^{-1} \wedge [(R_i \circ \varphi)/R'_i] = \phi^{fs}(\varphi) \wedge [\phi^{fs}(\varphi^{-1})]^{-1}, \tag{33}$$

$$\phi^{bb}(\varphi) = \bigwedge_{i\in I}[R_i\backslash(\varphi \circ R'_i)] \wedge [R'_i\backslash(\varphi^{-1} \circ R_i)]^{-1} = \phi^{bs}(\varphi) \wedge [\phi^{bs}(\varphi^{-1})]^{-1}, \tag{34}$$

$$\phi^{fbb}(\varphi) = \bigwedge_{i\in I}[(R'_i \circ \varphi^{-1})/R_i]^{-1} \wedge [R'_i\backslash(\varphi^{-1} \circ R_i)]^{-1} = \phi^{fs}(\varphi) \wedge [\phi^{bs}(\varphi^{-1})]^{-1}, \tag{35}$$

$$\phi^{bfb}(\varphi) = \bigwedge_{i\in I}[R_i\backslash(\varphi \circ R'_i)] \wedge [(R_i \circ \varphi)/R'_i] = \phi^{bs}(\varphi) \wedge [\phi^{fs}(\varphi^{-1})]^{-1}, \tag{36}$$

$$\begin{aligned}\phi^{rb}(\varphi) &= \bigwedge_{i\in I}[R_i\backslash(\varphi \circ R'_i)] \wedge [(R_i \circ \varphi)/R'_i] \wedge [(R'_i \circ \varphi^{-1})/R_i]^{-1} \wedge [R'_i\backslash(\varphi^{-1} \circ R_i)]^{-1} \\ &= \phi^{fs}(\varphi) \wedge [\phi^{bs}(\varphi^{-1})]^{-1} \wedge \phi^{bs}(\varphi) \wedge [\phi^{fs}(\varphi^{-1})]^{-1} = \phi^{fb}(\varphi) \wedge \phi^{bb}(\varphi),\end{aligned} \tag{37}$$

for any $\varphi \in \mathscr{R}(W,W')$. Notice that in the expression "$\phi^\theta(\alpha^{-1})$" ($\theta \in \{fs, bs\}$) a function from $\mathscr{R}(W',W)$ into itself is denoted by ϕ^θ.

The following theorem provides alternative forms of the second and third conditions in the definitions of simulations and bisimulations, using initial fuzzy relations π^θ, and the corresponding functions ϕ^θ for $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$. These forms are more suitable for the construction of algorithms that will be given in the sequel.

Theorem 4.5. *Let $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$ and let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two fuzzy Kripke models. A fuzzy relation $\varphi \in \mathscr{R}(W,W')$ satisfies the conditions (θ-2) and (θ-3) if and only if it satisfies*

$$\varphi \leqslant \phi^\theta(\varphi), \qquad \varphi \leqslant \pi^\theta. \tag{38}$$

Proof. We will prove only the case $\theta = fs$. The assertion concerning the case $\theta = bs$ follows by the duality, and according to Eqs. (30) and (33)-(37), all other assertions can be obtained by the first two.

Consider an arbitrary $\varphi \in \mathscr{R}(W,W')$. According to Lemma 4.3(b), φ satisfies the condition (*fs-3*) if and only if $\varphi^{-1} \leqslant V'_p/V_p = (V_p\backslash V'_p)^{-1}$, for all $p \in PV$, which is equivalent to $\varphi \leqslant V_p\backslash V'_p$, for all $p \in PV$. Hence, we have

$$\varphi \leqslant \bigwedge_{p\in PV}(V_p\backslash V'_p) = \pi^{fs}.$$

Therefore, φ satisfies (*fs-3*) if and only if $\varphi \leqslant \pi^{fs}$.

On the other hand, φ satisfies (*fs*-2) if and only if

$$\varphi^{-1}(w',w) \wedge R_i(w,u) \leqslant (R'_i \circ \varphi^{-1})(w',u),$$

for all $w, u \in W$, $w' \in W'$ and $i \in I$. According to the adjunction property (2), this is equivalent to

$$\varphi^{-1}(w',w) \leqslant \bigwedge_{u\in W}[R_i(w,u) \to (R'_i \circ \varphi^{-1})(w',u))] = ((R'_i \circ \varphi^{-1})/R_i)(w',w),$$

for all $w \in W$, $w' \in W'$ and $i \in I$, which is further equivalent to

$$\varphi(w,w') \leqslant \bigwedge_{i\in I}[(R'_i \circ \varphi^{-1})/R_i]^{-1}(w,w') = (\phi^{fs}(\varphi))(w,w'),$$

for all $w \in W$ and $w' \in W'$. Therefore, φ satisfies (*fs-3*) if and only if $\varphi \leqslant \phi^{fs}(\varphi)$.

Now, we conclude that a fuzzy relation $\varphi \in \mathscr{R}(W,W')$ satisfies (*fs-2*) and (*fs-3*) if and only if it satisfies (38) (for $\theta = fs$), which has to be proved. □

5. Testing the existence and computing the greatest simulations and bisimulations

In this section we provide a method for testing the existence of simulations and bisimulations between fuzzy Kripke models, and for computing the greatest ones, in the cases when they exist.

Let W and W' be non-empty sets of worlds and let $\phi : \mathscr{R}(W, W') \to \mathscr{R}(W, W')$ be an isotone function, i.e., let $\alpha \leqslant \beta$ implies $\phi(\alpha) \leqslant \phi(\beta)$, for all $\alpha, \beta \in \mathscr{R}(W, W')$. A fuzzy relation $\alpha \in \mathscr{R}(W, W')$ is called a *post-fixed point* of ϕ if $\alpha \leqslant \phi(\alpha)$, and is called a *fixed point* of ϕ if $\alpha = \phi(\alpha)$. The well-known Knaster-Tarski fixed point theorem (stated and proved in a more general context, for complete lattices) asserts that the set of all post-fixed points of ϕ form a complete lattice (cf. [48]). Moreover, for any fuzzy relation $\pi \in \mathscr{R}(W, W')$ we have that the set of all post-fixed points of ϕ contained in π is also a complete lattice. According to Theorem 4.5, our main task is to find an efficient procedure for computing the greatest post-fixed point of the function ϕ^θ contained in the fuzzy relation π^θ, for each $\theta = \{fs, bs, fb, bb, fbb, bfb, rb\}$.

Note that the set of all post-fixed points of an isotone function on a complete lattice less than or equal to a given element is always non-empty, because it contains the least element of this lattice. However, a trivial case may occur that this set consist only of that single element. In our case, since we are dealing with a complete lattice of fuzzy relations and isotone functions on it of the form ϕ^θ, the empty relation may be the only post-fixed point contained in the corresponding fuzzy relation π^θ, and in this case there is no any simulation/bisimulation of type θ. We remember that we defined simulations and bisimulations, as well as presimulations and prebisimulations, so that they must be non-empty.

If the set of all post-fixed points of the function ϕ^θ contained in π^θ includes at least one non-empty fuzzy relation, then the greatest post-fixed point of ϕ^θ contained in π^θ is non-empty, and we will see that it is the greatest presimulation/prebisimulation of type θ, but it is not necessary a simulation/bisimulation of this type. We will show that it can be easily tested whether this greatest presimulation/prebisimulation of type θ is a simulation/bisimulation of this type, by simply checking if it satisfies the condition (θ-1).

Therefore, our task is actually to find an efficient procedure for computing the greatest post-fixed point of ϕ^θ contained in π^θ, and to check if it is non-empty and if it satisfies the condition (θ-1).

Let $\phi : \mathscr{R}(W, W') \to \mathscr{R}(W, W')$ be an isotone function and $\pi \in \mathscr{R}(W, W')$. We define a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of fuzzy relations from $\mathscr{R}(W, W')$ by

$$\varphi_1 = \pi, \qquad \varphi_{k+1} = \varphi_k \wedge \phi(\varphi_k) \text{ for each } k \in \mathbb{N}. \tag{39}$$

The sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is obviously descending. If we denote by $\hat{\varphi}$ the greatest post-fixed point of ϕ contained in π, we can verify that

$$\hat{\varphi} \leqslant \bigwedge_{k\in\mathbb{N}} \varphi_k. \tag{40}$$

Now, two questions arise. First, under what conditions does the equality in (40) hold? Second, under what conditions is this sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ finite? If this sequence is finite, then it is not hard to show that there exists $k \in \mathbb{N}$ such that $\varphi_k = \varphi_m$, for every $m \geqslant k$, i.e., there exists $k \in \mathbb{N}$ such that the sequence stabilizes on k. We can recognize that the sequence has stabilized when we find the smallest $k \in \mathbb{N}$ such that $\varphi_k = \varphi_{k+1}$. In this case $\hat{\varphi} = \varphi_k$, and we have an algorithm which computes $\hat{\varphi}$ in a finite number of steps. Some conditions under which equality holds in (40) or the sequence is finite can be found in [28, 29].

The next two theorems are essentially proved in [29] (see also [14]), but for the sake of completeness we state them here.

A sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of fuzzy relations from $\mathscr{R}(W, W')$ is called *image-finite* if the set $\bigcup_{k\in\mathbb{N}} \mathrm{Im}(\varphi_k)$ is finite. Clearly, $\{\varphi_k\}_{k\in\mathbb{N}}$ is finite if and only if it is image-finite. Next, a function $\phi : \mathscr{R}(W, W') \to \mathscr{R}(W, W')$ is called *image-localized* if there exists a finite $K \subset H$ such that for each fuzzy relation $\varphi \in \mathscr{R}(W, W')$ we have

$$\mathrm{Im}(\phi(\varphi)) \subseteq \langle K \cup \mathrm{Im}(\varphi)\rangle, \tag{41}$$

where $\langle X\rangle$ stands for the subalgebra of $\mathscr{H}$ generated by the set $X \subseteq H$. Such K will be called a *localization set* of the function ϕ. A fuzzy Kripke model $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ is called *image-finite* if the relation R_i is image-finite, for every $i \in I$, it is called *domain-finite* if the relation R_i is domain-finite, for every $i \in I$, and it is called *degree-finite* if the relation R_i is degree-finite, for every $i \in I$.

Theorem 5.1. *Let the function ϕ be image-localized, let K be its localization set, let $\pi \in \mathscr{R}(W, W')$, and let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the sequence of fuzzy relations in $\mathscr{R}(W, W')$ defined by* (39). *Then*

$$\bigcup_{k\in\mathbb{N}} \mathrm{Im}(\varphi_k) \subseteq \langle K \cup \mathrm{Im}(\pi)\rangle. \tag{42}$$

If, moreover, $\langle K \cup \mathrm{Im}(\pi)\rangle$ is a finite subalgebra of $\mathscr{H}$, then the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is finite.

Theorem 5.2. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two image-finite fuzzy Kripke models.*
For any $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$ the function ϕ^θ is isotone and image-localized.

Proof. Let $\varphi_1, \varphi_2 \in \mathscr{R}(W, W')$ be fuzzy relation such that $\varphi_1 \leqslant \varphi_2$, and consider the following systems of fuzzy relation inequations:

$$\chi^{-1} \circ R_i \leqslant R'_i \circ \varphi_1^{-1}, \qquad \text{for every } i \in I, \tag{43}$$

$$\chi^{-1} \circ R_i \leqslant R'_i \circ \varphi_2^{-1}, \qquad \text{for every } i \in I, \tag{44}$$

where $\chi \in \mathscr{R}(W, W')$ is an unknown fuzzy relation. Using Lemma 4.3(b) and the definition of an inverse relation, it can be easily shown that the set of all solutions to system (43) (resp. (44)) form a principal ideal of $\mathscr{R}(W, W')$ generated by $\phi^{fs}(\varphi_1)$ (resp. $\phi^{fs}(\varphi_2)$). Since for every $i \in I$ we have that $R'_i \circ \varphi_1^{-1} \leqslant R'_i \circ \varphi_2^{-1}$, we conclude that every solution to (43) is a solution to (44). Consequently, $\phi^{fs}(\varphi_1)$ is a solution to (44), so $\phi^{fs}(\varphi_1) \leqslant \phi^{fs}(\varphi_2)$. Therefore, we proved that ϕ^{fs} is an isotone function.

Next, let $K = \bigcup_{i\in I}(\mathrm{Im}(R_i) \cup \mathrm{Im}(R'_i))$ and let $\varphi \in \mathscr{R}(W, W')$ be an arbitrary fuzzy relation. It is evident that $\mathrm{Im}(\phi^{fs}(\varphi)) \subseteq \langle K \cup \mathrm{Im}(\varphi)\rangle$, and since fuzzy relations R_i and R'_i are image-finite, for every $i \in I$, then K is also finite. This confirms that the function ϕ^{fs} is image-localized. □

Now we are ready for the main result of the paper. The next theorem provides algorithms for computing the greatest presimulations or prebisimulations of a given type between two fuzzy Kripke models and consequently gain the greatest simulations or bisimulations of a given type, when they exist.

Theorem 5.3. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be fuzzy Kripke models, let $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, and let a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of fuzzy relations from $\mathscr{R}(W, W')$ be defined by*

$$\varphi_1 = \pi^\theta, \qquad \varphi_{k+1} = \varphi_k \wedge \phi^\theta(\varphi_k) \text{ for each } k \in \mathbb{N}. \tag{45}$$

If $\langle \mathrm{Im}(\pi^\theta) \cup \bigcup_{i\in I}(\mathrm{Im}(R_i) \cup \mathrm{Im}(R'_i))\rangle$ is a finite subalgebra of $\mathscr{H}$, then the following is true:

(a) *the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is finite and descending, and there is the least natural number k such that $\varphi_k = \varphi_{k+1}$;*
(b) *if φ_k is non-empty, then it is the greatest fuzzy relation in $\mathscr{R}(W, W')$ which satisfies* (θ-2) *and* (θ-3)*, i.e., φ_k is the greatest presimulation/prebisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$;*
(c) *if φ_k is non-empty and satisfies* (θ-1)*, then it is the greatest fuzzy relation in $\mathscr{R}(W, W')$ which satisfies* (θ-1), (θ-2) *and* (θ-3)*, i.e., φ_k is the greatest simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$;*
(d) *if φ_k is empty or does not satisfy* (θ-1)*, then there is not any fuzzy relation in $\mathscr{R}(W, W')$ satisfying* (θ-1), (θ-2), *and* (θ-3)*, i.e., there is not any simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$.*

Proof. We will prove only the case $\theta = fs$. All other cases can be proved in a similar manner.

So, let $\langle \mathrm{Im}(\pi)^\theta \cup \bigcup_{i\in I}(\mathrm{Im}(R_i) \cup \mathrm{Im}(R'_i))\rangle$ be a finite subalgebra of $\mathscr{H}$.

(a) According to Theorems 5.2 and 5.1, the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is finite and descending, so there are $k, m \in \mathbb{N}$ such that $\varphi_k = \varphi_{k+m}$, whence $\varphi_{k+1} \leqslant \varphi_k = \varphi_{k+m} \leqslant \varphi_{k+1}$. Thus, there is $k \in \mathbb{N}$ such that $\varphi_k = \varphi_{k+1}$, and consequently, there is the least natural number having this property.

(b) By $\varphi_k = \varphi_{k+1} = \varphi_k \wedge \phi^{fs}(\varphi_k)$ we obtain that $\varphi_k \leqslant \phi^{fs}(\varphi_k)$, and also, $\varphi_k \leqslant \varphi_1 = \pi^{fs}$. Therefore, by Theorem 4.5 it follows that φ_k satisfies (*fs-2*) and (*fs-3*).

Let $\alpha \in \mathscr{R}(W, W')$ be an arbitrary fuzzy relation which satisfies (*fs-2*) and (*fs-3*). As we have already noted, α satisfies (*fs-3*) if and only if $\alpha \leqslant \pi^{fs} = \varphi_1$. Next, suppose that $\alpha \leqslant \varphi_n$, for some $n \in \mathbb{N}$. Then for every $i \in I$ we have that $\alpha^{-1} \circ R_i \leqslant R'_i \circ \alpha^{-1} \leqslant R'_i \circ \varphi_n^{-1}$, and according to Lemma 4.4(b), $\alpha^{-1} \leqslant (R'_i \circ \varphi_n^{-1})/R_i$, i.e., $\alpha \leqslant [(R'_i \circ \varphi_n^{-1})/R_i]^{-1} = \phi^{fs}(\varphi_n)$. Therefore, $\alpha \leqslant \varphi_n \wedge \phi^{fs}(\varphi_n) = \varphi_{n+1}$. Now, by induction we obtain that $\alpha \leqslant \varphi_n$, for every $n \in \mathbb{N}$, and hence, $\alpha \leqslant \varphi_k$. This means that φ_k is the greatest fuzzy relation in $\mathscr{R}(W, W')$ satisfying (*fs-2*) and (*fs-3*).

(c) This follows immediately from (b).

(d) Suppose that φ_k does not satisfy (*fs-1*). Let $\varphi \in \mathscr{R}(W, W')$ be an arbitrary fuzzy relation which satisfies (*fs-1*), (*fs-2*) and (*fs-3*). According to (b) of this theorem, $\varphi \leqslant \varphi_k$, so we have that $R_i \leqslant R'_i \circ \varphi^{-1} \leqslant R'_i \circ \varphi_k^{-1}$. But, this contradicts our starting assumption that φ_k does not satisfy (*fs-1*). Hence, we conclude that there is not any fuzzy relation in $\mathscr{R}(W, W')$ which satisfies (*fs-1*), (*fs-2*) and (*fs-3*). □

Algorithm 5.4. [*Testing the existence and computing the greatest simulations and bisimulation*] The input of this algorithm are two fuzzy Kripke models $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$. The algorithm decides whether there is a simulation or bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ of a given type $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, and when it exists, the output of the algorithm is the greatest simulation/bisimulation of type θ.

The procedure is to construct a sequence of fuzzy relations $\{\varphi_k\}_{k\in\mathbb{N}}$, in the following way:

(A1) In the first step we compute π^θ and we set $\varphi_1 = \pi^\theta$.

(A2) After the kth step let a fuzzy relation φ_k has been constructed.

(A3) In the next step we construct the fuzzy relation φ_{k+1} by means of the formula $\varphi_{k+1} = \varphi_k \wedge \phi^\theta(\varphi_k)$.

(A4) Simultaneously, we check whether $\varphi_{k+1} = \varphi_k$.

(A5) The first time we find a number k such that $\varphi_{k+1} = \varphi_k$, the procedure of constructing the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ terminates, and if φ_k is non-empty, then it is the greatest presimulation/prebisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ of type θ. If φ_k is empty, then there is not any presimulation/prebisimulation nor simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$;

(A6) If φ_k is non-empty, in the final step we check whether it satisfies (θ-1). If φ_k satisfies (θ-1), then it is the greatest simulation/bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ of type θ, and if φ_k does not satisfy (θ-1), then there is not any simulation/bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ of type θ.

If the underlying Heyting algebra $\mathscr{H}$ is locally finite, in the sense that each finitely generated subalgebra of $\mathscr{H}$ is finite, then the algorithm terminates in a finite number of steps, for arbitrary finite fuzzy Kripke models over $\mathscr{H}$. Inter alia, examples of locally finite Heyting algebras include Gödel algebras and linearly ordered Heyting algebras. On the other hand, if $\mathscr{H}$ is not locally finite, then the algorithm terminates in a finite number of steps under conditions determined by Theorems 5.1 and 5.3.

However, regardless of the local finiteness of the underlying Heyting algebra and the fulfillment of the conditions of Theorems 5.1 and 5.3, the conditions under which there exists the greatest simulation/bisimulation of a given type and the greatest simulation/bisimulation itself are characterized by the following theorem.

If the underlying Heyting algebra $\mathscr{H}$ satisfies condition (6), we have the following.

Theorem 5.5. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two finite fuzzy Kripke models, let $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the sequence of fuzzy relations from $\mathscr{R}(W, W')$ defined by* (45)*, and let*

$$\varphi = \bigwedge_{k\in\mathbb{N}} \varphi_k. \tag{46}$$

Then the following is true:

(a) *if φ is non-empty, then it is the greatest fuzzy relation in $\mathscr{R}(W, W')$ which satisfies (θ-2) and (θ-3), i.e., it is the greatest presimulation/prebisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$;*

(b) *if φ is non-empty and satisfies (θ-1), then it is the greatest fuzzy relation in $\mathscr{R}(W, W')$ which satisfies (θ-1), (θ-2) and (θ-3), i.e., it is the greatest simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$;*

(c) *if φ is empty or does not satisfy (θ-1), then there is not any fuzzy relation in $\mathscr{R}(W, W')$ which satisfies (θ-1), (θ-2) and (θ-3), i.e., there is not any simulation/bisimulation of type θ between $\mathfrak{M}$ and $\mathfrak{M}'$.*

Proof. Only the case $\theta = fs$ will be proved. All other cases can be proved similarly.

(a) For arbitrary $i \in I$, $w \in W$ and $w' \in W'$ we have that

$$\begin{aligned}\left(\bigwedge_{k\in\mathbb{N}}(R_i' \circ \varphi_k^{-1})\right)(w',w) &= \bigwedge_{k\in\mathbb{N}}(R_i' \circ \varphi_k^{-1})(w',w) = \bigwedge_{k\in\mathbb{N}}\left(\bigvee_{u'\in W'} R_i'(w',u') \wedge \varphi_k^{-1}(u',w)\right)\\ &= \bigvee_{u'\in W'}\left(\bigwedge_{k\in\mathbb{N}} R_i'(w',u') \wedge \varphi_k^{-1}(u',w)\right) \qquad \text{(by (7))}\\ &= \bigvee_{u'\in W'}\left(R_i'(w',u') \wedge \left(\bigwedge_{k\in\mathbb{N}} \varphi_k^{-1}(u',w)\right)\right) \qquad \text{(by (5))}\\ &= \bigvee_{u'\in W'}\left(R_i'(w',u') \wedge \varphi^{-1}(u',w)\right) = (R_i' \circ \varphi^{-1})(w',w),\end{aligned}$$

which means that

$$\bigwedge_{k\in\mathbb{N}} R_i' \circ \varphi_k^{-1} = R_i' \circ \varphi^{-1},$$

for every $i \in I$. The use of condition (7) is justified by the facts that W' is finite, and that $\{\varphi_k^{-1}(u',w)\}_{k\in\mathbb{N}}$ is a non-increasing sequence, so $\{R_i'(w',u') \wedge \varphi_k^{-1}(u',w)\}_{k\in\mathbb{N}}$ is also a non-increasing sequence.

Now, for all $k \in \mathbb{N}$ we have that

$$\varphi \leqslant \varphi_{k+1} \leqslant \phi^{fs}(\varphi_k) = [(R_i' \circ \varphi_k^{-1})/R_i]^{-1},$$

which is equivalent to

$$\varphi^{-1} \circ R_i \leqslant R_i' \circ \varphi_k^{-1}.$$

As the last inequation holds for every $k \in \mathbb{N}$ we have that

$$\varphi^{-1} \circ R_i \leqslant \bigwedge_{k\in\mathbb{N}} R_i' \circ \varphi_k^{-1} = R_i' \circ \varphi^{-1},$$

for every $i \in I$. Therefore, φ satisfies (*fs-2*). Moreover, $\varphi \leqslant \varphi_1 = \pi^{fs}$, so φ also satisfies (*fs-3*).

Next, let $\alpha \in \mathscr{R}(W, W')$ be an arbitrary fuzzy relation satisfying (*fs-2*) and (*fs-3*). According to Theorem 4.5, $\alpha \leqslant \phi^{fs}(\alpha)$ and $\alpha \leqslant \pi^{fs} = \varphi_1$. By induction, we can easily prove that $\alpha \leqslant \varphi_k$ for every $k \in \mathbb{N}$, therefore, $\alpha \leqslant \varphi$. This means that φ is the greatest fuzzy relation $\mathscr{R}(W, W')$ which satisfies (*fs-2*) and (*fs-3*).

The assertion (b) follows immediately from (a), whereas the assertion (c) can be proved in the same way as the assertion (d) of Theorem 5.3. □

According to the previous theorem, if there is the greatest presimulation/prebisimulation of type θ, it is equal to the infimum of the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ defined by formula (45). Computing that infimum requires computing all members of the sequence, which can only be effectively done when this sequence is finite, in a way described in Algorithm 5.4. However, what to do if this sequence is not finite, i.e., if Algorithm 5.4 fails to terminate in a finite number of steps? In such situations we could "approximate" fuzzy simulations and bisimulations with crisp simulations and bisimulations. We will show how Algorithm 5.4 can be modified to test the existence and compute the greatest crisp simulations and bisimulations. The modified algorithm always terminates in a finite number of steps, independently of the properties of the underlying structure of truth values. Also, in Section 6 many interesting examples are given concerning the crisp simulations and bisimulations from which the following conclusions are drawn. First, the greatest crisp simulations and bisimulations cannot be obtained simply by taking the crisp parts of the greatest fuzzy simulations and bisimulations. Second, there are cases in which there is a fuzzy simulation/bisimulation of a given type between two fuzzy Kripke models, but there is not any crisp simulation/bisimulation of this type between them.

Let W and W' be non-empty finite sets of worlds, and let $\mathscr{R}^c(W, W')$ denote the set of all crisp relations from $\mathscr{R}(W, W')$. For each fuzzy relation $\varphi \in \mathscr{R}(W, W')$ we have that $\varphi^c \in \mathscr{R}^c(W, W')$, where φ^c denotes *the crisp part* of a fuzzy relation φ, i.e., a function $\varphi^c : W \times W' \to \{0, 1\}$ defined by $\varphi^c(w, w') = 1$ if $\varphi(w, w') = 1$, and $\varphi^c(w, w') = 0$, if $\varphi(w, w') < 1$, for arbitrary $w \in W$ and $w' \in W'$. Equivalently, φ^c is considered as an ordinary crisp relation between W and W' given by $\varphi^c = \{(w, w') \in W \times W' \mid \varphi(w, w') = 1\}$.

For each function $\phi : \mathscr{R}(W, W') \to \mathscr{R}(W, W')$ we define a function $\phi^c : \mathscr{R}^c(W, W') \to \mathscr{R}^c(W, W')$ by

$$\phi^c(\varphi) = (\phi(\varphi))^c \qquad \text{for any} \qquad \varphi \in \mathscr{R}^c(W, W').$$

If ϕ is isotone, then it can be easily shown that φ^c is also an isotone function.

Theorem 5.6. *Let W and W' be non-empty finite sets, let $\phi : \mathscr{R}(W, W') \to \mathscr{R}(W, W')$ be an isotone function and let $\pi \in \mathscr{R}(W, W')$ be a given fuzzy relation. A crisp relation $\varrho \in \mathscr{R}^c(W, W')$ is the greatest crisp solution in $\mathscr{R}(W, W')$ to the system*

$$\chi \leqslant \phi(\chi), \qquad \chi \leqslant \pi, \tag{47}$$

if and only if it is the greatest solution in $\mathscr{R}^c(W, W')$ to the system

$$\xi \leqslant \phi^c(\xi), \qquad \xi \leqslant \pi^c, \tag{48}$$

where χ is an unknown fuzzy relation and ξ is an unknown crisp relation.

Furthermore, a sequence $\{\varrho_k\}_{k\in\mathbb{N}} \subseteq \mathscr{R}(W, W')$ defined by

$$\varrho_1 = \pi^c,\ \varrho_{k+1} = \varrho \wedge \phi^c(\varrho_k) \text{ for every } k \in \mathbb{N} \tag{49}$$

is a finite descending sequence of crisp relations, and the least member of this sequence is the greatest solution to the system (48) *in $\mathscr{R}^c(W, W')$.*

Proof. The proof of this theorem can be obtained simply by translating the proof of Theorem 5.8 from [28] to the case of relations between the two sets. □

Taking ϕ to be any of the functions ϕ^θ, for $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, Theorem 5.6 gives algorithms for deciding whether there is a crisp simulation/bisimulation of a given type between two fuzzy Kripke models, and computing the greatest one, when it exists. As it can be seen in Theorem 5.6, these algorithms always terminate in a finite number of steps, independently of the properties of the underlying structure of truth values.

It is worth noting that functions $(\phi^\theta)^c$, for all $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, can be characterized as follows:

$$(w, w') \in (\phi^{fs})^c(\varrho) \quad \Leftrightarrow \quad (\forall i \in I)(\forall u \in W) R_i(w, u) \leqslant (R'_i \circ \varrho^{-1})(w', u)$$
$$(w, w') \in (\phi^{bs})^c(\varrho) \quad \Leftrightarrow \quad (\forall i \in I)(\forall u \in W) R_i(u, w) \leqslant (\varrho \circ R'_i)(u, w')$$
$$(\phi^{fb})^c(\varrho) = (\phi^{fs})^c(\varrho) \wedge [(\phi^{fs})^c(\varrho^{-1})]^{-1}$$
$$(\phi^{bb})^c(\varrho) = (\phi^{bs})^c(\varrho) \wedge [(\phi^{bs})^c(\varrho^{-1})]^{-1}$$
$$(\phi^{fbb})^c(\varrho) = (\phi^{fs})^c(\varrho) \wedge [(\phi^{bs})^c(\varrho^{-1})]^{-1}$$
$$(\phi^{bfb})^c(\varrho) = (\phi^{bs})^c(\varrho) \wedge [(\phi^{fs})^c(\varrho^{-1})]^{-1}$$
$$(\phi^{rb})^c(\varrho) = (\phi^{fs})^c(\varrho) \wedge [(\phi^{bs})^c(\varrho^{-1})]^{-1} \wedge (\phi^{bs})^c(\varrho) \wedge [(\phi^{fs})^c(\varrho^{-1})]^{-1}$$

for all $\varrho \in \mathscr{R}^c(W, W')$, $w \in W$ and $w' \in W'$.

6. Computational examples for testing the existence and computing the greatest simulations and bisimulations

In this section we give examples which demonstrate the application of algorithms and clarify relationships between different types of simulations and bisimulations.

It is generally known that every linearly ordered Heyting algebra is a Gödel algebra (cf. [19]) and every Gödel algebra is a Heyting algebra with the Dummett condition $(x \to y) \vee (y \to x) = 1$ (cf. [11]). Also, every Boolean algebra is a Heyting algebra, with $A \to B$ given by $\neg A \vee B$, or $\neg\neg A = A$. Therefore, several examples are on the standard Gödel modal logic over $[0,1]$, while the last example is on the Boolean algebra of all subsets of some set A.

In the sequel, for any $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$, by φ^θ we will denote the greatest simulation/bisimulation of type θ between two given fuzzy Kripke models, if it exists. On the other hand, by φ^θ_* we will denote the greatest fuzzy relation satisfying (θ-2) and (θ-3). It can be empty, but if it is non-empty, it is the greatest presimulation/prebisimulation of type θ. Analogously, ϱ^θ will denote the greatest crisp simulation/bisimulation of type θ, if it exists, and ϱ^θ_* the greatest crisp relation satisfying (θ-2) and (θ-3). If it is non-empty, it is the greatest crisp presimulation/prebisimulation of type θ.

Example 6.1. Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two fuzzy Kripke models over the Gödel structure, where $W = \{u, v, w\}$, $W' = \{u', v'\}$ and set $I = \{1\}$. Fuzzy relations R_1, R'_1 and fuzzy sets V_p, V_q, V'_p and V'_q are represented by the following fuzzy matrices and vectors:

$$R_1 = \begin{bmatrix} 1 & 0 & 0.9 \\ 1 & 0.1 & 0.5 \\ 1 & 0 & 1 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 1 \\ 0.2 \\ 1 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 1 \\ 0.7 \\ 1 \end{bmatrix}, \tag{50}$$

$$R'_1 = \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix}, \qquad V'_p = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \qquad V'_q = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix}. \tag{51}$$

Using algorithms for testing the existence of simulations and bisimulations between fuzzy Kripke models $\mathfrak{M}$ and $\mathfrak{M}'$ and computing the greatest ones, we have:

$$\varphi^{fs}_* = \varphi^{fs} = \begin{bmatrix} 1 & 0.2 \\ 1 & 1 \\ 1 & 0.2 \end{bmatrix}, \qquad \varphi^{bs}_* = \varphi^{bs} = \begin{bmatrix} 1 & 0.2 \\ 1 & 1 \\ 1 & 0.2 \end{bmatrix},$$

$$\varphi^{fb}_* = \varphi^{fb} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \\ 1 & 0.2 \end{bmatrix}, \qquad \varphi^{bb}_* = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

$$\varphi^{fbb}_* = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \qquad \varphi^{bfb}_* = \varphi^{bfb} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \\ 1 & 0.2 \end{bmatrix}, \qquad \varphi^{rb}_* = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

while φ^{bb}, φ^{fbb} and φ^{rb} do not exist, since φ^{bb}_*, φ^{fbb}_* and φ^{rb}_* do not satisfy (bb-1), (fbb-1) and (rb-1), respectively.

Algorithms for testing the existence and computing crisp simulations and bisimulations yield:

$$\varrho^{fs}_* = \varrho^{fs} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \varrho^{bs}_* = \varrho^{bs} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix},$$

while ϱ^{fb}_*, ϱ^{bb}_*, ϱ^{fbb}_*, ϱ^{bfb}_* and ϱ^{rb}_* are empty, so ϱ^{fb}, ϱ^{bb}, ϱ^{fbb}, ϱ^{bfb} and ϱ^{rb} do not exist. Therefore, we have that there are no the greatest crisp fb- and bfb-bisimulations, regardless of the fact that there are the greatest fuzzy bisimulations of these types.

If we consider the reverse fuzzy Kripke models for $\mathfrak{M}$ and $\mathfrak{M}'$, we have the opposite situation. Namely, in this case there are no fb- and bfb-bisimulations, while there are the greatest fs- and bs-simulations, as well as the greatest bb- and fbb-bisimulations. Since regular bisimulations are self-dual, there is not any regular bisimulation even between the reverse fuzzy Kripke models.

The following example illustrates the situation where there are all five types of bisimulations, and they are mutually identical, which also holds for all types of crisp bisimulations.

Example 6.2. Let us replace R_1, V_p and V_q in (50) with

$$R_1 = \begin{bmatrix} 0.9 & 1 & 1 \\ 0.4 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 0.9 \\ 0.4 \\ 0.4 \end{bmatrix}, \tag{52}$$

and R'_1, V'_p and V'_q in (51) with

$$R'_1 = \begin{bmatrix} 0.9 & 1 \\ 0.4 & 0.5 \end{bmatrix}, \qquad V'_p = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \qquad V'_q = \begin{bmatrix} 0.9 \\ 0.4 \end{bmatrix}. \tag{53}$$

Then, we have:

$$\varphi^{fs}_* = \varphi^{fs} = \begin{bmatrix} 1 & 0.3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \varphi^{bs}_* = \varphi^{bs} = \begin{bmatrix} 1 & 0.3 \\ 0.9 & 1 \\ 0.9 & 1 \end{bmatrix},$$

$$\varphi^{fb}_* = \varphi^{fb} = \varphi^{bb}_* = \varphi^{bb} = \varphi^{fbb}_* = \varphi^{fbb} = \varphi^{bfb}_* = \varphi^{bfb} = \varphi^{rb}_* = \varphi^{rb} = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \\ 0.3 & 1 \end{bmatrix},$$

and also:

$$\varrho^{fs}_* = \varrho^{fs} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \varrho^{bs}_* = \varrho^{bs} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\varrho^{fb}_* = \varrho^{fb} = \varrho^{bb}_* = \varrho^{bb} = \varrho^{fbb}_* = \varrho^{fbb} = \varrho^{bfb}_* = \varrho^{bfb} = \varrho^{rb}_* = \varrho^{rb} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The next example concerns simulations and bisimulations between fuzzy Kripke models with two fuzzy relations, i.e., it concerns a modal language with two quadruples of modal operators.

Example 6.3. Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two fuzzy Kripke models over the Gödel structure, where $W = \{u, v, w\}$, $W' = \{u', v'\}$ and set $I = \{1, 2\}$. Fuzzy relations R_1, R_2, R'_1, R'_2 and fuzzy sets V_p, V_q, V'_p and V'_q are represented by the following fuzzy matrices and vectors:

$$R_1 = \begin{bmatrix} 0.7 & 0.6 & 0.6 \\ 1 & 0.6 & 0.6 \\ 1 & 0.5 & 0.5 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0.8 & 0.7 & 0.7 \\ 0.5 & 0.9 & 0.9 \\ 0.5 & 0.9 & 0.9 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 0.9 \\ 0.8 \\ 0.8 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 0.8 \\ 0.4 \\ 0.4 \end{bmatrix},$$

$$R'_1 = \begin{bmatrix} 0.7 & 0.6 \\ 1 & 0.6 \end{bmatrix}, \qquad R'_2 = \begin{bmatrix} 0.8 & 0.7 \\ 0.5 & 0.9 \end{bmatrix}, \qquad V'_p = \begin{bmatrix} 0.9 \\ 0.8 \end{bmatrix}, \qquad V'_q = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix}.$$

Algorithms for testing the existence and computing simulations and bisimulations between fuzzy Kripke models $\mathfrak{M}$ and $\mathfrak{M}'$ yield:

$$\varphi_*^{fs} = \varphi^{fs} = \begin{bmatrix} 1 & 0.4 \\ 0.7 & 1 \\ 0.7 & 1 \end{bmatrix}, \quad \varphi_*^{bs} = \varphi^{bs} = \begin{bmatrix} 1 & 0.4 \\ 0.8 & 1 \\ 0.8 & 1 \end{bmatrix}, \quad \varphi_*^{fb} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}, \quad \varphi_*^{bb} = \varphi^{bb} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \\ 0.4 & 1 \end{bmatrix},$$

$$\varphi_*^{fbb} = \varphi^{fbb} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \\ 0.4 & 1 \end{bmatrix}, \quad \varphi_*^{bfb} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}, \quad \varphi_*^{rb} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \\ 0.4 & 0.5 \end{bmatrix},$$

and φ_*^{fb}, φ_*^{bfb} and φ_*^{rb} do not satisfy (fb-1), (bfb-1) and (rb-1), respectively, which means that φ^{fb}, φ^{bfb} and φ^{rb} do not exist.

On the other hand, algorithms for testing the existence and computing crisp simulations and bisimulations yield:

$$\varrho_*^{fs} = \varrho^{fs} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \varrho_*^{bs} = \varrho^{bs} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \varrho_*^{bb} = \varrho^{bb} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \varrho_*^{fbb} = \varrho^{fbb} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

In this case, ϱ_*^{fb}, ϱ_*^{bfb} and ϱ_*^{rb} are empty, so there are no ϱ^{fb}, ϱ^{bfb} and ϱ^{rb}.

The following example shows what the simulations and bisimulations look like between a fuzzy Kripke model $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and itself. We give this example to clearly see all variations and differences between various types of simulations and bisimulations.

Example 6.4. Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model over the Gödel structure, where $W = \{u, v, w\}$ and set $I = \{1\}$. A fuzzy relation R_1 and fuzzy sets V_p, V_q, are represented by the following fuzzy matrices and vectors:

$$R_1 = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.4 & 0.8 & 1 \\ 1 & 0.3 & 0.8 \end{bmatrix}, \quad V_p = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \end{bmatrix}, \quad V_q = \begin{bmatrix} 0.3 \\ 0.7 \\ 0.8 \end{bmatrix}. \tag{54}$$

If we set $\mathfrak{M}' = \mathfrak{M}$, then we have:

$$\varphi_*^{fs} = \varphi^{fs} = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 0.3 & 1 & 0.1 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \quad \varphi_*^{bs} = \varphi^{bs} = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 0.3 & 1 & 0.1 \\ 0.3 & 0.7 & 1 \end{bmatrix},$$

$$\varphi_*^{fb} = \varphi^{fb} = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}, \quad \varphi_*^{bb} = \varphi^{bb} = \begin{bmatrix} 1 & 0.3 & 0.1 \\ 0.3 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix},$$

$$\varphi_*^{fbb} = \varphi^{fbb} = \begin{bmatrix} 1 & 0.3 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}, \quad \varphi_*^{bfb} = \varphi^{bfb} = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.3 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}, \quad \varphi_*^{rb} = \varphi^{rb} = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}.$$

On the other hand, all crisp simulations and bisimulations are equal to the equality relation (identity matrix).

The last example of this section shows what the simulations and bisimulations look like between two fuzzy Kripke models where underlying structure is Boolean algebra. It is especially interesting that the Boolean algebra in this example is not linearly ordered.

Example 6.5. Let $A = \{a, b, c\}$ be an arbitrary set of three elements and let $\mathcal{P}(A)$ be the power set of A. Structure $(\mathcal{P}(A), \cup, \cap, ', \emptyset, A)$ with operations of union, intersection, and complementation, and the distinguished subsets $\emptyset$ and A, is called the *Boolean algebra of all subsets of A*, or the *power set algebra* on A.

Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ be two fuzzy Kripke models over the power set algebra on A, where $W = \{u, v, w\}$, $W' = \{u', v'\}$ and set $I = \{1\}$. Fuzzy relations R_1, R'_1 and fuzzy sets V_p, V_q, V'_p and V'_q are represented by the following fuzzy matrices and vectors:

$$R_1 = \begin{bmatrix} A & \emptyset & A \\ \{a,b\} & \{b,c\} & \{a,c\} \\ A & \emptyset & A \end{bmatrix}, \qquad V_p = \begin{bmatrix} A \\ \{b\} \\ A \end{bmatrix}, \qquad V_q = \begin{bmatrix} A \\ \{b,c\} \\ A \end{bmatrix},$$

$$R'_1 = \begin{bmatrix} A & \{b\} \\ A & \{b,c\} \end{bmatrix}, \qquad V'_p = \begin{bmatrix} A \\ \{b\} \end{bmatrix}, \qquad V'_q = \begin{bmatrix} A \\ \{b,c\} \end{bmatrix}.$$

Algorithms for testing the existence and computing simulations and bisimulations between fuzzy Kripke models $\mathfrak{M}$ and $\mathfrak{M}'$ yield:

$$\varphi^{fs}_* = \varphi^{fs} = \begin{bmatrix} A & \{b\} \\ A & A \\ A & \{b\} \end{bmatrix}, \quad \varphi^{bs}_* = \varphi^{bs} = \begin{bmatrix} A & \{b\} \\ A & A \\ A & \{b\} \end{bmatrix}, \quad \varphi^{fb}_* = \varphi^{fb} = \begin{bmatrix} A & \{b\} \\ \{b\} & A \\ A & \{b\} \end{bmatrix}, \quad \varphi^{bb}_* = \begin{bmatrix} \{a,b\} & \{b\} \\ \{b\} & A \\ \{a,b\} & \{b\} \end{bmatrix},$$

$$\varphi^{fbb}_* = \begin{bmatrix} \{a,b\} & \{b\} \\ \{b\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix}, \quad \varphi^{bfb}_* = \varphi^{bfb} = \begin{bmatrix} A & \{b\} \\ \{b\} & A \\ A & \{b\} \end{bmatrix}, \quad \varphi^{rb}_* = \begin{bmatrix} \{a,b\} & \{b\} \\ \{b\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix},$$

and φ^{bb}_*, φ^{fbb}_* and φ^{rb}_* do not satisfy (bb-1), (fbb-1) and (rb-1), respectively, which means that φ^{bb}, φ^{fbb} and φ^{rb} do not exist.

On the other hand, algorithms for testing the existence and computing crisp simulations and bisimulations yield:

$$\varrho^{fs}_* = \varrho^{fs} = \begin{bmatrix} A & \emptyset \\ A & A \\ A & \emptyset \end{bmatrix}, \qquad \varrho^{bs}_* = \varrho^{bs} = \begin{bmatrix} A & \emptyset \\ A & A \\ A & \emptyset \end{bmatrix},$$

while ϱ^{fb}_*, ϱ^{bb}_*, ϱ^{fbb}_*, ϱ^{bfb}_* and ϱ^{rb}_* are empty, so there are no ϱ^{fb}, ϱ^{bb}, ϱ^{fbb}, ϱ^{bfb} and ϱ^{rb}, similar like in Example 6.1.

7. Afterset and foreset fuzzy Kripke models

In this section, we present several ways to reduce the number of worlds of a fuzzy Kripke model while preserving its semantic properties. In other words, we provide a construction of a reduced fuzzy Kripke model which is $\Phi_{I,\mathscr{H}}$-equivalent, $\Phi^{+}_{I,\mathscr{H}}$-equivalent or $\Phi^{-}_{I,\mathscr{H}}$-equivalent to the original fuzzy Kripke model.

The following theorem was proved in [55] (see also [31]).

Theorem 7.1. *Let Q be a fuzzy quasi-order on a set W and E the natural fuzzy equivalence of Q. Then*

(a) For arbitrary $w, u \in W$ the following conditions are equivalent:

(i) $E(w, u) = 1$;

(ii) $E_w = E_u$;

(iii) $Q_w = Q_u$;

(iv) $Q^w = Q^u$.

(b) Functions $Q_w \mapsto E_w$ of W/Q to W/E, and $Q_w \mapsto Q^w$ of W/Q to $W\backslash Q$ are bijective functions.

If W is a finite set with n members, then a fuzzy quasi-order Q on W is viewed as an $n \times n$ fuzzy matrix with entries in $\mathscr{H}$ (it is usually identified with that matrix, which is called a *fuzzy quasi-order matrix*). In that case Q-aftersets are row vectors, whereas Q-foresets are column vectors of this matrix. The previous theorem says that the ith and jth row vectors of this matrix are equal if and only if its ith and jth column vectors are equal, and vice versa. Moreover, we have that a fuzzy quasi-order Q is a fuzzy order if and only if all its row vectors are different, or equivalently, if and only if all its column vectors are different.

Let $\mathfrak{F} = (W, \{R_i\}_{i\in I})$ be a fuzzy Kripke frame over $\mathscr{H}$ and let Q be a fuzzy quasi-order on W. For each $i \in I$ we can define a fuzzy relation $R_i^{W/Q} : W/Q \times W/Q \to H$ by

$$R_i^{W/Q}(Q_u, Q_v) = \bigvee_{w,w'\in W} Q(u,w) \wedge R_i(w,w') \wedge Q(w',v), \tag{55}$$

or equivalently

$$R_i^{W/Q}(Q_u, Q_v) = (Q \circ R_i \circ Q)(u,v) = Q_u \circ R_i \circ Q^v, \tag{56}$$

for all $u, v \in W$. According to the statement (a) of Theorem 7.1, $R_i^{W/Q}$ is well-defined, for each $i \in I$, and we have that $\mathfrak{F}/Q = (W/Q, \{R_i^{W/Q}\}_{i\in I})$ is a fuzzy Kripke frame, called the *afterset fuzzy Kripke frame* of $\mathfrak{F}$ w.r.t. Q.

In addition, if $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ is a fuzzy Kripke model, then we define the fuzzy functions $R_i^{W/Q}$ as in (55), for every propositional variable $p \in PV$ we define a fuzzy set $V_p^{W/Q} \in \mathscr{F}(W/Q)$ by

$$V_p^{W/Q}(Q_w) = \bigvee_{u\in W} V_p(u) \wedge Q(u,w) = (V_p \circ Q)(w) = V_p \circ Q^w, \tag{57}$$

for any $w \in W$, and we define a function $V^{W/Q} : (W/Q) \times (PV \cup \overline{H}) \to H$ by

$$V^{W/Q}(Q_w, p) = V_p^{W/Q}(Q_w) \quad \text{and} \quad V^{W/Q}(Q_w, \bar{t}) = t,$$

for all $w \in W$, $p \in PV$ and $\bar{t} \in \overline{H}$. We inductively extend $V^{W/Q}$ to a function $V^{W/Q} : (W/Q) \times \Phi_{I,\mathscr{H}} \to H$ as in (V1)-(V6), and for each $A \in \Phi_{I,\mathscr{H}}$ we define a fuzzy set $V_A^{W/Q} \in \mathscr{F}(W/Q)$ by $V_A^{W/Q}(Q_w) = V^{W/Q}(Q_w, A)$, for each $A \in \Phi_{I,\mathscr{H}}$. Then we have that $\mathfrak{M}/Q = (W/Q, \{R_i^{W/Q}\}_{i\in I}, V^{W/Q})$ is a fuzzy Kripke model, which is called the *afterset fuzzy Kripke model* of $\mathfrak{M}$ w.r.t. Q. If E is a fuzzy equivalence, then $\mathfrak{M}/E$ will be called the *factor fuzzy Kripke model* of $\mathfrak{M}$ w.r.t. E.

In the same way, using foresets instead of aftersets, we can define the *foreset fuzzy Kripke model* of $\mathfrak{M}$ w.r.t. Q. However, this does not give anything new because the afterset and the foreset fuzzy Kripke models of $\mathfrak{M}$ w.r.t. Q are isomorphic.

The following theorem can be regarded as a counterpart of the well-known Second Isomorphism Theorem from algebra (cf. [8] §2.6). The proof of this theorem can be obtained directly from the proof of Theorem 3.3 from [55], so it is omitted.

Theorem 7.2. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model and let P and Q be fuzzy quasi-orders on $\mathfrak{M}$ such that $P \leqslant Q$. Then a fuzzy relation Q/P on W/P defined by*

$$Q/P(P_w, P_u) = Q(w,u), \qquad \textit{for all } w, u \in W, \tag{58}$$

is a fuzzy quasi-order on W/P and fuzzy Kripke models $\mathfrak{M}/Q$ and $(\mathfrak{M}/P)/(Q/P)$ are isomorphic.

Remark 7.3. For any given fuzzy quasi-order Q on a fuzzy Kripke model $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$, the rule $w \mapsto Q_w$ defines a surjective function of W onto W/Q. This means that the afterset fuzzy Kripke model $\mathfrak{M}/Q$ has smaller or equal number of worlds than the fuzzy Kripke model $\mathfrak{M}$.

Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model. It is easy to see that for any $\theta \in \{fs, bs, fb, bb, fbb, bfb, rb\}$ the equality relation on W satisfies (θ-1), (θ-2) and (θ-3), i.e., it is a θ-simulation/bisimulation on $\mathfrak{M}$ (between $\mathfrak{M}$ and itself). It follows that the union of all θ-simulations/bisimulations on $\mathfrak{M}$ is non-empty, and it is also a θ-simulation/bisimulation, i.e., it is *the greatest θ-simulation/bisimulation on* $\mathfrak{M}$. We can also easily verify that the greatest θ-simulation (for $\theta \in \{fs, bs\}$) and the greatest θ-bisimulation (for $\theta \in \{fbb, bfb\}$) are fuzzy quasi-orders, while the greatest θ-bisimulation (for $\theta \in \{fb, bb, rb\}$) is a fuzzy equivalence. This emphasizes the importance of studying θ-simulations that are fuzzy quasi-orders, which will be called *θ-simulation fuzzy quasi-orders* (for $\theta \in \{fs, bs\}$), as well as of studying θ-bisimulations that are fuzzy equivalences, which will be called *θ-bisimulation fuzzy equivalences* (for $\theta \in \{fb, bb, rb\}$).

In the following text, special attention will be paid to forward and backward simulation fuzzy quasi-orders and forward and backward bisimulation fuzzy equivalences on a Kripke model.

The following two theorems establish connections between a model $\mathfrak{M}$ and its afterset model $\mathfrak{M}/Q$, that can be regarded as counterparts of the well-known First Isomorphism Theorem from algebra.

Theorem 7.4. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model, let Q be a fuzzy quasi-order on W, and let $\mathfrak{M}/Q = (W/Q, \{R_i^{W/Q}\}_{i\in I}, V^{W/Q})$ be the afterset fuzzy Kripke model with respect to Q. Then the following is valid:*

(A) *A fuzzy relation $\varphi \in \mathscr{R}(W, W/Q)$ defined by*

$$\varphi(u, Q_v) = Q(u, v), \qquad \text{for all } u, v \in W, \tag{59}$$

is a backward simulation between $\mathfrak{M}$ and $\mathfrak{M}/Q$.

(B) *If Q is a forward simulation on $\mathfrak{M}$, then φ is a forward simulation between $\mathfrak{M}$ and $\mathfrak{M}/Q$.*

Proof. (A) We first notice that φ is a well-defined function, in the sense that for all $u, v_1, v_2 \in W$ such that $Q_{v_1} = Q_{v_2}$ we have that $\varphi(u, Q_{v_1}) = \varphi(u, Q_{v_2})$. Indeed, according to Theorem 7.1 we have that $Q^{v_1} = Q^{v_2}$ and

$$\varphi(u, Q_{v_1}) = Q(u, v_1) = Q^{v_1}(u) = Q^{v_2}(u) = Q(u, v_2) = \varphi(u, Q_{v_2}).$$

Further, for arbitrary $u, v \in W$, $p \in PV$ and $i \in I$

$$V_p(u) \leqslant (Q \circ V_p \circ Q)(u) = \bigvee_{w\in W} Q(u, w) \wedge V_p \circ Q(w) = \bigvee_{w\in W} \varphi(u, Q_w) \wedge V_p^{W/Q}(Q_w) = (\varphi \circ V_p^{W/Q})(u), \tag{60}$$

$$\begin{aligned}(R_i \circ \varphi)(u, Q_v) &= \bigvee_{w\in W} R_i(u, w) \wedge \varphi(w, Q_v) = \bigvee_{w\in W} R_i(u, w) \wedge Q(w, v) = (R_i \circ Q)(u, v)\\ &\leqslant (Q \circ Q \circ R_i \circ Q)(u, v) = \bigvee_{w\in W} Q(u, w) \wedge (Q \circ R_i \circ Q)(w, v)\\ &= \bigvee_{Q_w\in W/Q} \varphi(u, Q_w) \wedge R_i^{W/Q}(Q_w, Q_v) = (\varphi \circ R_i^{W/Q})(u, Q_v),\end{aligned} \tag{61}$$

$$(V_p \circ \varphi)(Q_v) = \bigvee_{w\in W} V_p(w) \wedge \varphi(w, Q_v) = \bigvee_{w\in W} V_p(w) \wedge Q(w, v) = (V_p \circ Q)(v) = V_p^{W/Q}(Q_v). \tag{62}$$

Note that the inequalities in (60) and (61) follow from the fact that $\alpha \leqslant \alpha \circ S$ and $\alpha \leqslant S \circ \alpha$, for each fuzzy relation or fuzzy set α, and each reflexive fuzzy relation S on a given set. Therefore, φ is a backward simulation between $\mathfrak{M}$ and $\mathfrak{M}/Q$.

(B) For arbitrary $u, v \in W$, $p \in PV$ and $i \in I$ we have

$$(V_p^{W/Q} \circ \varphi^{-1})(u) = \bigvee_{Q_w \in W/Q} V_p^{W/Q}(Q_w) \wedge \varphi^{-1}(Q_w, u) = \bigvee_{w \in W} (V_p \circ Q)(w) \wedge Q^{-1}(w, u)$$

$$= (V_p \circ Q \circ Q^{-1})(u) \geqslant V_p(u) \qquad \text{(due to the transitivity of } Q \circ Q^{-1}), \tag{63}$$

$$(\varphi^{-1} \circ R_i)(Q_v, u) = \bigvee_{w \in W} \varphi^{-1}(Q_v, w) \wedge R_i(w, u) = \bigvee_{w \in W} Q^{-1}(v, w) \wedge R_i(w, u) = (Q^{-1} \circ R_i)(v, u), \tag{64}$$

$$(R_i^{W/Q} \circ \varphi^{-1})(Q_v, u) = \bigvee_{Q_w \in W/Q} R_i^{W/Q}(Q_v, Q_w) \wedge \varphi^{-1}(Q_w, u)$$

$$= \bigvee_{w \in W} (Q \circ R_i \circ Q)(v, w) \wedge Q^{-1}(w, u) = (Q \circ R_i \circ Q \circ Q^{-1})(v, u), \tag{65}$$

$$(\varphi^{-1} \circ V_p)(Q_v) = \bigvee_{w \in W} \varphi^{-1}(Q_v, w) \wedge V_p(w) = \bigvee_{w \in W} Q^{-1}(v, w) \wedge V_p(w)$$

$$= (Q^{-1} \circ V_p)(v) = (V_p \circ Q)(v) = V_p^{W/Q}(Q_v). \tag{66}$$

From (63) and (66) it immediately follows that φ satisfies (fs-1) and (fs-3). With the additional assumption that Q is a forward simulation, and due to reflexivity of Q, (64) and (65) yield

$$(\varphi^{-1} \circ R_i)(u, Q_v) = (Q^{-1} \circ R_i)(u, v) \leqslant (R_i \circ Q^{-1})(u, v) \leqslant (Q \circ R_i \circ Q \circ Q^{-1})(u, v) = (R_i^{W/Q} \circ \varphi^{-1})(u, Q_v).$$

Therefore, φ satisfies (fs-2), so it is a forward simulation. □

Remark 7.5. If we define $V_p^{W/Q} \in \mathscr{F}(W/Q)$ and $\varphi \in \mathscr{R}(W, W/Q)$ by

$$V_p^{W/Q}(Q_v) = (Q \circ V_p)(v), \qquad \varphi(u, Q_v) = Q^{-1}(u, v) = Q(v, u), \qquad \text{for all } u, v \in W,\ p \in PV, \tag{67}$$

then without any additional assumption we have that φ is a forward simulation between $\mathfrak{M}$ and $\mathfrak{M}/Q$, and with the additional assumption that Q^{-1} is a backward simulation on $\mathfrak{M}$ we get that φ is a backward simulation between $\mathfrak{M}$ and $\mathfrak{M}/Q$. This can be easily shown, in a similar way as in the proof of Theorem 7.4.

Theorem 7.6. *Let $\mathfrak{M} = (W, \{R_i\}_{i \in I}, V)$ be a fuzzy Kripke model, let E be a fuzzy equivalence on W, and let $\mathfrak{M}/E = (W/E, \{R_i^{W/E}\}_{i \in I}, V^{W/E})$ be the afterset fuzzy Kripke model with respect to E.*

(A) *A fuzzy relation $\varphi \in \mathscr{R}(W, W/E)$ defined by*

$$\varphi(u, E_v) = E(u, v), \qquad \text{for all } u, v \in W, \tag{68}$$

is both a forward and a backward simulation between $\mathfrak{M}$ and $\mathfrak{M}/E$.

(B) *The following conditions are equivalent:*

(i) *E is a forward (resp. backward) bisimulation fuzzy equivalence on $\mathfrak{M}$;*

(ii) *φ is a forward (resp. backward) bisimulation between $\mathfrak{M}$ and $\mathfrak{M}/E$;*

(iii) *φ is a backward-forward (resp. forward-backward) bisimulation between $\mathfrak{M}$ and $\mathfrak{M}/E$.*

Proof. (A) Since $E = E^{-1}$ and $E \circ V_p = V_p \circ E$, for each $p \in PV$, it follows directly from Theorem 7.4 and Remark 7.5 that φ is both a forward and a backward simulation.

(B) We will prove only the assertions that refer to forward bisimulations. Claims concerning backward bisimulations can be proved in a similar way.

(i)⇒(ii) and (i)⇒(iii). Suppose that E is a forward bisimulation. This means that $E \circ R_i \leqslant R_i \circ E$ and $E \circ V_p = V_p \circ E \leqslant V_p$, for all $i \in I$ and $p \in PV$. According to (A) we have that φ is a forward and backward simulation, so it remains to prove that φ^{-1} is a forward simulation.

For arbitrary $u, v \in W$, $p \in PV$ and $i \in I$ we have

$$V_p^{W/E}(E_v) = (V_p \circ E)(v) = \bigvee_{w\in W} V_p(w) \wedge E(w,v) = \bigvee_{w\in W} V_p(w) \wedge \varphi(w,E_v) = (V_p \circ \varphi)(v), \tag{69}$$

$$\begin{aligned}(\varphi \circ R_i^{W/E})(u,E_v) &= \bigvee_{E_w\in W/E} \varphi(u,E_w) \circ R_i^{W/E}(E_w,E_v) = \bigvee_{w\in W} E(u,w) \wedge (E \circ R_i \circ E)(w,v)\\ &= (E \circ E \circ R_i \circ E)(u,v) = (E \circ R_i \circ E)(u,v) \leqslant (R_i \circ E \circ E)(u,v) = (R_i \circ E)(u,v)\\ &= \bigvee_{w\in W} R_i(u,w) \wedge E(w,v) = \bigvee_{w\in W} R_i(u,w) \wedge \varphi(w,E_v) = (R_i \circ \varphi)(u,E_v),\end{aligned} \tag{70}$$

$$\begin{aligned}(\varphi \circ V_p^{W/E})(u) &= \bigvee_{E_w\in W/E} \varphi(u,E_w) \wedge V_p^{W/E}(E_w) = \bigvee_{w\in W} E(u,w) \wedge (V_p \circ E)(w)\\ &= (E \circ E \circ V_p \circ E)(u) = (E \circ V_p)(u) \leqslant V_p(u).\end{aligned} \tag{71}$$

Thus, φ^{-1} is a forward simulation, whence we get that φ is a forward bisimulation, and also a backward-forward bisimulation. In the same way we prove the assertion that refers to backward bisimulations.

(ii)⇒(i) and (iii)⇒(i). Suppose that φ is a forward bisimulation or a backward-forward bisimulation, i.e., that φ^{-1} is a forward simulation. According to (71) we get $E \circ V_p = \varphi \circ V_p^{W/E} \leqslant V_p$, for each $p \in PV$, and according to (70) we get

$$(E \circ R_i \circ E)(u,v) = (\varphi \circ R_i^{W/E})(u,E_v) \leqslant (R_i \circ \varphi)(u,E_v) = (R_i \circ E)(u,v),$$

for all $u, v \in W$ and $i \in I$. From there we conclude that $E \circ R_i \circ E \leqslant R_i \circ E$, which yields

$$E \circ R_i \leqslant E \circ R_i \circ E \leqslant R_i \circ E.$$

Therefore, E is a forward bisimulation. □

The following theorems provide conditions under which the factor Kripke models $\mathfrak{M}$ and $\mathfrak{M}/E$ are $\Phi_{I,\mathscr{H}}$-equivalent, $\Phi^+_{I,\mathscr{H}}$-equivalent and $\Phi^-_{I,\mathscr{H}}$-equivalent, respectively. They are proven under the assumption that the underlying complete Heyting algebra $\mathscr{H}$ is linearly ordered.

Theorem 7.7. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a image-finite fuzzy Kripke model over a linearly ordered Heyting algebra, let E be a forward bisimulation fuzzy equivalence on $\mathfrak{M}$, and $\mathfrak{M}/E = (W/E, \{R_i^{W/E}\}_{i\in I}, V^{W/E})$ be the factor fuzzy Kripke model with respect to E. A fuzzy relation $\varphi \in \mathscr{R}(W, W/E)$ defined by*

$$\varphi(u,E_v) = E(u,v), \text{for all } u,v \in W, \tag{72}$$

is a forward bisimulation and the following is true:

$$\varphi(u,Q_v) \leqslant \bigwedge_{A\in\Phi^+_{I,\mathscr{H}}} V_A(u) \leftrightarrow V_A^{W/E}(E_v), \text{ for all } u,v \in W. \tag{73}$$

Consequently, $\mathfrak{M}$ and $\mathfrak{M}/E$ are $\Phi^+_{I,\mathscr{H}}$-equivalent fuzzy Kripke models.

Proof. The fact that φ is a forward bisimulation follows from Theorem 7.6. By induction on complexity of a formula $A \in \Phi^+_{I,\mathscr{H}}$ we will prove that

$$\varphi(u,E_v) \leqslant V_A(u) \leftrightarrow V_A^{W/E}(E_v), \text{ for all } u,v \in W \text{ and every } A \in \Phi^+_{I,\mathscr{H}}. \tag{74}$$

Induction basis: If $A = p \in PV$, then from the fact that φ is forward bisimulation we have

$$\varphi^{-1} \circ V_p \leqslant V_p^{W/E}, \qquad \varphi \circ V_p^{W/E} \leqslant V_p,$$

and according to Lemma 4.3, it follows

$$\varphi^{-1} \leqslant V_p^{W/E}/V_p = (V_p\backslash V_p^{W/E})^{-1}, \qquad \varphi \leqslant V_p/V_p^{W/E},$$

whence

$$\varphi \leqslant V_p\backslash V_p^{W/E}, \qquad \varphi \leqslant V_p/V_p^{W/E},$$

i.e.,

$$\varphi \leqslant (V_p\backslash V_p^{W/E}) \wedge (V_p/V_p^{W/E}) = V_p \leftrightarrow V_p^{W/E}.$$

Therefore, (74) holds for any propositional variable p, and it trivially holds for any truth constant $\bar{t}$.

Induction step: a) Let $A = B \wedge C$ and let (74) hold for B and C, i.e., $\varphi \leqslant V_B \leftrightarrow V_B^{W/E}$ and $\varphi \leqslant V_C \leftrightarrow V_C^{W/E}$. This yields

$$\varphi \leqslant (V_B \leftrightarrow V_B^{W/E}) \wedge (V_C \leftrightarrow V_C^{W/E}).$$

Using the property of Heyting algebras $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \leqslant (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$, we get

$$\begin{aligned}\varphi(u, E_v) &\leqslant (V(u,B) \leftrightarrow V_B^{W/E}(E_v)) \wedge (V(u,C) \leftrightarrow V_C^{W/E}(E_v))\\ &\leqslant (V(u,B) \wedge V(u,C)) \leftrightarrow (V_B^{W/E}(E_v) \wedge V_C^{W/E}(E_v))\\ &= V(u, B \wedge C) \leftrightarrow V_{B\wedge C}^{W/E}(E_v)\\ &= V_A(u) \leftrightarrow V_A^{W/E}(E_v),\end{aligned}$$

for all $u \in W$ and $E_v \in W/E$, and we conclude that (74) holds for $A = B \wedge C$.

b) Let A be of the form $B \to C$ and let (74) hold for B and C. In a similar way as a), using the property of Heyting algebras $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \leqslant (x_1 \to x_2) \leftrightarrow (y_1 \to y_2)$, we prove that (74) also holds for A.

c) Let $A = \Diamond_i B$ and (74) let hold for B, i.e.,

$$\varphi \leqslant V_B \leftrightarrow V_B^{W/E} = (V_B\backslash V_B^{W/E}) \wedge (V_B/V_B^{W/E}).$$

Then it follows that

$$\varphi \leqslant V_B\backslash V_B^{W/E} \quad \text{and} \quad \varphi^{-1} \leqslant \left(V_B\backslash V_B^{W/E}\right)^{-1} = V_B^{W/E}/V_B,$$

and according to Lemma 4.3 we finally get $\varphi^{-1} \circ V_B \leqslant V_B^{W/E}$. Now we have

$$\begin{aligned}\varphi^{-1} \circ V_A = \varphi^{-1} \circ R_i \circ V_B &\leqslant R_i^{W/E} \circ \varphi^{-1} \circ V_B \qquad \text{according to } (fb\text{-}2)\\ &\leqslant R_i^{W/E} \circ V_B^{W/E} = V_A^{W/E},\end{aligned}$$

for every $i \in I$. Hence, from $\varphi^{-1} \circ V_A \leqslant V_A^{W/E}$ we can conclude that $\varphi^{-1} \leqslant V_A^{W/E}/V_A = (V_A\backslash V_A^{W/E})^{-1}$, whence $\varphi \leqslant V_A\backslash V_A^{W/E}$. In a similar way we can conclude that $\varphi \leqslant V_A/V_A^{W/E}$, which means that

$$\varphi \leqslant (V_A\backslash V_A^{W/E}) \wedge (V_A/V_A^{W/E}) = V_A \leftrightarrow V_A^{W/E}.$$

Therefore, we have proved that (74) holds for $A = \Diamond_i B$.

d) Suppose that $A = \Box_i B$ and (74) holds for B. In a similar way as in c), from $\varphi \leqslant V_B \leftrightarrow V_B^{W/E}$, we conclude

$$\varphi^{-1} \circ V_B \leqslant V_B^{W/E}, \qquad \varphi \circ V_B^{W/E} \leqslant V_B.$$

Since underlying structure is linearly ordered, values $\varphi(u, Q_v) = \varphi^{-1}(E_v, u)$, $V_A(u)$ and $V_A^{W/E}(E_v)$ can be compared with each other for every $u \in W$, $E_v \in W/E$, therefore, case analysis can be used.

If $\varphi^{-1}(Q_v, u) \leqslant V_A(u) \wedge V_A^{W/E}(E_v)$ and $V_A(u) \neq V_A^{W/E}(E_v)$, then

$$\varphi(u, E_v) = \varphi^{-1}(E_v, u) \leqslant V_A(u) \wedge V_A^{W/E}(E_v) = V_A(u) \leftrightarrow V_A^{W/E}(E_v).$$

In case $V_A(u) = V_A^{W/E}(E_v)$ we have that $V_A(u) \leftrightarrow V_A^{W/E}(E_v) = 1$, which gives $\varphi(u, E_v) \leqslant V_A(u) \leftrightarrow V_A^{W/E}(E_v)$.

Hence, we only need to consider case where $\varphi^{-1}(E_v, u) > V_A(u) \wedge V_A^{W/E}(E_v)$. Without loss of generality, we can assume that $\varphi^{-1}(E_v, u) > V_A(u)$, and then we have:

$$\begin{aligned} V_A(u) &= \varphi^{-1}(E_v, u) \wedge V_A(u) \\ &= \varphi^{-1}(E_v, u) \wedge \bigwedge_{w \in W} (R_i(u, w) \to V_B(w)) \\ &= \bigwedge_{w \in W} \Big[\varphi^{-1}(E_v, u) \wedge \Big(R_i(u, w) \to V_B(w)\Big)\Big] \qquad \text{(property (5))} \\ &= \bigwedge_{w \in W} \Big[\varphi^{-1}(E_v, u) \wedge \Big(\varphi^{-1}(E_v, u) \wedge R_i(u, w) \to V_B(w)\Big)\Big] \qquad \text{(property (9))} \\ &= \varphi^{-1}(E_v, u) \wedge \bigwedge_{w \in W} \Big[\varphi^{-1}(E_v, u) \wedge R_i(u, w) \to V_B(w)\Big] \qquad \text{(property (5))} \end{aligned} \tag{75}$$

Since the relation φ is a forward bisimulation, it satisfies (*fb-2*), i.e.

$$\varphi^{-1} \circ R_i \leqslant R_i^{W/E} \circ \varphi^{-1}, \qquad \text{for every } i \in I.$$

Next, since $R_i^{W/E}$ is image-finite, for any $w \in W$ we can find $E_z \in W/E$ such that

$$\varphi^{-1}(E_v, u) \wedge R_i(u, w) \leqslant R_i^{W/E}(E_v, E_z) \wedge \varphi^{-1}(E_z, w),$$

and it follows

$$\Big(\varphi^{-1}(E_v, u) \wedge R_i(u, w)\Big) \to V_B(w) \geqslant \Big(\varphi^{-1}(E_z, w) \wedge R_i^{W/E}(E_v, E_z)\Big) \to V_B(w).$$

Now, two cases need to be analyzed. First, if $V_B(w) = V_B^{W/E}(E_z)$, then

$$\Big(\varphi^{-1}(E_z, w) \wedge R_i^{W/E}(E_v, E_z)\Big) \to V_B(w) \geqslant R_i^{W/E}(E_v, E_z) \to V_B(w) = R_i^{W/E}(E_v, E_z) \to V_B^{W/E}(E_z).$$

On the other hand, if $V_B(w) \neq V_B^{W/E}(E_z)$, then by the induction hypothesis we have that

$$\varphi^{-1}(E_z, w) \leqslant (V_B(w) \leftrightarrow V_B^{W/E}(E_z)) \leqslant V_B(w).$$

Thus,

$$\Big(\varphi^{-1}(E_z, w) \wedge R_i^{W/E}(E_v, E_z)\Big) \to V_B(w) = 1 \geqslant R_i^{W/E}(E_v, E_z) \to V_B^{W/E}(E_z).$$

In both cases, we have shown that for any $w \in W$, we can find E_z such that

$$\Big(\varphi^{-1}(E_v, u) \wedge R_i(u, w)\Big) \to V_B(w) \geqslant R_i^{W/E}(E_v, E_z) \to V_B^{W/E}(E_z).$$

Therefore,

$$\bigwedge_{w \in W} \Big(\varphi^{-1}(E_v, u) \wedge R_i(u, w)\Big) \to V_B(w) \geqslant \bigwedge_{z \in W} R_i^{W/E}(E_v, E_z) \to V_B^{W/E}(E_z) = V_A^{W/E}(E_v)$$

and using (75) we conclude:

$$V_A(u) \geqslant \varphi^{-1}(E_v, u) \wedge V_A^{W/E}(E_v).$$

Because of the assumption that $\varphi^{-1}(E_v, u) > V_A(u)$, we have

$$V_A(u) \geqslant V_A^{W/E}(E_v) \text{ and } \varphi^{-1}(E_v, u) > V_A^{W/E}(E_v).$$

Since $\varphi^{-1}(E_v, u) > V_A^{W/E}(E_v)$, by the same reasoning we can prove that $V_A^{W/E}(E_v) \geqslant V_A(u)$. Hence, we have $V_A(u) = V_A^{W/E}(E_z)$, and since $\varphi(u, E_v) = \varphi^{-1}(E_v, u)$ it follows

$$\varphi(u, E_v) \leqslant V_A(u) \leftrightarrow V_A^{W/E}(E_v) = 1$$

when $\varphi^{-1}(E_v, u) > V_A(u) \wedge V_A^{W/E}(E_v)$. This completes the proof of the theorem. □

In a similar way we prove the following two theorems.

Theorem 7.8. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a domain-finite fuzzy Kripke model over a linearly ordered Heyting algebra, let E be a backward bisimulation fuzzy equivalence on W, and let $\mathfrak{M}/E = (W/E, \{R_i^{W/E}\}_{i\in I}, V^{W/E})$ be the factor fuzzy Kripke model with respect to E. A fuzzy relation $\varphi \in \mathscr{R}(W, W/E)$ defined by*

$$\varphi(u, E_v) = E(u, v), \text{for all } u, v \in W, \tag{76}$$

is a backward bisimulation and the following is true:

$$\varphi(u, E_v) \leqslant \bigwedge_{A\in\Phi^{-}_{I,\mathscr{H}}} V_A(u) \leftrightarrow V_A^{W/E}(E_v). \tag{77}$$

Consequently, $\mathfrak{M}$ and $\mathfrak{M}/E$ are $\Phi^{-}_{I,\mathscr{H}}$-equivalent fuzzy Kripke models.

Proof. This follows from the previous theorem since a backward bisimulation between two models is a forward bisimulation between the reverse models. □

Theorem 7.9. *Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a degree-finite fuzzy Kripke model over a linearly ordered Heyting algebra, let E be a regular bisimulation fuzzy equivalence on W, and let $\mathfrak{M}/E = (W/E, \{R_i^{W/E}\}_{i\in I}, V^{W/E})$ be the factor fuzzy Kripke model with respect to E. A fuzzy relation $\varphi \in \mathscr{R}(W, W/E)$ defined by*

$$\varphi(u, E_v) = E(u, v), \text{for all } u, v \in W, \tag{78}$$

is a regular bisimulation and the following is true:

$$\varphi(u, E_v) \leqslant \bigwedge_{A\in\Phi_{I,\mathscr{H}}} V_A(u) \leftrightarrow V_A^{W/E}(E_v). \tag{79}$$

Consequently, $\mathfrak{M}$ and $\mathfrak{M}/E$ are $\Phi_{I,\mathscr{H}}$-equivalent fuzzy Kripke models.

Proof. This follows immediately from the previous two theorems. □

8. Computational examples for reductions of fuzzy Kripke models

In this section we provide examples which demonstrate the application of theorems from the previous section in the state reduction of fuzzy Kripke models. As in Section 6, several examples are based on the standard Gödel modal logic over the real unit interval $[0, 1]$, while the last example is on the Boolean algebra.

As we already said in the previous section, the greatest bisimulation of type $\theta \in \{fb, bb, rb\}$ on a fuzzy Kripke model $\mathfrak{M}$ is a fuzzy equivalence, which will be denoted by E^θ, while the greatest bisimulation of type $\theta \in \{fbb, bfb\}$ on $\mathfrak{M}$ is a fuzzy quasi-order, which will be denoted by Q^θ.

The following example illustrates a situation where E^{fb} reduces the number of worlds of the model, but none of the other bisimulations do so.

Example 8.1. Let $\mathfrak{M} = (W, \{R_1\}, V)$ be the fuzzy Kripke model from Example 6.1, i.e., let the fuzzy relation R_1 and fuzzy sets V_p, V_q, be represented by the following fuzzy matrix and vectors:

$$R_1 = \begin{bmatrix} 1 & 0 & 0.9 \\ 1 & 0.1 & 0.5 \\ 1 & 0 & 1 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 1 \\ 0.2 \\ 1 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 1 \\ 0.7 \\ 1 \end{bmatrix}. \tag{80}$$

Using algorithms for computing the greatest bisimulations on the fuzzy Kripke model $\mathfrak{M}$ we have:

$$E^{fb} = \begin{bmatrix} 1 & 0.2 & 1 \\ 0.2 & 1 & 0.2 \\ 1 & 0.2 & 1 \end{bmatrix}, \qquad E^{bb} = \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.1 \\ 0.5 & 0.1 & 1 \end{bmatrix}, \qquad E^{rb} = \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.1 \\ 0.5 & 0.1 & 1 \end{bmatrix},$$

$$Q^{fbb} = \begin{bmatrix} 1 & 0.2 & 1 \\ 0.1 & 1 & 0.1 \\ 0.5 & 0.2 & 1 \end{bmatrix}, \qquad Q^{bfb} = \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.2 & 1 & 0.2 \\ 1 & 0.1 & 1 \end{bmatrix}.$$

Hence, E^{fb} is a forward bisimulation fuzzy quasi-order with two different aftersets, and we have:

$$E^{fb} \circ R_1 \circ E^{fb} = \begin{bmatrix} 1 & 0.2 & 1 \\ 1 & 0.2 & 1 \\ 1 & 0.2 & 1 \end{bmatrix}, \qquad V_p \circ E^{fb} = V_p = \begin{bmatrix} 1 \\ 0.2 \\ 1 \end{bmatrix}, \qquad V_q \circ E^{fb} = V_q = \begin{bmatrix} 1 \\ 0.7 \\ 1 \end{bmatrix}.$$

Now, from (56) and (57) we get the related afterset model $\mathfrak{M}/E^{fb} = (W/E^{fb}, \{R_1^{W/E^{fb}}\}, V^{W/E^{fb}})$ where

$$R_1^{W/E^{fb}} = \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix}, \qquad V_p^{W/E^{fb}} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \qquad V_q^{W/E^{fb}} = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix},$$

i.e., we get the model with a smaller number of worlds isomorphic to the model $\mathfrak{M}'$ from Example 6.1. According to Theorem 7.7 we have that the models $\mathfrak{M}$ and $\mathfrak{M}/E^{fb}$ are $\Phi^{+}_{I,\mathscr{H}}$-equivalent.

On the other hand, E^{bb}, E^{rb}, Q^{fbb} and Q^{bfb} are fuzzy equivalences and fuzzy quasi-orders whose equivalence classes and aftersets are all different (such fuzzy equivalences and fuzzy quasi-orders are called fuzzy equalities and fuzzy orders, respectively). For that reason, they cannot reduce the number of worlds of the model.

What we can also conclude from there is that the greatest forward-backward bisimulation and the greatest backward-forward bisimulation are not necessarily fuzzy equivalences.

If we consider the reverse model $\mathfrak{M}^{-1} = (W, \{R_1\}^{-1}, V)$, then we have that the greatest backward bisimulation on $\mathfrak{M}^{-1}$ reduces the number of worlds of this model, and in this case the related afterset model is $\Phi^{-}_{I,\mathscr{H}}$-equivalent to $\mathfrak{M}^{-1}$, but other types of bisimulations on $\mathfrak{M}^{-1}$ cannot reduce its number of worlds.

Example 8.2. Let $\mathfrak{M} = (W, \{R_1\}, V)$ be the fuzzy Kripke model from Example 6.2, i.e., let the fuzzy relation R_1 and fuzzy sets V_p and V_q be given as follows:

$$R_1 = \begin{bmatrix} 0.9 & 1 & 1 \\ 0.4 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 0.9 \\ 0.4 \\ 0.4 \end{bmatrix}. \tag{81}$$

Using algorithms for computing the greatest bisimulations on the fuzzy Kripke model $\mathfrak{M}$ we have:

$$E^{fb} = E^{bb} = E^{rb} = Q^{fbb} = Q^{bfb} = \begin{bmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 1 \\ 0.3 & 1 & 1 \end{bmatrix}.$$

Let us denote all these fuzzy equivalences by E. Then, we have:

$$E \circ R_1 \circ E = \begin{bmatrix} 0.9 & 1 & 1 \\ 0.4 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}, \qquad V_p \circ E = V_p = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}, \qquad V_q \circ E = V_q = \begin{bmatrix} 0.9 \\ 0.4 \\ 0.4 \end{bmatrix},$$

and from (56) and (57) we get the related factor fuzzy Kripke model $\mathfrak{M}/E = (W/E, \{R_1^{W/E}\}, V^{W/E})$, where

$$R_1^{W/E} = \begin{bmatrix} 0.9 & 1 \\ 0.4 & 0.5 \end{bmatrix}, \qquad V_p^{W/E} = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \qquad V_q^{W/E} = \begin{bmatrix} 0.9 \\ 0.4 \end{bmatrix},$$

i.e., the model $\mathfrak{M}'$ from Example 6.2 with smaller number of states.

Also, according to Theorem 7.9, the models $\mathfrak{M}$ and $\mathfrak{M}/E$ are $\Phi_{I,\mathscr{H}}$-equivalent. Clearly, these models are also $\Phi^{+}_{I,\mathscr{H}}$-equivalent and $\Phi^{-}_{I,\mathscr{H}}$-equivalent.

The next example illustrates a situation where no type of bisimulation can reduce the number of worlds of a model.

Example 8.3. Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model over the Gödel structure $[0,1]$, where $W = \{u, v, w\}$ and set $I = \{1\}$. Fuzzy relation R_1 and fuzzy sets V_p, V_q, are represented by the following fuzzy matrix and vectors:

$$R_1 = \begin{bmatrix} 0.7 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.9 \\ 1 & 0.4 & 0.8 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.5 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 0.3 \\ 0.7 \\ 0.7 \end{bmatrix}. \tag{82}$$

Using the algorithms for computing the greatest bisimulations on the fuzzy Kripke model $\mathfrak{M}$ we have:

$$E^{fb} = E^{bb} = E^{rb} = Q^{fbb} = Q^{bfb} = \begin{bmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{bmatrix}.$$

Clearly, this fuzzy equivalence is a fuzzy equality, i.e., all its equivalence classes are different. This means that the number of worlds of the related factor fuzzy Kripke model is the same as the number of worlds of the original fuzzy Kripke model $\mathfrak{M}$.

The following example illustrates a situation where all three types of bisimulation fuzzy equivalences can reduce the number of worlds of a fuzzy Kripke model, but provide factor fuzzy Kripke models with different number of worlds.

Example 8.4. Let $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ be a fuzzy Kripke model over the Gödel structure $[0,1]$, where $W = \{u, v, w, z\}$ and set $I = \{1\}$. Fuzzy relation R_1 and fuzzy sets V_p and V_q are represented by the following fuzzy matrix and vectors:

$$R_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0.1 & 0 & 0 \\ 0.9 & 0.5 & 1 & 1 \\ 0.9 & 0.5 & 1 & 1 \end{bmatrix}, \qquad V_p = \begin{bmatrix} 1 \\ 0.2 \\ 1 \\ 1 \end{bmatrix}, \qquad V_q = \begin{bmatrix} 1 \\ 0.7 \\ 1 \\ 1 \end{bmatrix}.$$

Using algorithms for computing the greatest bisimulations on the fuzzy Kripke model $\mathfrak{M}$ we have:

$$E^{fb} = \begin{bmatrix} 1 & 0.1 & 0.5 & 0.5 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.5 & 0.1 & 1 & 1 \\ 0.5 & 0.1 & 1 & 1 \end{bmatrix}, \qquad E^{bb} = \begin{bmatrix} 1 & 0.2 & 1 & 1 \\ 0.2 & 1 & 0.2 & 0.2 \\ 1 & 0.2 & 1 & 1 \\ 1 & 0.2 & 1 & 1 \end{bmatrix}, \qquad E^{rb} = \begin{bmatrix} 1 & 0.1 & 0.5 & 0.5 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.5 & 0.1 & 1 & 1 \\ 0.5 & 0.1 & 1 & 1 \end{bmatrix},$$

$$Q^{fbb} = \begin{bmatrix} 1 & 0.1 & 0.5 & 0.5 \\ 0.2 & 1 & 0.2 & 0.2 \\ 1 & 0.1 & 1 & 1 \\ 1 & 0.1 & 1 & 1 \end{bmatrix}, \qquad Q^{bfb} = \begin{bmatrix} 1 & 0.2 & 1 & 1 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.5 & 0.2 & 1 & 1 \\ 0.5 & 0.2 & 1 & 1 \end{bmatrix}.$$

Clearly, E^{fb} and E^{rb} provide factor fuzzy Kripke models with 3 worlds, whereas E^{bb} provides the factor fuzzy Kripke model with 2 worlds. However, the factor model with respect to E^{fb} cannot be further reduced by the greatest forward bisimulation, but it can be easily verified that it can be reduced by the backward bisimulation, which again provides a factor model with 2 worlds.

The last example illustrates a situation where the fuzzy Kripke model is over partially ordered Boolean algebra and hence none of the theorems 7.7, 7.8 and 7.9 do not hold. Still, in this example E^{fb} reduces the number of worlds of the model, but none of the other bisimulations do so.

Example 8.5. Let $\mathfrak{M} = (W, \{R_1\}, V)$ be the fuzzy Kripke model from Example 6.5, i.e., let the fuzzy relation R_1 and fuzzy sets V_p, V_q, be represented by the following fuzzy matrix and vectors:

$$R_1 = \begin{bmatrix} A & \emptyset & A \\ \{a,b\} & \{b,c\} & \{a,c\} \\ A & \emptyset & A \end{bmatrix}, \qquad V_p = \begin{bmatrix} A \\ \{b\} \\ A \end{bmatrix}, \qquad V_q = \begin{bmatrix} A \\ \{b,c\} \\ A \end{bmatrix}. \tag{83}$$

Using algorithms for computing the greatest bisimulations on the fuzzy Kripke model $\mathfrak{M}$ we have:

$$E^{fb} = \begin{bmatrix} A & \{b\} & A \\ \{b\} & A & \{b\} \\ A & \{b\} & A \end{bmatrix}, \qquad E^{bb} = \begin{bmatrix} A & \{b\} & \{a,b\} \\ \{b\} & A & \{b\} \\ \{a,b\} & \{b\} & A \end{bmatrix}, \qquad E^{rb} = \begin{bmatrix} A & \{b\} & \{a,b\} \\ \{b\} & A & \{b\} \\ \{a,b\} & \{b\} & A \end{bmatrix},$$

$$Q^{fbb} = \begin{bmatrix} A & \{b\} & \{a,b\} \\ \{b\} & A & \{b\} \\ A & \{b\} & A \end{bmatrix}, \qquad Q^{bfb} = \begin{bmatrix} A & \{b\} & A \\ \{b\} & A & \{b\} \\ \{a,b\} & \{b\} & A \end{bmatrix}.$$

Hence, E^{fb} is a forward bisimulation fuzzy quasi-order with two different aftersets and we have:

$$E^{fb} \circ R_1 \circ E^{fb} = \begin{bmatrix} A & \{b\} & A \\ A & \{b,c\} & A \\ A & \{b\} & A \end{bmatrix}, \qquad V_p \circ E^{fb} = V_p = \begin{bmatrix} A \\ \{b\} \\ A \end{bmatrix}, \qquad V_q \circ E^{fb} = V_q = \begin{bmatrix} A \\ \{b,c\} \\ A \end{bmatrix}.$$

Now, from (56) and (57) we get the related afterset model $\mathfrak{M}/E^{fb} = (W/E^{fb}, \{R_1^{W/E^{fb}}\}, V^{W/E^{fb}})$ where

$$R_1^{W/E^{fb}} = \begin{bmatrix} A & \{b\} \\ A & \{b,c\} \end{bmatrix}, \qquad V_p^{W/E^{fb}} = \begin{bmatrix} A \\ \{b\} \end{bmatrix}, \qquad V_q^{W/E^{fb}} = \begin{bmatrix} A \\ \{b,c\} \end{bmatrix},$$

i.e., we get the model with a smaller number of worlds isomorphic to the model $\mathfrak{M}'$ from Example 6.5. However, since the underlying structure is not linearly ordered, we cannot apply Theorem 7.7.

On the other hand, E^{bb}, E^{rb}, Q^{fbb} and Q^{bfb} are fuzzy equivalences and fuzzy quasi-orders whose equivalence classes and aftersets are all different and for that reason, they cannot reduce the number of worlds of the model.

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