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Minimax inequalities for functions with noncompact domain and diagonal *GC*-quasiconcavity and their applications

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Abstract. This paper introduces the notion of diagonal *GC*-quasiconcavity which generalizes the notions of quasiconcavity, *CF*-quasiconcavity, diagonal transfer quasiconcavity, *C*-quasiconcavity, diagonal *C*-concavity, and diagonal *C*-quasiconcavity. We first establish some theorems for the existence of α-equilibrium of minimax inequalities for functions with noncompact domain and diagonal *GC*-quasiconcavity in topological spaces without linear structure. Next, we apply these results to characterize the existence of saddle points and solutions to the complementarity problem. Finally, we derive some intersection theorems and their equivalent forms.

1. Introduction

Minimax inequalities play a key role in proving many existence problems in non-linear analysis and applied mathematics, especially in optimization problems, variational inequalities, saddle points, intersection points, maximal elements, fixed points, and complementarity problems. In the framework of Hausdorff topological vector spaces, Fan [15] established a classical minimax inequality for the functions satisfying lower semicontinuity and quasiconcavity. Since then, many authors have generalized and extended minimax inequalities by weakening the quasiconcavity/semicontinuity of functions; see, for example, Ha [17], Zhou and Chen [36], Chang and Zhang [10], Tian [29], Tian and Zhou [30], Baye et al. [7], Forgö [16], Yuan et al. [35], Kim and Lee [19], Kim and Kum [20], Kim and Lee [21], Lu and Tang [24], Hou [18], Chang [11], Nasri and Sosa [25], Cotrina and Zúñiga [12], Agarwal et al. [1], Balaj and Khamsi [4], Agarwal et al. [2], Balaj [5, 6], Scalzo [28], Castellani and Giuli [9], and Cotrina and Svensson [13]. It is worth noticing that Kim [22] introduced the notion of diagonal *C*-quasiconcavity, which unifies the notions of various concavity in most of the literature. Further, based on this notion, Kim [22] obtained two minimax inequalities in topological spaces without linear structure. In contrast to the perspective of the aforementioned authors, Tian [31] recently proposed a single condition named γ-recursive transfer lower semicontinuity, which can characterize the existence of γ-equilibrium of minimax inequalities in arbitrary topological spaces.

Motivated by the above observations, in this paper, we investigate the existence of α-equilibrium of minimax inequalities for functions with noncompact domain and diagonal *GC*-quasiconcavity in topological

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spaces without linear structure. As applications, we establish a theorem on the existence of saddle points, a theorem on the existence of solutions to the complementarity problem, and some intersection theorems with their equivalent forms. The results presented in this paper generalize and extend the corresponding results in the literature.

The rest of this paper is organized as follows. In the next section, we state notation, definitions and a lemma for later use. In Section 3, by utilizing the notion of diagonal GC-quasiconcavity, we prove the existence of α-equilibrium of minimax inequalities for functions with noncompact domain in topological spaces without linear structure. Section 4 provides sufficient conditions for the existence of saddle points and solutions to the complementarity problem. In Section 5, as applications of minimax inequalities obtained in Section 3, we establish some intersection theorems and their equivalent forms in topological spaces without linear structure. Section 6 concludes this paper.

2. Preliminaries

In this section, we begin with notation, definitions, and a lemma. We denote the set of all real numbers as $\mathbb{R}$. For a nonempty set X, we denote by 2^X the family of all subsets of X. If X is a topological space and $A \subseteq B \subseteq X$, then we shall use $\text{int}_B A$ to denote the relative interior of A in B. For two nonempty sets X and Y, a set-valued mapping $H : X \to 2^Y$ means that H assigns a unique set $H(x) \subseteq Y$ for every $x \in X$. Let $\delta_n = \text{co}\{e_0, e_1, \ldots, e_n\}$ be the standard n-dimensional simplex whose vertices is $\{e_0, e_1, \ldots, e_n\}$, where $\text{co}\{e_0, e_1, \ldots, e_n\}$ denotes the convex hull of $\{e_0, e_1, \ldots, e_n\}$ and e_i is the $(i+1)$th unit vector in $\mathbb{R}^{n+1}$. For all $t = \{t_0, t_1, \ldots, t_n\} \in \delta_n$, let $\Omega(t) := \{i \in \{0, 1, \ldots, n\} | t_i \neq 0\}$.

Now, we introduce the following two definitions which are more general than the notions of the diagonal C-quasiconcavity and α-C-quasiconcavity (respectively, C-quasiconvexity and α-C-quasiconvexity) introduced by Kim [22].

Definition 2.1. Let X be a topological space and Y be a nonempty subset of X. A function $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ is said to be diagonally GC-quasiconcave (for short, DGCQCV) in the second variable with respect to Y if for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$ $(n \geq 1)$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^Y$ such that (i) $f(x, x) \geq \min\{f(x, y_i) | i \in \Omega(t)\}$ for all $t \in \delta_n$ and all $x \in \sigma_n(t)$, and (ii) for any continuous mapping $\beta : Y \to \delta_n$, the composition $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point, i.e., there exists $t^* \in \delta_n$ such that $t^* \in (\beta \circ \sigma_n)(t^*)$. We say that f is diagonally GC-quasiconvex (for short, DGCQCX) in the second variable with respect to Y if $-f$ is DGCQCV in the second variable with respect to Y.

Example 2.2. Let Y be a nonempty subset of a topological space X. Suppose that $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ is diagonally C-quasiconcave on Y (for short, DCQCV) (see Kim [22]), i.e., for any $\{y_0, y_1, \ldots, y_n\} \subseteq X$ $(n \geq 1)$, there exists a continuous mapping $\sigma_n : \delta_n \to Y$ such that $f(\sigma_n(t), \sigma_n(t)) \geq \min\{f(\sigma_n(t), y_i) | i \in \Omega(t)\}$ for all $t \in \delta_n$. Then, we can conclude that f is DGCQCV in the second variable with respect to Y. Indeed, for any continuous mapping $\beta : Y \to \delta_n$, it follows from Brouwer fixed point theorem that the composition $\beta \circ \sigma_n : \delta_n \to \delta_n$ has a fixed point.

Remark 2.3. It follows from Example 2.2 and remarks of Kim [22] that DGCQCV condition includes the diagonal quasiconcavity in Zhou and Chen [36] and Chang and Zhang [10], the diagonal transfer quasiconcavity in Baye et al. [7], the CF-quasiconcavity in Forgö [16], the C-concavity in Kim and Lee [19] and Kim and Lee [21], the C-quasiconcavity in Hou [18], and the pair-concavity in Chang [11] as special cases.

Definition 2.4. Let X be a topological space and Y be a nonempty subset of X. A function $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ is said to be α-diagonally GC-quasiconcave (for short, α-DGCQCV) in the second variable with respect to Y for some $\alpha \in \mathbb{R}$ if for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$ $(n \geq 1)$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^Y$ such that (i) $\alpha \geq \min\{f(x, y_i) | i \in \Omega(t)\}$ for all $t \in \delta_n$ and all $x \in \sigma_n(t)$, and (ii) for any continuous mapping $\beta : Y \to \delta_n$, the composition $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point. If the function $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ is DGCQCV in the second variable with respect to Y, then it must be α-DGCQCV in the second variable with respect to Y by taking $\alpha = \sup_{y \in Y} f(y, y)$. In addition, f is called diagonally α-GC-quasiconvex (for short, α-DGCQCX) in the second variable with respect to Y if $-f$ is $(-\alpha)$-DGCQCV in the second variable with respect to Y.

Example 2.5. Let Y be a nonempty subset of a topological space X. Suppose that $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ is the α-diagonally C-quasiconcave (for short, α-DCQCV) function introduced by Kim [22], i.e., for any $\{y_0, y_1, \ldots, y_n\} \subseteq X$ ($n \geq 1$), there exists a continuous mapping $\sigma_n : \delta_n \to Y$ such that $\alpha \geq \min\{f(\sigma_n(t), y_i) | i \in \Omega(t)\}$ for all $t \in \delta_n$. Then, it is easy to verify that f is α-DGCQCV in the second variable with respect to Y. In fact, for any continuous mapping $\beta : Y \to \delta_n$, by Brouwer fixed point theorem, the composition $\beta \circ \sigma_n : \delta_n \to \delta_n$ has a fixed point.

Definition 2.6. ([Border [8]) Let X be a nonempty subset of a topological space E such that $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{+\infty}$ is a nondecreasing sequence of nonempty compact subsets of E, i.e., $K_n \subseteq K_{n+1}$ for every $n \in \{1, 2, \ldots\}$. A sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is called escaping from X relative to $\{K_n\}_{n=1}^{+\infty}$ if for each $n \in \{1, 2, \ldots\}$ there exists an $M \in \{1, 2, \ldots\}$ such that $x_k \notin K_n$ for every $k \geq M$.

Definition 2.7. Let $\alpha \in \mathbb{R}$ and $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ be a function, where E is a topological space and Y is a nonempty set. Let X be a nonempty subset of E. We call f α-transfer lower (respectively, upper) semicontinuous in the first variable with respect to X if for any $(x, y) \in X \times Y$ with $f(x, y) > \alpha$ (respectively, $f(x, y) < \alpha$), there exist $\widetilde{y} \in Y$ and an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) > \alpha$ (respectively, $f(z, \widetilde{y}) < \alpha$) for every $z \in V(x) \bigcap X$. A function $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ is said to be transfer lower (respectively, upper) semicontinuous in the first variable with respect to X if f is α-transfer lower (respectively, upper) semicontinuous with respect to X for every $\alpha \in \mathbb{R}$.

Remark 2.8. It is easy to check that if a function $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ is α-transfer lower (respectively, upper) semicontinuous in the first variable with respect to E (see Definition 3 in Lin [23]), then f is α-transfer lower (respectively, upper) semicontinuous in the first variable with respect to any $X \subseteq E$. Note that Definition 2.7 coincides with Definition 3 in Lin [23] when $X = E$. Unlike Definition 3 in Lin [23], Y in Definition 2.7 does not require topological properties.

Inspired by the work of Reny [27] and Prokopovych [26], we give the following definition of payoff weak security.

Definition 2.9. Let E be a topological space, Y be a nonempty set, and X be a nonempty subset of E. A function $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ is said to be payoff weakly secure in the first variable with respect to X if for any $(x, y) \in X \times Y$ and any $\varepsilon > 0$, there exist $\widetilde{y} \in Y$ and an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) \geq f(x, y) - \varepsilon$ for every $z \in V(x) \bigcap X$.

Remark 2.10. It is easy to see that if a function $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ is payoff secure in the first variable with respect to E, then f is payoff weakly secure in the first variable with respect to any $X \subseteq E$.

The following lemma illustrates the equivalence of transfer lower continuity and payoff weak security.

Lemma 2.11. *Let E be a topological space, Y be a nonempty set, and X be a nonempty subset of E. Let $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ be a function. Then, the following three assertions are equivalent to each other:*

(i) f is payoff weakly secure in the first variable with respect to X;
(ii) f is transfer lower semicontinuous in the first variable with respect to X;
(iii) for each $\alpha \in \mathbb{R}$, $\bigcup_{y \in Y} \{x \in X | f(x, y) > \alpha\} = \bigcup_{y \in Y} int_X \{x \in X | f(x, y) > \alpha\}$.

Proof. (i) $\Rightarrow$ (ii) For any $\alpha \in \mathbb{R}$ and any $(x, y) \in X \times Y$ with $f(x, y) > \alpha$, let $0 < \varepsilon < f(x, y) - \alpha$. Then, it follows from Definition 2.5 that there exist $\widetilde{y} \in Y$ and an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) \geq f(x, y) - \varepsilon > \alpha$ for every $z \in V(x) \bigcap X$, which implies that f is transfer lower semicontinuous with respect to X.

(ii) ⇒ (i) For each $(x, y) \in X \times Y$ and each $\varepsilon > 0$, since f is transfer lower semicontinuous in the first variable with respect to X, there exist $\widetilde{y} \in Y$ and an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) > f(x, y) - \varepsilon$ for all $z \in V(x) \bigcap X$, which implies that f is payoff weakly secure in the first variable with respect to X.

(ii) ⇒ (iii) Let $\alpha \in \mathbb{R}$ be arbitrarily given. It is clear that $\bigcup_{y \in Y} \mathrm{int}_X\{x \in X | f(x, y) > \alpha\} \subseteq \bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\}$. Next, we show that

$$\bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\} \subseteq \bigcup_{y \in Y} \mathrm{int}_X\{x \in X | f(x, y) > \alpha\}.$$

Let $x \in \bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\}$. Then, there exists $y \in Y$ such that $f(x, y) > \alpha$. By Definition 2.7, there exist $\widetilde{y} \in Y$ and an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) > \alpha$ for every $z \in V(x) \bigcap X$, which implies that $V(x) \bigcap X \subseteq \{x \in X | f(x, \widetilde{y}) > \alpha\}$, i.e., $x \in \mathrm{int}_X\{x \in X | f(x, \widetilde{y}) > \alpha\}$. Thus, $\bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\} \subseteq \bigcup_{y \in Y} \mathrm{int}_X\{x \in X | f(x, y) > \alpha\}$. Therefore, we have

$$\bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\} = \bigcup_{y \in Y} \mathrm{int}_X\{x \in X | f(x, y) > \alpha\}.$$

(iii) ⇒ (ii) For any $\alpha \in \mathbb{R}$ and any $(x, y) \in X \times Y$ such that $f(x, y) > \alpha$, we have $x \in \bigcup_{y \in Y}\{x \in X | f(x, y) > \alpha\}$. By (iii), there exists $\widetilde{y} \in Y$ such that $x \in \mathrm{int}_X\{x \in X | f(x, \widetilde{y}) > \alpha\}$. Therefore, there exists an open neighborhood $V(x)$ of x in E such that $f(z, \widetilde{y}) > \alpha$ for every $z \in V(x) \bigcap X$, which implies that f is transfer lower semicontinuous in the first variable with respect to X. □

Remark 2.12. Let E be a topological space, Y be a nonempty set, and X be a nonempty subset of E. Let $f : E \times Y \to \mathbb{R} \bigcup \{\pm\infty\}$ be a function. By using a method similar to that used to prove Lemma 2.11, we can prove that f is transfer upper semicontinuous in the first variable with respect to X if and only if $\bigcup_{y \in Y}\{x \in X | f(x, y) < \alpha\} = \bigcup_{y \in Y} \mathrm{int}_X\{x \in X | f(x, y) < \alpha\}$ for every $\alpha \in \mathbb{R}$.

3. Minimax inequalities in topological spaces

In this section, by using α-DGCQCV (DGCQCX) condition, we give some new results on the existence of α-equilibrium of minimax inequalities for functions with noncompact domain in topological spaces without linear structure.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let $f, g : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ be two functions such that the following assumptions hold:*

(i) $f(x, y) \leq g(x, y)$ for every $(x, y) \in X \times X$;
(ii) g is α-DGCQCV in the second variable with respect to K_n for all $n \in \{1, 2, \ldots\}$;
(iii) $f|_{X \times K_n}$ is α-transfer lower semicontinuous in the first variable with respect to K_n for all $n \in \{1, 2, \ldots\}$;
(iv) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $f(x_n, y_n) > \alpha$.

Then, f possesses an α-equilibrium $\widehat{x} \in X$ of minimax inequality, i.e., $f(\widehat{x}, y) \leq \alpha$ for all $y \in X$.

Proof. We first prove that for each $n \in \{1, 2, \ldots\}$, there exists $x_n \in K_n$ such that $f(x_n, y) \leq \alpha$ for every $y \in K_n$. Let $n \in \{1, 2, \ldots\}$ be fixed arbitrarily. In order to get this conclusion, we need to show that

$$\bigcap_{y \in K_n}\{x \in K_n | f(x, y) \leq \alpha\} \neq \emptyset.$$

Suppose that $\bigcap_{y \in K_n}\{x \in K_n | f(x, y) \leq \alpha\} = \emptyset$. Then, we have

$$\bigcup_{y \in K_n}\{x \in K_n | f(x, y) > \alpha\} = K_n.$$

By (iii) and Lemma 2.11, we have

$$\begin{aligned} K_n &= \bigcup_{y\in K_n}\{x\in K_n | f(x,y)>\alpha\} \\ &= \bigcup_{y\in K_n}\mathrm{int}_{K_n}\{x\in K_n | f(x,y)>\alpha\}. \end{aligned}$$

By the compactness of K_n, it is known that there exists $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$ such that $K_n = \bigcup_{i=0}^{k(n)} \mathrm{int}_{K_n}\{x \in K_n | f(x, y_i) > \alpha\}$, where $k(n)$ is a positive integer. Let $\{\beta_i\}_{i=0}^{k(n)}$ be the continuous partition of utility subordinated to the open cover $\{\mathrm{int}_{K_n}\{x \in K_n : f(x, y_i) > \alpha\}\}_{i=0}^{k(n)}$ of K_n, i.e., for each $i \in \{0, 1, \ldots, k(n)\}$, $\beta_i : K_n \to [0, 1]$ is a continuous function such that $\sum_{i=1}^{k(n)} \beta_i(x) = 1$ for every $x \in K_n$ and $\beta_i(x) > 0$ implies that $x \in \mathrm{int}_{K_n}\{x \in K_n | f(x, y_i) > \alpha\}$. Further, we define a continuous mapping $\beta : K_n \to \delta_{k(n)}$ by $\beta(x) = (\beta_0(x), \beta_1(x), \ldots, \beta_{k(n)}(x))$ for all $x \in K_n$.

For the above finite subset $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, it follows from (ii) that there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that

$$\alpha \geq \min\{g(x, y_i) | i \in \Omega(t)\}, \ \ \forall t = (t_0, t_1, \ldots, t_{k(n)}) \in \delta_{k(n)} \text{ and } x \in \sigma_{k(n)}(t), \tag{3.1}$$

and for the continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point $t^* = (t_0^*, t_1^*, \ldots, t_{k(n)}^*) \in \delta_{k(n)}$, i.e., $(t_0^*, t_1^*, \ldots, t_{k(n)}^*) \in (\beta \circ \sigma_{k(n)})(t^*)$. Let $x^* \in \sigma_{k(n)}(t^*)$ such that $t^* = \beta(x^*)$. Then, by (3.1) and (i), we have

$$\alpha \geq \min\{g(x^*, y_i) | i \in \Omega(t^*)\} \geq \min\{f(x^*, y_i) | i \in \Omega(t^*)\}. \tag{3.2}$$

Note that for every $i \in \Omega(t^*)$, $t_i^* \neq 0$ implies that $\beta_i(x^*) > 0$. Consequently, for all $i \in \Omega(t^*)$, we obtain

$$\begin{aligned} x^* &\in \mathrm{int}_{K_n}\{x \in K_n | f(x, y_i) > \alpha\} \\ &\subseteq \{x \in K_n | f(x, y_i) > \alpha\}. \end{aligned}$$

Thus, $f(x^*, y_i) > \alpha$, together with (3.2), yields the following contradiction:

$$\alpha \geq \min\{g(x^*, y_i) | i \in \Omega(t^*)\} \geq \min\{f(x^*, y_i) | i \in \Omega(t^*)\} > \alpha.$$

Till now, we have reached the conclusion that for each $n \in \{1, 2, \ldots\}$, there exists $x_n \in K_n$ such that $f(x_n, y) \leq \alpha$ for every $y \in K_n$.

Next, we prove that there exists $\widehat{x} \in X$ such that $f(\widehat{x}, y) \leq \alpha$ for every $y \in X$. We use the contradiction method to prove that the sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is not escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, for which we assume that $\{x_n\}_{n=1}^{\infty} \subseteq X$ is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$. Then, by (iv), there exists $n_0 \in \{1, 2, \ldots\}$ and $y_{n_0} \in K_{n_0}$ such that $f(x_{n_0}, y_{n_0}) > \alpha$. But from the foregoing conclusion, $x_{n_0} \in K_{n_0}$ means $f(x_{n_0}, y_{n_0}) \leq \alpha$, which contradicts $f(x_{n_0}, y_{n_0}) > \alpha$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{K_n\}_{n=1}^{\infty}$. By Definition 2.6, it is known that there exist $n_1 \in \{1, 2, \ldots\}$ and a subsequence of $\{x_n\}_{n=1}^{\infty}$ which lies entirely in K_{n_1}. Since K_{n_1} is a compact subset of a Hausdorff topological space E, there exist a subsequence $\{x_{n_{l_i}}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ in K_{n_1} and $\widehat{x} \in K_{n_1}$ such that $x_{n_{l_i}} \to \widehat{x}$ as $i \to \infty$. Now, we claim that $f(\widehat{x}, y) \leq \alpha$ for all $y \in X$. Suppose to the contrary that there exists $\widehat{y} \in X$ such that $f(\widehat{x}, \widehat{y}) > \alpha$. Since $\widehat{y} \in X$, there exists $n_2 \geq n_1$ such that $\widehat{y} \in K_{n_2}$. Since $\{K_n\}_{n=1}^{+\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E, we have $\widehat{x} \in K_{n_1} \subseteq K_{n_2}$. By (iii) and Lemma 2.11 again, there exists $\widetilde{y} \in K_{n_2}$ such that

$$\widehat{x} \in \mathrm{int}_{K_{n_2}}\{x \in K_{n_2} | f(x, \widetilde{y}) > \alpha\}.$$

Since $x_{n_{l_i}} \to \widehat{x}$ as $i \to \infty$, it follows that there exists i_0 such that

$$x_{n_{l_0}} \in \mathrm{int}_{K_{n_2}}\{x \in K_{n_2} | f(x, \widetilde{y}) > \alpha\} \text{ and } n_{l_0} \geq n_2.$$

Hence, we see that $f(x_{n_{l_0}}, \widetilde{y}) > \alpha$. Note that $K_n \subseteq K_{n+1}$ for every $n \in \{1, 2, \ldots\}$. Then, we have $\widetilde{y} \in K_{n_2} \subseteq K_{n_{l_0}}$ and thus, $f(x_{n_{l_0}}, \widetilde{y}) \leq \alpha$ which contradicts $f(x_{n_{l_0}}, \widetilde{y}) > \alpha$. Therefore, we must have $f(\widehat{x}, y) \leq \alpha$ for all $y \in X$. □

Remark 3.2. (1) Theorem 3.1 generalizes Theorem 2.2 of Yuan et al. [35] in the following aspects: (a) from topological vector spaces to Hausdorff topological spaces without linear structure. We believe that the topological vector space involved in Theorem 2.2 of Yuan et al. [35] should have the Hausdorff separation property. Indeed, as can be seen from the proof of this theorem, if the Hausdorff separation property is dropped, then there will be no guarantee that the limit of a convergent subsequence falls within some compact set; (b) concerns on the existence of the more general α-equilibrium of minimax inequality instead of the existence of 0-equilibrium of minimax inequality; (c) the assumptions are weaker: the lower semicontinuity of f and the quasiconcavity of g are replaced by α-transfer lower semicontinuity of f and α-DGCQCV condition of g, respectively.

(2) Similar to the above analysis, we see that Theorem 3.1 also extends Theorem 2.1 of Yu [34], Theorem 3.1 of Tan and Yu [32], and Theorem 3.1 of Ding [14] in several aspects.

(3) By Lemma 2.11, (iii) of Theorem 3.1 can be replaced by one of the following two conditions.

(iii)′ $f|_{X\times K_n}$ is payoff weakly secure in the first variable with respect to K_n for all $n \in \{1, 2, \ldots\}$.

(iii)′′ $\bigcup_{y\in K_n}\{x \in K_n | f(x, y) > \alpha\} = \bigcup_{y\in K_n} \mathrm{int}_{K_n}\{x \in K_n | f(x, y) > \alpha\}$.

(4) The following (iv)′ implies (iv) of Theorem 3.1.

(iv)′ For any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq X$ such that $y_n \in K_n$ and $\overline{\lim}_{n\to\infty} f(x_n, y_n) > \alpha$.

As a direct consequence of Theorem 3.1, we have the following Ky Fan minimax inequality in topological spaces without linear structure.

Corollary 3.3. *Let $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let $f, g : X \times X \to \mathbb{R}\bigcup\{\pm\infty\}$ be two functions and $\alpha = \sup_{x\in X} g(x, x)$. Suppose that the following assumptions are fulfilled:*

(i) $f(x, y) \le g(x, y)$ for every $(x, y) \in X \times X$;

(ii) g is α-DGCQCV in the second variable with respect to K_n for all $n \in \{1, 2, \ldots\}$;

(iii) $f|_{X\times K_n}$ is α-transfer lower semicontinuous in the first variable with respect to K_n for all $n \in \{1, 2, \ldots\}$;

(iv) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $f(x_n, y_n) > \alpha$.

Then, the Ky Fan minimax inequality $\inf_{x\in X} \sup_{x\in X} f(x, y) \le \sup_{x\in X} g(x, x)$ holds.

The following Theorems 3.4 and 3.8 for the existence of α-equilibrium of minimax inequalities are established in the case where no escaping sequences are introduced. Since their proofs are similar to the first half of the proof of Theorem 3.1, we omit them here.

Theorem 3.4. *Let X be a nonempty subset of a Hausdorff topological space E, K be a nonempty compact subset of X, and $\alpha \in \mathbb{R}$. Let $f, g : X \times X \to \mathbb{R}\bigcup\{\pm\infty\}$ be two functions such that the following assumptions hold:*

(i) $f(x, y) \le g(x, y)$ for every $(x, y) \in X \times X$;

(ii) g is α-DGCQCV in the second variable with respect to K;

(iii) f is α-transfer lower semicontinuous in the first variable with respect to K.

Then, f possesses an α-equilibrium $\widehat{x} \in K$ of minimax inequality, i.e., $f(\widehat{x}, y) \le \alpha$ for all $y \in X$.

Remark 3.5. Theorem 3.4 generalizes and extends Theorem 1 of Kim [22] in the following aspects: (a) the minimax inequality in Theorem 3.4 is more general than that in Theorem 1 of Kim [22]; (b) from DCQCV condition to DGCQCV condition; (c) the assumption on f to be lower semicontinuous is replaced by α-transfer lower semicontinuity, the latter being a weaker assumption.

If $\alpha = 0$ in Theorem 3.4, then we have the following result on the existence of 0-equilibrium of minimax inequality.

Corollary 3.6. *Let X be a nonempty subset of a Hausdorff topological space E and K be a nonempty compact subset of X. Let $f, g : X \times X \to \mathbb{R} \bigcup\{\pm\infty\}$ be two functions such that the following assumptions hold:*

(i) $f(x, y) \le g(x, y)$ for every $(x, y) \in X \times X$;
(ii) g is 0-DGCQCV in the second variable with respect to K;
(iii) f is 0-transfer lower semicontinuous in the first variable with respect to K.

Then, f possesses a 0-equilibrium $\widehat{x} \in K$ of minimax inequality, i.e., $f(\widehat{x}, y) \le 0$ for all $y \in X$.

Remark 3.7. Corollary 3.6 generalizes and extends Theorem 3.1 of Yang and Pu [33] in the following aspects: (a) from Hausdorff topological vector spaces to Hausdorff topological spaces without linear structure; (b) from compact set to noncompact set; (c) from one function to two functions; (d) from the generalized quasiconcavity to 0-DGCQCV condition. Indeed, (ii) and (iii) of Theorem 3.1 of Yang and Pu [33] imply (ii) of Corollary 3.6 in the case that $f = g$ and $K = X$; (e) the assumption that f is lower semicontinuous is replaced by the weaker assumption that f is 0-transfer lower semicontinuous.

Theorem 3.8. *Let X be a nonempty normal subspace of a topological space E and K be a nonempty compact subset of X. Let $\alpha \in \mathbb{R}$ and let $f : X \times X \to \mathbb{R} \bigcup\{\pm\infty\}$ be a function such that the following conditions hold:*

(i) f is α-DGCQCX in the second variable with respect to X;
(ii) $f|_{X\times K}$ is α-transfer upper semicontinuous in the first variable with respect to K;
(iii) for each $y \in X \setminus K$, $f(x, y) \ge \alpha$ if and only if $x \in K$.

Then, $-f$ possesses a $(-\alpha)$-equilibrium $\widehat{x} \in X$ of minimax inequality, i.e., $f(\widehat{x}, y) \ge \alpha$ for all $y \in X$.

By Theorem 3.8, we have the following minimax inequality.

Corollary 3.9. *Let X be a nonempty normal subspace of a topological space E and K be a nonempty compact subset of X. Let $f : X \times X \to \mathbb{R} \bigcup\{\pm\infty\}$ be a function and $\alpha = \inf_{x\in X} f(x, x)$ such that the following conditions hold:*

(i) f is α-DGCQCX in the second variable with respect to X;
(ii) $f|_{X\times K}$ is α-transfer upper semicontinuous in the first variable with respect to K;
(iii) for each $y \in X \setminus K$, $f(x, y) \ge \alpha$ if and only if $x \in K$.

Then, the miniamx inequality $\sup_{x\in X} \inf_{y\in X} f(x, y) \ge \inf_{x\in X} f(x, x)$ holds.

Remark 3.10. Corollary 3.9 generalizes and extends Theorem 2 of Kim [22] in the following aspects: (a) from DCQCX condition to DGCQCX condition; (b) the assumption on f to be lower semicontinuous is replaced by α-transfer upper semicontinuity, the latter being a weaker assumption; (c) the assumption on X to be a paracompact subset of a Hausdorff topological space is replaced by the weaker assumption that X is normal subspace of a topological space.

4. Saddle points and complementarity problem

In this section, by using Theorem 3.1, we present a theorem on the existence of saddle points in topological spaces without linear structure and a theorem on the existence of solutions to the complementarity problem in Hausdorff topological vector spaces.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$ and $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let $f : X\times X \to \mathbb{R} \bigcup\{\pm\infty\}$ be a function such that the following assumptions hold:*

(i) for each $n \in \{1, 2, \ldots\}$, f is α-DGCQCV in the second variable with respect to K_n and f is α-DGCQCX in the first variable with respect to K_n;
(ii) for each $n \in \{1, 2, \ldots\}$, $f|_{X\times K_n}$ is α-transfer lower semicontinuous in the first variable with respect to K_n and $f|_{K_n\times X}$ is α-transfer upper semicontinuous in the second variable with respect to K_n;

(iii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ *with* $x_n \in K_n$ *for every* $n \in \{1, 2, \dots\}$*, which is escaping from* X *relative to* $\{K_n\}_{n=1}^{\infty}$*, there exist* $n \in \{1, 2, \dots\}$ *and* $y_n \in K_n$ *such that* $f(x_n, y_n) > \alpha$*;*

(iv) for any sequence $\{y_n\}_{n=1}^{\infty} \subseteq X$ *with* $y_n \in K_n$ *for every* $n \in \{1, 2, \dots\}$*, which is escaping from* X *relative to* $\{K_n\}_{n=1}^{\infty}$*, there exist* $n \in \{1, 2, \dots\}$ *and* $x_n \in K_n$ *such that* $f(x_n, y_n) < \alpha$*.*

Then, f *has a saddle point* $(\widehat{x}, \widehat{y}) \in X \times X$*, i.e.,* $f(\widehat{x}, y) \leq \alpha = f(\widehat{x}, \widehat{y}) \leq f(x, \widehat{y})$ *for every* $(x, y) \in X \times X$*. In particular, the minimax inequality* $\sup_{y\in X} \inf_{x\in X} f(x, y) = \inf_{x\in X} \sup_{y\in X} f(x, y)$ *holds.*

Proof. Since for each $n \in \{1, 2, \dots\}$, f is α-DGCQCV in the second variable with respect to K_n, $f|_{X\times K_n}$ is α-transfer lower semicontinuous in the first variable with respect to K_n, and (iii) is satisfied, we see by Theorem 3.1 that there exists a point $\widehat{x}$ such that $f(\widehat{x}, y) \leq \alpha$ for every $y \in X$. Now, let $h(y, x) = -f(x, y)$ for every $(y, x) \in X \times X$. By the fact that f is α-DGCQCX in the first variable with respect to K_n for all $n \in \{1, 2, \dots\}$, we can infer that h is $(-\alpha)$-DGCQCV in the second variable with respect to K_n. Since $f|_{K_n\times X}$ is α-transfer upper semicontinuous in the second variable with respect to K_n for all $n \in \{1, 2, \dots\}$, it is easy to see that $h|_{X\times K_n}$ is $(-\alpha)$-transfer lower semicontinuous in the first variable with respect to K_n. By (iv), for any sequence $\{y_n\}_{n=1}^{\infty} \subseteq X$ with $y_n \in K_n$ for every $n \in \{1, 2, \dots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \dots\}$ and $x_n \in K_n$ such that $h(y_n, x_n) > -\alpha$. Therefore, by Theorem 3.1 again for h, we have $h(\widehat{y}, x) = -f(x, \widehat{y}) \leq -\alpha$ and so that, $f(x, \widehat{y}) \geq \alpha$ for all $x \in X$. Thus, we have $f(\widehat{x}, y) \leq \alpha = f(\widehat{x}, \widehat{y}) \leq f(x, \widehat{y})$ for every $(x, y) \in X \times X$, i.e., f has a saddle point $(\widehat{x}, \widehat{y}) \in X \times X$, which implies that $\inf_{x\in X} \sup_{y\in X} f(x, y) \leq \sup_{y\in X} \inf_{x\in X} f(x, y)$. The reverse minimax inequality $\inf_{x\in X} \sup_{y\in X} f(x, y) \geq \sup_{y\in X} \inf_{x\in X} f(x, y)$ clearly holds. Therefore, we have $\inf_{x\in X} \sup_{y\in X} f(x, y) = \sup_{y\in X} \inf_{x\in X} f(x, y)$. □

Remark 4.2. Compared with Theorem 3 of Kim [22], X in Theorem 4.1 does not need to be compact. Moreover, the semicontinuity and quasiconcavity of the function in Theorem 4.1 are weaker than the corresponding conditions of the function in Theorem 3 of Kim [22].

Let E be a Hausdorff topological vector space, E^* be the dual space of E, X be a convex cone of E, X^* be the polar cone of E^*, and $\xi : X \to E^*$ be a mapping. The complementarity problem can be summarized as finding a point $\widehat{x} \in X$ such that $\xi(\widehat{x}) \in X^*$ and $\langle \xi(\widehat{x}), \widehat{x} \rangle = 0$.

Now, we are ready to apply Theorem 3.1 to the complementarity problem in Hausdorff topological vector spaces.

Theorem 4.3. *Let* $X = \bigcup_{n=1}^{+\infty} K_n$*, where* X *is a convex cone of a Hausdorff topological vector space* E *and* $\{K_n\}_{n=1}^{\infty}$ *is a nondecreasing sequence of nonempty compact convex subsets of a Hausdorff topological space* E*. Let* $\xi : X \to E^*$ *be a mapping such that the function* $f : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ *defined by* $f(x, y) = \langle \xi(x), y - x \rangle$ *for every* $(x, y) \in X \times X$*, satisfies the following assumptions:*

(i) $f|_{X\times K_n}$ *is* 0*-transfer lower semicontinuous in the first variable with respect to* K_n *for all* $n \in \{1, 2, \dots\}$*;*

(ii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ *with* $x_n \in K_n$ *for every* $n \in \{1, 2, \dots\}$*, which is escaping from* X *relative to* $\{K_n\}_{n=1}^{\infty}$*, there exist* $n \in \{1, 2, \dots\}$ *and* $y_n \in K_n$ *such that* $f(x_n, y_n) > 0$*.*

Then, there exists a point $\widehat{x} \in X$ *such that* $\xi(\widehat{x}) \in X^*$ *and* $\langle \xi(\widehat{x}), \widehat{x} \rangle = 0$*.*

Proof. For each $n \in \{1, 2, \dots\}$ and each $\{y_0, y_1, \dots, y_{k(n)}\} \subseteq K_n$, let us define a continuous mapping $\sigma_{k(n)} : \delta_{k(n)} \to K_n$ by $\sigma_{k(n)}(t_0, t_1, \dots, t_{k(n)}) = \sum_{j=0}^{k(n)} t_j y_j = \sum_{i\in\Omega(t)} t_i y_i \in \mathrm{co}\{y_i | i \in \Omega(t)\}$ for all $t = (t_0, t_1, \dots, t_{k(n)}) \in \delta_{k(n)}$. Since each $f(y, y) = 0$ and f is a linear function in y, we have the following:

$$\begin{aligned} 0 &= f(\sigma_{k(n)}(t), \sigma_{k(n)}(t)) \\ &= \langle \xi(\sigma_{k(n)}(t)), \sum_{i\in\Omega(t)} t_i y_i - \sigma_{k(n)}(t) \rangle \\ &= \sum_{i\in\Omega(t)} t_i f(\sigma_{k(n)}(t), y_i). \end{aligned}$$

Thus, there exists $i_0 \in \Omega(t)$ such that $f(\sigma_{k(n)}(t), y_{i_0}) \leq 0$ and so that, we get

$$0 \geq \min\{f(\sigma_{k(n)}(t_0, t_1, \ldots, t_{k(n)}), y_i) | i \in \Omega(t)\}, \quad \forall t \in \delta_{k(n)}.$$

Moreover, for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, it follows from Brouwer fixed point theorem that the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to \delta_{k(n)}$ has a fixed point. This shows that f is 0-DGCQCV in the second variable with respect to K_n for every $n \in \{1, 2, \ldots\}$. Therefore, by Theorem 3.1 with $f = g$ and $\alpha = 0$, there exists a point $\widehat{x} \in X$ such that $f(\widehat{x}, y) \leq 0$ for every $y \in X$, i.e., $\langle \xi(\widehat{x}), y - \widehat{x} \rangle \leq 0$. By Lemma 1 due to Allen [3], we have $\xi(\widehat{x}) \in X^*$ and $\langle \xi(\widehat{x}), \widehat{x} \rangle = 0$. □

5. Intersection theorems

In this section, we first use Theorem 3.1 to establish an intersection theorem and its equivalent forms in topological spaces without linear structure. Next, we use Theorems 3.4 and 3.8 to obtain two intersection theorems without introducing the escaping sequences.

Theorem 5.1. *Let $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let $F, G : X \to 2^X$ be two set-valued mappings such that the following assumptions hold:*

(i) for each $y \in X$, $F(y) \subseteq G(y)$;

(ii) for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$, and for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point;

(iii) for each $n \in \{1, 2, \ldots\}$, $\bigcup_{y \in K_n} \{x \in K_n | x \notin G(y)\} = \bigcup_{y \in K_n} int_{K_n} \{x \in K_n | x \notin G(y)\}$;

(iv) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $x_n \notin G(y_n)$.

Then, $\bigcap_{y \in X} G(y) \neq \emptyset$.

Proof. Define two functions $f, g : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ by setting, for every $(x, y) \in X \times X$,

$$f(x, y) = \begin{cases} \alpha, & x \in G(y), \\ +\infty, & x \notin G(y), \end{cases}$$

$$g(x, y) = \begin{cases} \alpha, & x \in F(y), \\ +\infty, & x \notin F(y), \end{cases}$$

where α is an arbitrary given real number. Then, we have the following:

(a) By (i), we have $f(x, y) \leq g(x, y)$ for every $(x, y) \in X \times X$.

(b) By (ii), for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$. Hence, by the definition of g, we have

$$\alpha \geq \min\{g(x, y_j) | j \in \Omega(t)\} = \alpha.$$

Furthermore, for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point. Therefore, g is α-DGCQCV in the second variable with respect to K_n for all $n \in \{1, 2, \ldots\}$.

(c) Since $\{x \in X | x \notin G(y)\} = \{x \in X | f(x, y) > \alpha\}$ for every $y \in X$, it follows from (iii) that $\bigcup_{y \in K_n} \{x \in K_n | f(x, y) > \alpha\} = \bigcup_{y \in K_n} \mathrm{int}_{K_n} \{x \in K_n | f(x, y) > \alpha\}$. Therefore, by Lemma 2.11, $f|_{X \times K_n}$ is α-transfer lower semicontinuous in the first variable with respect to K_n for all $n \in \{1, 2, \ldots\}$.

(d) For any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $x_n \notin G(y_n)$, which, together with the definition of f, leads to $f(x_n, y_n) = +\infty > \alpha$.

From the above arguments, we see that all the conditions of Theorem 3.1 are satisfied. Thus, there exists a point $\widehat{x} \in X$ such that $f(\widehat{x}, y) \leq \alpha$ for all $y \in X$, which means $\bigcap_{y \in X} G(y) \neq \emptyset$. □

The following theorem can be regarded as a maximal element version of the above intersection theorem.

Theorem 5.2. *Let $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let $H, L : X \to 2^X$ be two set-valued mappings such that the following assumptions hold:*

(i) for each $x \in X$, $L(x) \subseteq H(x)$;
(ii) for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $y_j \notin H(x)$, and for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point;
(iii) for each $n \in \{1, 2, \ldots\}$, $\bigcup_{y \in K_n} \{x \in K_n | x \in L^{-1}(y)\} = \bigcup_{y \in K_n} int_{K_n} \{x \in K_n | x \in L^{-1}(y)\}$;
(iv) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $y_n \in L(x_n)$.

Then, there exists a point $x^* \in X$ such that $L(x^*) = \emptyset$.

Proof. Define two set-valued mappings $F, G : X \to 2^X$ by $F(y) = X \setminus H^{-1}(y)$ and $G(y) = X \setminus L^{-1}(y)$ for every $y \in X$. Then, by Theorem 5.1, we have $\bigcap_{y \in X} G(y) \neq \emptyset$. Taking $x^* \in \bigcap_{y \in X} G(y)$ and combining the definition of G, we see that $y \notin L(x^*)$ for all $y \in X$, which implies that $L(x^*) = \emptyset$. □

Remark 5.3. By Remark 3.2 and the fact that Theorems 2.2 and 2.2′ of Yuan et al. [35] are equivalent, we see that Theorem 5.2 generalizes Theorem 2.2′ of Yuan et al. [35] in several aspects.

We give the geometric form of Theorem 5.2 as follows.

Theorem 5.4. *Let $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let M, Q be two nonempty subsets of $X \times X$ such that the following assumptions hold:*

(i) $M \subseteq Q$ and for each $n \in \{1, 2, \ldots\}$, $\bigcup_{y \in K_n} \{x \in K_n | (x, y) \in M\} = \bigcup_{y \in K_n} int_{K_n} \{x \in K_n | (x, y) \in M\}$;
(ii) or each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $(x, y_j) \notin Q$, and for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point;
(iii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $(x_n, y_n) \in M$.

Then, there exists a point $x^* \in X$ such that $(x^*, y) \notin M$ for every $y \in X$.

Proof. Define two set-valued mappings $H, L : X \to 2^X$ by $H(x) = \{y \in X | (x, y) \in Q\}$ and $L(x) = \{y \in X | (x, y) \in M\}$ for every $x \in X$. Then, we have the following:

(a) It follows from (i) that for each $x \in X$, $L(x) \subseteq H(x)$ and for each $n \in \{1, 2, \ldots\}$, $\bigcup_{y \in K_n} \{x \in K_n | y \in L(x)\} = \bigcup_{y \in K_n} \mathrm{int}_{K_n} \{x \in K_n | y \in L(x)\}$.

(b) By (ii), for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $y_j \notin H(x)$, and for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point.

(c) By (iii), for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $y_n \in L(x_n)$.

The above statements ensure that all the requirements of Theorem 5.2 are fulfilled. Therefore, by Theorem 5.2, there exists a point $x^* \in X$ such that $L(x^*) = \emptyset$, which means that $(x^*, y) \notin M$ for every $y \in X$. □

The following section theorem can be seen as the dual form of Theorem 5.4.

Theorem 5.5. *Let $X = \bigcup_{n=1}^{+\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonempty compact subsets of a Hausdorff topological space E. Let P, A be two nonempty subsets of $X \times X$ such that the following assumptions hold:*

(i) $A \subseteq P$ and for each $n \in \{1, 2, \ldots\}$, $\bigcup_{y \in K_n} \{x \in K_n | (x, y) \notin P\} = \bigcup_{y \in K_n} int_{K_n} \{x \in K_n | (x, y) \notin P\}$;

(ii) for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, there is $j \in \Omega(t)$ with $(x, y_j) \in A$, and for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point;

(iii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $(x_n, y_n) \notin P$.

Then, there exists a point $x^* \in X$ such that $\{x^*\} \times X \subseteq P$.

Proof. Let $M = X \times X \setminus P$ and $Q = X \times X \setminus A$. Then, it follows from Theorem 5.4 that there exists a point $x^* \in X$ such that $(x^*, y) \notin M$ for every $y \in X$, i.e., $\{x^*\} \times X \subseteq P$. □

Theorem 5.6. *Theorems 3.1, 5.1, 5.2, 5.4 and 5.5 are equivalent.*

Proof. We have showed Theorem 3.1 ⇒ Theorem 5.1, Theorem 5.1 ⇒ Theorem 5.2, Theorem 5.2 ⇒ Theorem 5.4, and Theorem 5.4 ⇒ Theorem 5.5. Now, we prove that Theorem 5.5 ⇒ Theorem 3.1. In fact, let $P = \{(x, y) \in X \times X | f(x, y) \leq \alpha\}$ and $A = \{(x, y) \in X \times X | g(x, y) \leq \alpha\}$. By (i) of Theorem 3.1, we have $A \subseteq P$. By Definition 2.4 and (ii) of Theorem 3.1, one can see that for each $n \in \{1, 2, \ldots\}$ and each $\{y_0, y_1, \ldots, y_{k(n)}\} \subseteq K_n$, there exists a nonempty-valued set-valued mapping $\sigma_{k(n)} : \delta_{k(n)} \to 2^{K_n}$ such that for each $t \in \delta_{k(n)}$ and each $x \in \sigma_{k(n)}(t)$, $\alpha \geq \min\{g(x, y_i) | i \in \Omega(t)\}$, from which we can infer that there exists $j \in \Omega(t)$ such that $g(x, y_j) \leq \alpha$ and thus, $(x, y_j) \in A$. Furthermore, for any continuous mapping $\beta : K_n \to \delta_{k(n)}$, the composition $\beta \circ \sigma_{k(n)} : \delta_{k(n)} \to 2^{\delta_{k(n)}}$ has a fixed point. This shows that (ii) of Theorem 5.5 is satisfied. By using (iii) of Theorem 3.1 and Lemma 2.11, we have $\bigcup_{y \in K_n} \{x \in K_n | f(x, y) > \alpha\} = \bigcup_{y \in K_n} \mathrm{int}_{K_n} \{x \in K_n | f(x, y) > \alpha\}$ for all $n \in \{1, 2, \ldots\}$. By the definition of P, we deduce that $\bigcup_{y \in K_n} \{x \in K_n | (x, y) \notin P\} = \bigcup_{y \in K_n} \mathrm{int}_{K_n} \{x \in K_n | (x, y) \notin P\}$ for all $n \in \{1, 2, \ldots\}$. Finally, by (iv) of Theorem 3.1 and the definition of P, we know that for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ with $x_n \in K_n$ for every $n \in \{1, 2, \ldots\}$, which is escaping from X relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \{1, 2, \ldots\}$ and $y_n \in K_n$ such that $(x_n, y_n) \notin P$. Therefore, by Theorem 5.5, there exists a point $x^* \in X$ such that $\{x^*\} \times X \subseteq P$, i.e., $f(x^*, y) \leq \alpha$ for all $y \in X$. □

By Theorems 3.4 and 3.8, we have the following two intersection theorems without introducing the escaping sequences.

Theorem 5.7. *Let X be a nonempty subset of a Hausdorff topological space E and K be a nonempty compact subset of X. Let $F, G : X \to 2^X$ be two set-valued mappings such that the following assumptions hold:*

(i) for each $y \in X$, $F(y) \subseteq G(y)$;

(ii) for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^K$ such that for each $t \in \delta_n$ and each $x \in \sigma_n(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$, and for any continuous mapping $\beta : K \to \delta_n$, the mapping $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point;

(iii) $\bigcup_{y \in X} \{x \in K | x \notin G(y)\} = \bigcup_{y \in X} int_K \{x \in K | x \notin G(y)\}$.

Then, $\bigcap_{y \in X} G(y) \neq \emptyset$.

Proof. Let α be an arbitrary given real number. Define two functions $f, g : X \times X \to \mathbb{R} \bigcup \{\pm\infty\}$ by setting, for each $(x, y) \in X \times X$,

$$f(x, y) = \begin{cases} \alpha, & x \in G(y), \\ +\infty, & x \notin G(y), \end{cases}$$

$$g(x, y) = \begin{cases} \alpha, & x \in F(y), \\ +\infty, & x \notin F(y). \end{cases}$$

By (i)-(iii), one can see that f and g satisfy (i)-(iii) of Theorem 3.4. Therefore, by Theorem 3.4, there exists a point $\widehat{x} \in X$ such that $f(\widehat{x}, y) \leq \alpha$ for all $y \in X$, which means $\bigcap_{y \in X} G(y) \neq \emptyset$. □

Theorem 5.8. *Let X be a nonempty subset of a Hausdorff topological space E and K be a nonempty compact subset of X. Let $F : X \to 2^X$ be a set-valued mapping such that the following assumptions hold:*

(i) for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^X$ such that for each $t \in \delta_n$ and each $x \in \sigma_n(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$, and for any continuous mapping $\beta : X \to \delta_n$, the mapping $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point;

(ii) $\bigcup_{y\in K}\{x \in K | x \notin F(y)\} = \bigcup_{y\in K} int_K\{x \in K | x \notin F(y)\}$;

(iii) for each $y \in X \setminus K$, $x \in F(y)$ if and only if $x \in K$.

Then, $\bigcap_{y\in X} F(y) \neq \emptyset$.

Proof. Let α be an arbitrary given real number. Define a function $f : X \times X \to \mathbb{R} \bigcup\{\pm\infty\}$ by setting, for every $(x, y) \in X \times X$,

$$f(x,y) = \begin{cases} \alpha, & x \in F(y), \\ -\infty, & x \notin F(y). \end{cases}$$

Then, we get the following:

(a) By (i), for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^X$ such that for each $t \in \delta_n$ and each $x \in \sigma_n(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$. By the definition of f, we have

$$\alpha \leq \max\{f(x, y_j) | j \in \Omega(t)\} = \alpha.$$

Furthermore, for any continuous mapping $\beta : X \to \delta_n$, the composition $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point. Therefore, f is α-DGCQCX in the second variable with respect to X.

(b) Since $\{x \in X | x \notin F(y)\} = \{x \in X | f(x, y) < \alpha\}$ for every $y \in X$, it follows from (ii) that $\bigcup_{y\in K}\{x \in K | f(x, y) < \alpha\} = \bigcup_{y\in K} \mathrm{int}_K\{x \in K | f(x, y) < \alpha\}$. Therefore, by Remark 2.12, $f|_{X\times K}$ is α-transfer upper semicontinuous in the first variable with respect to K.

(c) By (iii) and the definition of f, we see that for each $y \in X \setminus K$, $f(x, y) \geq \alpha$ if and only if $x \in K$.

By the above arguments, we see that all the requirements of Theorem 3.8 are satisfied. Thus, by Theorem 3.8, there exists a point $\widehat{x} \in X$ such that $f(\widehat{x}, y) \geq \alpha$ for all $y \in X$, which means $\bigcap_{y\in X} F(y) \neq \emptyset$. □

When $X = K$, Theorem 5.7 reduces to the following corollary.

Corollary 5.9. *Let X be a nonempty compact subset of a Hausdorff topological space E and $F, G : X \to 2^X$ be two set-valued mappings such that the following assumptions hold:*

(i) for each $y \in X$, $F(y) \subseteq G(y)$;

(ii) for each $\{y_0, y_1, \ldots, y_n\} \subseteq X$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^X$ such that for each $t \in \delta_n$ and each $x \in \sigma_n(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$, and for any continuous mapping $\beta : X \to \delta_n$, the mapping composition $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point;

(iii) $\bigcup_{y\in X}\{x \in X | x \notin G(y)\} = \bigcup_{y\in X} int_X\{x \in X | x \notin G(y)\}$.

Then, $\bigcap_{y\in X} G(y) \neq \emptyset$.

Remark 5.10. Corollary 5.9 generalizes and extends Theorem 2.1 of Yang and Pu [33] in the following aspects: (a) from one set-valued mapping to two set-valued mappings; (b) from Hausdorff topological vector spaces to Hausdorff topological spaces without linear structure; (c) (ii) of Corollary 5.9 is weaker than the assumption that F is a generalized KKM-mapping. In fact, let F be a generalized KKM-mapping. Then, by Definition 2.1 of Yang and Pu [33], there exists a continuous mapping $\sigma_n : \delta_n \to X$ such that for each $t \in \delta_n$, there is $j \in \Omega(t)$ with $x = \sigma_n(t) \in F(y_j)$. Furthermore, by Brouwer fixed point theorem, we see that, for any continuous mapping $\beta : X \to \delta_n$, the composition mapping $\beta \circ \sigma_n : \delta_n \to \delta_n$ has a fixed point. Therefore, (ii) of Corollary 5.9 is satisfied; (d) the assumption on F to be a set-valued mapping with closed values is replaced by (iii) of Corollary 5.9, the latter being weaker; (e) the assumption that X has the fixed point property has been dropped. Indeed, we can prove that the conclusion of Theorem 2.1 of Yang and Pu [33] still holds without this condition.

Corollary 5.11. *Let X be a nonempty compact subset of a Hausdorff topological space E and let M, Q be two nonempty subsets of $X \times X$ such that the following assumptions hold:*

(i) $M \subseteq Q$;

(ii) *for each* $\{y_0, y_1, \ldots, y_n\} \subseteq X$, *there exists a nonempty-valued set-valued mapping* $\sigma_n : \delta_n \to 2^X$ *such that for each* $t \in \delta_n$ *and each* $x \in \sigma_n(t)$, *there is* $j \in \Omega(t)$ *with* $(x, y_j) \notin Q$, *and for any continuous mapping* $\beta : X \to \delta_n$, *the mapping composition* $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ *has a fixed point;*

(iii) $\bigcup_{y\in X}\{x \in X|(x, y) \in M\} = \bigcup_{y\in X} int_X\{x \in X|(x, y) \in M\}$.

Then, there exists a point $x^* \in X$ *such that* $(x^*, y) \notin M$ *for every* $y \in X$.

Proof. Define two set-valued mappings $F, G : X \to 2^X$ by $F(y) = \{x \in X|(x, y) \notin Q\}$ and $G(y) = \{x \in X|(x, y) \notin M\}$ for every $y \in X$. By (i), we have $F(y) \subseteq G(y)$ for every $y \in X$. By (ii), we see that for any $\{y_0, y_1, \ldots, y_n\} \subseteq X$, there exists a nonempty-valued set-valued mapping $\sigma_n : \delta_n \to 2^X$ such that for each $t \in \delta_n$ and each $x \in \sigma_n(t)$, there is $j \in \Omega(t)$ with $x \in F(y_j)$, and for any continuous mapping $\beta : X \to \delta_n$, the mapping composition $\beta \circ \sigma_n : \delta_n \to 2^{\delta_n}$ has a fixed point. By (iii), we have $\bigcup_{y\in X}\{x \in X|x \notin G(y)\} = \bigcup_{y\in X} \mathrm{int}_X\{x \in X|x \notin G(y)\}$. Therefore, by Corollary 5.9, we have $\bigcap_{y\in X} G(y) \neq \emptyset$, which implies that exists a point $x^* \in X$ such that $(x^*, y) \notin M$ for every $y \in X$. □

Remark 5.12. (1) It is easy to see that Corollary 5.11 is equivalent to Corollary 5.9. In addition, Corollary 5.11 generalizes Theorem 3.3 of Yang and Pu [33] in the following aspects: (a) from one subset of $X \times X$ to two subsets of $X \times X$; (b) from Hausdorff topological vector spaces to Hausdorff topological spaces without linear structure; (c) (ii) of Corollary 5.11 is wreaker than (ii) of Theorem 3.3 of Yang and Pu [33]; (d) (iii) of Corollary 5.11 is weaker than (i) of Theorem 3.3 of Yang and Pu [33]; (e) the assumption that X has the fixed point property has been removed.

(2) Corollary 5.11 also revises and generalizes Theorem 4 of Kim [22]. Indeed, if the condition that $(x, x) \notin B$ for all $x \in X$ is removed from Theorem 4 of Kim [22], then the conclusion of this theorem still holds. From this, we can further see that the condition that $(x, x) \in N$ for all $x \in X$ of Theorem 5 of Kim [22] and the condition that $x \notin G(x)$ for all $x \in X$ of Theorem 6 of Kim [22] can be removed. Finally, we point out that condition (3) of Theorem 7 of Kim [22] is incorrectly set. This is because, according to the previous analysis, when conditions (1), (2) and (4) of Theorem 7 of Kim [22] are satisfied, we can deduce the conclusion that there exists a point $x^* \in X$ such that $(x^*, y) \notin A$ for all $y \in X$, which contradicts condition (3) of this theorem.

6. Conclusions

The main purpose of this paper is to investigate the existence of α-equilibrium of minimax inequalities for functions with noncompact domain in topological spaces without linear structure. We have achieved this goal by replacing usual quasiconcavity (quasiconvexity) with DGCQCV (DGCQCX) condition. Subsequently, by using the existence results of solutions for minimax inequalities, we prove the existence of saddle points in topological spaces without linear structure, and the existence of solutions to the complementarity problem in topological vector spaces. Finally, by using our results, we establish some intersection theorems in topological spaces and their equivalent forms. In our view, future research on the existence of α-equilibrium of minimax inequalities based on further weakening transfer lower semicontinuity and DGCQCV (DGCQCX) condition should be of interest. On this basis, we study the existence of Nash equilibria for generalized discontinuous games in topological spaces without linear structure.

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References

[1] R.P. Agarwal, M. Balaj, D. O'Regan, Intersection theorems for weak KKM set-valued mappings in the finite dimensional setting, Topol. Appl. 262 (2019) 64–79.
[2] R.P. Agarwal, M. Balaj, D. O'Regan, Weak KKM set-valued mappings in hyperconvex metric spaces, Filomat 35 (2021) 157–167.
[3] G. Allen, Variational inequalities, complementarity problems, and duality theorems, J. Math. Anal. Appl. 58 (1977) 1–10.
[4] M. Balaj, M.A. Khamsi, Common fixed point theorems for set-valued mappings in normed spaces, RACSAM 113 (2019) 1893–1905.
[5] M. Balaj, Existence results for quasi-equilibrium problems under a weaker equilibrium condition, Oper. Res. Lett. 49 (2021) 333–337.
[6] M. Balaj, Intersection theorems for generalized weak KKM set-valued mappings with applications in optimization, Math. Nach. 294 (2021) 1262–1276.
[7] M. Baye, G.Q. Tian, J.X. Zhou, Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs, Rev. Econom. Stud. 60 (1993) 935–948.
[8] K.C. Border, Fixed point theorems with applications to economics and game theory, Cambridge University Press, Cambridge, 1985.
[9] M. Castellani, M. Giuli, A generalized Ky Fan minimax inequality on finite-dimensional spaces, J. Optim. Theory Appl. 190 (2021) 343–357.
[10] S.S. Chang, Y. Zhang, Generalized KKM theorem and variational inequality and its applications, J. Math. Anal. Appl. 159 (1991) 208–223.
[11] S.Y. Chang, Inequalities and Nash equilibria, Nonlinear Anal. 73 (2010) 2933–2940.
[12] J. Cotrina, J. Zúñiga, A note on quasi-equilibrium problems, Oper. Res. Lett. 46 (2018) 138–140.
[13] J. Cotrina, A. Svensson, The finite intersection property for equilibrium problems, J. Glob. Optim. 79 (2021) 941–957.
[14] X.P. Ding, Pareto equilibria of muliticriteria games without compactness, continuity, and concavity, Appl. Math. Mech. 17 (1996) 847–854.
[15] K. Fan, A minimax inequality and its applications, in "Inequalities III" (O. Shisha, Ed.), Academic Press, pp. 103-113. San Diego, 1972.
[16] F. Forgö, On the existence of Nash-equilibrium in *n*-person generalized concave games, in: S. Komlósi, T. Rapcsák, S. Schaible (Eds.), Generalized Convexity: Lecture Notes in Economics and Mathematical Systems, vol. 405, Springer-Verlag, Berlin, 1994.
[17] C.W. Ha, A non-compact minimax theorem, Pacific J. Math. 97 (1981) 115–117.
[18] J.C. Hou, Characterization of the existence of a pure-strategy Nash equilibrium, Appl. Math. Lett. 22 (2009) 689–692.
[19] W.K. Kim, K.H. Lee, On the existence of Nash equilibrium in *N*-person games with *C*-concavity, Computers and Math. Appl. 44 (2002) 1219–1228.
[20] W.K. Kim, S. Kum, Existence of Nash equilibria with *C*-concavity, Nonlinear Anal. 63 (2005) 1857–1865.
[21] W.K. Kim, K.H. Lee, Nash equilibrium and minimax theorem with *C*-concavity, J. Math. Anal. Appl. 328 (2007) 1206–1216.
[22] W.K. Kim, Generalized *C*-concave conditions and their applications, Acta Math. Hungar. 130 (2011) 140–154.
[23] L.J. Lin, Applications of a fixed point theorem in *G*-convex spaces, Nonlinear Anal. 46 (2001) 601–608.
[24] H.S. Lu, D.S. Tang, An intersection theorem in *L*-convex spaces with applications, J. Math. Anal. Appl. 312 (2005) 343-356.
[25] M. Nasri, W. Sosa, Equilibrium problems and generalized Nash games, Optimization 60 (2011) 1161–1170.
[26] P. Prokopovych, On equilibrium existence in payoff secure games, Econom. Theory 48 (2011) 5–16.
[27] P. Reny, On the existence of pure and mixed strategy equilibria in discontinuous games, Econometrica 67 (1999) 1029–1056.
[28] V. Scalzo, Equilibrium existence in games: slight single deviation property and Ky Fan minimax inequality, J. Math Econ. 82 (2019) 197–201.
[29] G.Q. Tian, Generalizations of the FKKM theorem and Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity, J. Math. Anal. Appl. 170 (1992) 457–471.
[30] G.Q. Tian, J.X. Zhou, Quasi-inequalities without the concavity assumption, J. Math. Anal. Appl. 172 (1993) 289–299.
[31] G.Q. Tian, Full characterizations of minimax inequality, fixed point theorem, saddle point theorem, and KKM principle in arbitrary topological spaces, J. Fixed Point Theory Appl. 19 (2017) 1679–1693.
[32] K.K. Tan, J. Yu, New minimax inequality with applications to existence theorems of equilibrium points, J. Optim. Theory Appl. 82 (1994) 105–120.
[33] Z. Yang, Y.J. Pu, Generalized Knaster-Kuratowski-Mazurkiewicz theorem without convex hull, J. Optim. Theory Appl. 154 (2012) 17–29.
[34] J. Yu, On Nash equilibrium in *N*-person games over relefexive Banach spaces, J. Optim. Theory Appl. 73 (1992) 211–214.
[35] G.X.Z. Yuan, G. Isac, K.K. Tan, J. Yu, The study of minimax inequalities, abstract economics and applictions to variational inequalities and Nash equilibria, Acta Appl. Math. 54 (1998) 135–166.
[36] J.X. Zhou, G. Chen, Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, J. Math. Anal. Appl. 132 (1988) 213–225.